Superalgebras for the Penning Trap

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The hamiltonian describing a single fermion in a Penning trap is shown to be supersymmetric in certain cases. The supersymmetries of interest occur when the ratio of the cyclotron frequency to the axial frequency is $3/2$ and the particle has anomalous magnetic moment $4/3$ or $2/3$. At these supersymmetric points, the spectrum shows uniformly spaced crossed levels. The associated superalgebras are $\text{su}(2|1)$ and $\text{su}(1|1)$. The phase space for this problem has an $\text{osp}(2|6)$ structure and contains all the degeneracy superalgebras.
1. Introduction

The Penning trap [1, 2] is an impressive tool for precision spectroscopy of charged particles. High-precision measurements conducted on particles in a Penning trap include a comparison of the anomalous magnetic moments for the electron and positron to a precision of $10^{-12}$ [3], a measurement of the charge-to-mass ratio for protons and antiprotons to $10^{-10}$ [4], and a search for time dependence in the anomaly frequency of a trapped electron [5]. Comparable precisions have been attained in measurements of the mass ratio of the proton to the electron [6], the masses of molecular ions [7], and bounds on the anisotropy of space [8]. Recent theoretical investigations indicate that Penning-trap experiments can constrain Lorentz and CPT violation at the level of $10^{-20}$ in the context of a general standard-model extension [9]. Numerous other applications of Penning traps exist [10].

In the present paper, we investigate the symmetries of the hamiltonian describing a single charged fermionic particle confined in a Penning trap with hyperbolic electrodes. The symmetry depends on the relative values of the magnetic and electric fields and on the gyromagnetic ratio of the trapped particle. For certain values of these parameters, superalgebras [11] arise.

There are relatively few physical manifestations of superalgebras. One arises in nuclear physics [12]. Another exists in atomic systems [13, 14], where a broken quantum-mechanical supersymmetry has been shown to underly the properties of the chemical elements. It has recently been suggested that a supersymmetry also exists in the context of traps [15]. In this case, a radial supersymmetry for the trap wave functions provides a description of a small cloud of particles in a trap via an effective single-particle formalism. The associated parallels between traps and atoms in the context of quantum-mechanical supersymmetry have been studied in some detail [16]. Some other results in quantum-mechanical supersymmetry are reviewed in [17].

The supersymmetries discussed in this paper for the Penning trap are of a different type. The idea is to consider the full hamiltonian written in terms of creation
and annihilation operators. The (anti)commutation relations satisfied by quadratic combinations of these operators define the superalgebras relevant to the problem.

In section 2, the relevant features of the Penning trap are reviewed and some definitions are given. The relative strengths of the trapping fields required for degeneracies to occur are discussed in section 3. The central algebra common to all cases is given in section 4, and each of the five relevant superalgebras are presented in turn in sections 5 to 9. Section 10 summarizes and discusses the results.

2. The Penning Trap

In most situations, the dynamics of a particle in a Penning trap is dominated by its interaction with a uniform magnetic field $\mathbf{B}$. For convenience, we work in cylindrical coordinates $(\rho, \phi, z)$ with $\mathbf{B} = B\hat{z}$. A suitable choice of vector potential is $\mathbf{A} = (B\rho/2)\hat{\phi}$.

The quadrupole electric field of the trap is produced by electrodes in one of several possible forms [18, 19]. We restrict attention to the case with electrode surfaces given in cylindrical coordinates by the expressions

$$z^2 = \frac{\rho^2}{2} \pm d^2,$$

where $d$ is a constant. The upper equation is a hyperboloid of two sheets and describes the endcap surfaces, which intersect the $z$ axis at $z = \pm d$ and have potential $V/2$. The remaining electrode surface has potential $-V/2$ and has shape determined by the lower sign in Eq. (1). It is a hyperboloid of one sheet encircling the $z$ axis with waist radius $\sqrt{2}d$ in the $z = 0$ plane. The electrostatic potential is

$$\phi(\rho, \phi, z) = \frac{V}{2d^2}(z^2 - \rho^2/2)$$

in the trapping region.

Let the trapped particle have charge $q$ and mass $m$. We assume that $q$ and $V$ have the same sign, thereby ensuring axial trapping. Defining the axial frequency $\omega_z = (qV/md^2)^{1/2}$ and the cyclotron frequency $\omega_c = |qB|/m$, the hamiltonian for $q > 0$ is

$$\tilde{H} = -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{8}m\Omega^2 \rho^2 + \frac{1}{2}m\omega_z^2 z^2 + \frac{1}{2}\hbar \omega_c i \frac{\partial}{\partial \phi},$$
where $\Omega = (\omega^2 - 2\omega_z^2)^{1/2}$. For $q < 0$, the last term would be negative. The algebraic structure of the problem turns out to be independent of the sign of $q$, and to avoid carrying two signs in the expressions that follow, we restrict attention to the case $q > 0$.

Equation (3) separates by defining $\Psi(\rho, \phi, z) \equiv (r_0/\rho)^{1/2}W(\rho)\Theta(\theta, z)$, where $r_0 = (\hbar/m\omega_c)^{1/2}$. The equation in $\rho$ is

$$\begin{cases} \frac{-\hbar^2}{2m} \frac{d^2}{d\rho^2} + \frac{\hbar^2}{2m} \frac{(\hat{M}^2 - \frac{1}{4})}{\rho^2} + \frac{1}{8}m\Omega^2 \rho^2 - \left[ E - (\hat{K} + \frac{1}{2})\hbar\omega_z + \frac{1}{2}\hat{M}\hbar\omega_z \right] \end{cases} W(\rho) = 0 \quad ,$$

where $\hat{M}$ and $\hat{K}$ are separation constants taking values $\hat{M} = 0, \pm 1, \pm 2, \ldots$ and $\hat{K} = 0, 1, 2, \ldots$. The energy eigenvalues $E$ for this problem are

$$E_{N,\hat{K},\hat{M}} = \frac{\hbar}{2} \left[ \Omega N + 2\omega_z \hat{K} - \omega_c \hat{M} + (\Omega + \omega_z) \right] \quad ,$$

where $N$ takes values $N = |\hat{M}|, |\hat{M}|+2, |\hat{M}|+4, \ldots$. The full solution to the stationary problem $\tilde{H}\Psi = E\Psi$ involves generalized Laguerre and Hermite polynomials,

$$\Psi_{N,\hat{K},\hat{M}}(\rho, \phi, z) = C_{N,\hat{K},|\hat{M}|}(\frac{\rho}{r_0})^{|\hat{M}|}\exp\left[ -\frac{k}{4} \left( \frac{\rho}{r_0} \right)^2 - \frac{1}{2} \left( \frac{z}{s_0} \right)^2 + i\hat{M}\phi \right] \times L_{\frac{N/2-|\hat{M}|}{2}}(\frac{k}{2} \left( \frac{\rho}{r_0} \right)^2) H_{\frac{\hat{K}}{2}} \left( \frac{z}{s_0} \right) \quad ,$$

where $k = \Omega/\omega_c$, $s_0 = (\hbar/m\omega_z)^{1/2}$, and the normalization coefficient is

$$C_{N,\hat{K},|\hat{M}|} = \left[ \frac{\sqrt{k}}{r_0^2 s_0 2^{K}\pi^{3/2}} \left( \frac{k}{2} \right)^{|\hat{M}|+1/2} \frac{\Gamma \left( \frac{N}{2} - |\hat{M}| \right) + 1}{\Gamma \left( \frac{N}{2} + |\hat{M}| \right) + 1} \frac{\Gamma \left( \frac{N}{2} + |\hat{M}| \right) + 1}{\Gamma \left( \frac{N}{2} + |\hat{M}| + 1 \right) \Gamma \left( \hat{K} + 1 \right)} \right]^{1/2} \quad .$$

For the special case $k = 0$, the coefficient of the $\rho^2$ term in Eq. (4) would vanish and the above solutions would change. We exclude this case because it does not allow long-term confinement. In the initial stages of trapping before significant cooling has occurred, the motion of the particle can be understood classically. The possible trajectories are either circles about the central axis or curves that exit the trap. The former are unstable to radial perturbations. We therefore restrict attention to the range of values $0 < k \leq 1$, or, equivalently, $0 < \Omega \leq \omega_c$. 

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The hamiltonian $\tilde{H}$ can be expressed in terms of creation and annihilation operators. A transformation of the phase space yields six dimensionless operators

$$\begin{align*}
a, a^\dagger &= \frac{\hbar}{\sqrt{2k}} (\pm \partial_x + i \partial_y) + \sqrt{\frac{k}{8 r_0}} (x \pm iy), \\
b, b^\dagger &= \frac{\hbar}{\sqrt{2k}} (\mp \partial_x + i \partial_y) - \sqrt{\frac{k}{8 r_0}} (x \mp iy), \\
c, c^\dagger &= \pm \frac{s_0}{\sqrt{2}} \partial_z + \frac{1}{\sqrt{2s_0}} z.
\end{align*}$$

(8)

They commute with each other except for the cases

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \quad [c, c^\dagger] = 1.$$

(9)

The transformation (8) preserves the canonical properties of the phase space, including the commutation relations for the momentum and position operators. Therefore, it is symplectic [24].

The symplectic transformation casts the hamiltonian into the form

$$\tilde{H} = \hbar \omega_+ (a^\dagger a + \frac{1}{2}) - \hbar \omega_-(b^\dagger b + \frac{1}{2}) + \hbar \omega_z (c^\dagger c + \frac{1}{2}),$$

(10)

where $\omega_+ = (\omega_c + \Omega)/2$ and $\omega_- = (\omega_c - \Omega)/2$ are called the modified cyclotron frequency and the magnetron frequency, respectively. The negative sign in Eq. (10) reveals an inverted oscillator in the system, which in principle could lead to an instability in the presence of radiation. However, in practical situations this energy loss is controlled by ensuring $\omega_+ \gg \omega_-$, so particles may be trapped “indefinitely” [18].

For particles with spin 1/2, a term $H'$ must be added to the hamiltonian (3),

$$H' \equiv -\vec{\mu} \cdot \vec{B} = -\frac{g}{4} \hbar \omega_c \sigma_3,$$

(11)

where $g$ is the Landé factor relating the spin to the magnetic dipole moment and $\sigma_3$ is the third Pauli matrix. The operators $f \equiv (\sigma_1 + i \sigma_2)/2$ and $f^\dagger \equiv (\sigma_1 - i \sigma_2)/2$ have one nonzero anticommutation relation,

$$\{f, f^\dagger\} \equiv ff^\dagger + f^\dagger f = 1,$$

(12)

and they provide a formalism for describing the spin degree of freedom. The additional term in the hamiltonian is $H' = \hbar \omega_g (f^\dagger f - \frac{1}{2})$, where $\omega_g = |g| \omega_c/2$. The sign of this term assumes $gg > 0$. 

4
Combining the bosonic and fermionic degrees of freedom we obtain the full hamiltonian $H \equiv \tilde{H} + H'$ in operator form:

$$H = \hbar \omega_+ (a^\dagger a + \frac{1}{2}) - \hbar \omega_- (b^\dagger b + \frac{1}{2}) + \hbar \omega_z (c^\dagger c + \frac{1}{2}) + \hbar \omega_g (f^\dagger f - \frac{1}{2}) \quad .$$ (13)

The basis states for this problem can be denoted by $|N_a, N_b, N_c, N_f\rangle$, where $N_a, N_b, N_c \in \{0, 1, 2, \ldots\}$ are the eigenvalues of the number operators $a^\dagger a, b^\dagger b$ and $c^\dagger c$, and where $N_f \in \{0, 1\}$ is the eigenvalue of $f^\dagger f$.

The energy eigenvalues of the system follow from Eq. (13):

$$E(N_a, N_b, N_c, N_f; \omega_c, \omega_z, g)/\hbar \equiv \omega_+(N_a + \frac{1}{2}) - \omega_-(N_b + \frac{1}{2}) + \omega_z (N_c + \frac{1}{2}) + \omega_g (N_f - \frac{1}{2}) \quad .$$ (14)

The quantum numbers used here are related to the ones in Eq. (5) by $N_a = (N - \hat{M})/2$, $N_b = (N + \hat{M})/2$, and $N_c = \hat{K}$.

The relative values of the frequencies in equation (14) play an important part in the superalgebra structures considered below. To this end, it is useful to define the ratio $\sigma$ of the cyclotron and axial frequencies,

$$\sigma \equiv \frac{\omega_c}{\omega_z} = \left(\frac{qB^2d^2}{mV}\right)^{1/2} \quad .$$ (15)

This parameter contains information about the relative values of $B$ and $V$. For experiments with single trapped electrons, typical values [18] are $d \simeq 0.3$ cm, $B \simeq 6$ T, and $V \simeq 10$ V, giving $\sigma \simeq 3 \times 10^3$. In this limit of $\sigma \gg 1$, the motion of the trapped particle is dominated by its interaction with the magnetic field, and Eq. (14) becomes

$$\lim_{\sigma \to \infty} E(N_a, N_b, N_c, N_f; \omega_c, \omega_z, g) = \hbar \omega_c \left[ (N_a + \frac{1}{2}) g N_f - \frac{1}{2} \left( \frac{2 \epsilon}{q}\right) \right] \quad .$$ (16)

For experiments with single trapped protons, typical values [18] are $d \simeq 0.1$ cm, $B \simeq 5$ T, and $V \simeq 50$ V, giving the lower value $\sigma \simeq 8$. As $\sigma$ is decreased, the confining effect of the magnetic field is weakened, and trapping becomes impractical when $\sigma = \sqrt{2}$. This corresponds to the excluded case $k = 0$. Exceptional measurement precisions are possible: for trapped protons, cyclotron-frequency precisions are at the
90 parts per trillion level [4], making it feasible to probe minuscule effects such as Lorentz violation [20].

3. Degeneracy superalgebras and frequency equalities

The algebraic structures that arise for the single-particle Penning trap are superalgebras because both fermionic and bosonic operators are involved. We focus on degeneracy superalgebras formed from operators that commute with the Hamiltonian, thereby linking degenerate eigenstates.

All the symmetries we consider are based on the Hamiltonian (13). Superalgebras arise for special values of the two parameters $g$ and $\sigma$, which in turn determine the four characteristic frequencies $\omega_\pm$, $\omega_z$, and $\omega_g$ up to an overall factor. As an illustrative example, consider the case of $g = 2/3$ and $\sigma = 3/2$. The Penning-trap Hamiltonian is

$$H/\hbar = (a^\dagger a + c^\dagger c + 1) - \frac{1}{2}(b^\dagger b - f^\dagger f + 1),$$

and there are two distinct frequencies, $\omega_+ = \omega_z = 2\omega_\pm = 2\omega_g$. The generator $b^\dagger f$ increases $N_b$ by one unit while decreasing $N_f$ by the same amount. It commutes with the Hamiltonian because of the equality of $\omega_g$ and $\omega_-$.

In the most general case, $\omega_\pm$, $\omega_z$, and $\omega_g$ are distinct. There are four generators constructed from quadratic combinations of creation and annihilation operators that commute with the Hamiltonian: $a^\dagger a$, $b^\dagger b$, $c^\dagger c$, and $f^\dagger f$. They generate an abelian algebra $u(1) \times u(1) \times u(1) \times u(1)$ and form a complete set of commuting operators. Their interpretation as constants of the motion is considered in the next section. The generators of this abelian algebra commute with the Hamiltonian and with any other degeneracy operators regardless of the values of $g$ and $\sigma$. Therefore, all the degeneracy superalgebras considered below contain this four-dimensional central algebra.

Even with four distinct frequencies, degeneracies in the energies can occur. Consider the case of $\sigma = 9/4$ and $g = 2/3$. The corresponding Hamiltonian is

$$H/\hbar = 2(a^\dagger a + \frac{1}{2}) - \frac{1}{4}(b^\dagger b + \frac{1}{2}) + (c^\dagger c + \frac{1}{2}) + \frac{3}{4}(f^\dagger f - \frac{1}{2}).$$

The point is that the associated frequencies are all rational multiples of each other. By taking combinations higher than quadratic in the creation or annihilation operators,
generators can be constructed that commute with the Hamiltonian. Take, for example, the operator $a^\dagger c^2$. It increases $N_a$ by one unit and decreases $N_c$ by two units. This ensures commutation with the Hamiltonian because the associated frequencies $\omega_+$ and $\omega_z$ are in the ratio $2 : 1$. Other generators that commute with this Hamiltonian are $a(c^\dagger)^2$, $(b^\dagger)c^c$, $ab^8$, $a^\dagger(b^\dagger)^8$, and $bcf^\dagger$. A detailed study of the algebraic structures associated with cubic and higher combinations of creation or annihilation operators lies beyond the scope of the present work.

Next, consider the case of three distinct frequencies. For a superalgebra to arise, $\omega_g$ must be equated with another frequency. We give a few examples. For $\sigma = 9/4$ and $g = 2/9$, we find that the ratio $\omega_+ : \omega_- : \omega_z : \omega_g$ is $8 : 1 : 4 : 1$, so that $\omega_g = \omega_-$. For $\sigma = 11/6$ and $g = 18/11$, the frequency ratio is $9 : 2 : 6 : 9$, so that $\omega_g = \omega_+$. For $\sigma = 9/4$ and $g = 8/9$, the frequency ratio is $8 : 1 : 4 : 4$, so that $\omega_g = \omega_z$. The superalgebras that arise are all isomorphic and are discussed in section 5.

Next, consider ways in which the single-particle Penning-trap system can have two distinct characteristic frequencies in a rational ratio. Of these, we focus on the simplest possible ratio, $2 : 1$. It turns out that there are only two cases. One arises for $g = 2/3$ and $\sigma = 3/2$ and the corresponding Hamiltonian is given in Eq. (17). This case is considered in section 6. The other arises for $\sigma = 3/2$ and $g = 4/3$. It is the intersection point of the curves $\omega_+$, $\omega_z$, and $\omega_g$ as functions of $\sigma$, and is illustrated in Figure 1. For this case, the frequencies are $\omega_+ = \omega_z = \omega_g = 2\omega_-$ and the associated supersymmetries are considered in detail in section 7.

It is not possible to equate all four frequencies to yield a single characteristic frequency for the system. This can be seen in Figure 1, which shows that $\omega_z$ cannot equal $\omega_-$. The two cases with two distinct characteristic frequencies are special. They represent the largest possible superalgebras that can be constructed from quadratic generators for the single-particle Penning trap. Both cases have $\sigma = 3/2$, but differ in
the values of $g$.

4. Constants of the motion for the supersymmetric configuration

For the supersymmetric point $\sigma = 3/2$, the hamiltonian can be written in terms of four constants of the motion $H_\rho, H_\phi, H_z, \text{and } H_f$ to be defined below:

$$H = H_\rho + H_\phi + H_z + H_f \quad .$$

These operators have simple physical interpretations.

The first one is the energy operator of a harmonic oscillator in the $xy$ plane with frequency $\omega_z/4$:

$$H_\rho \equiv -\frac{\hbar^2}{2m} \left( \partial_\rho^2 + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \partial_\phi^2 \right) + \frac{1}{2} m \left( \frac{\omega_z}{4} \right)^2 \rho^2$$

$$= \frac{\hbar \omega_z}{4} \left( a^\dagger a + b^\dagger b + 1 \right) \quad .$$

(20)

The operator $H_\phi$ is a rotational energy about the $z$ axis:

$$H_\phi \equiv \frac{1}{2} \hbar \omega_c i \partial_\phi = -\frac{1}{2} \omega_c L_z \quad ,$$

where $\omega_c = 3\omega_z/2$. This term has negative eigenvalues for $L_z$ in the $+z$ direction. This is consistent with the presence of an inverted harmonic oscillator in the Penning trap. The angular momentum about the $z$ axis can be expressed in terms of the creation and annihilation operators [21] as

$$L_z = \hbar (b^\dagger b - a^\dagger a) \quad .$$

(22)

The operator $H_z$ is the energy operator of a harmonic oscillator with frequency $\omega_z$ on the $z$ axis:

$$H_z \equiv -\frac{\hbar^2}{2m} \partial_z^2 + \frac{1}{2} m \omega_z^2 z^2 = \hbar \omega_z \left( c^\dagger c + \frac{1}{2} \right) \quad .$$

(23)

The operator $H_f$ is the energy operator for the splitting between the two spin projections onto the $z$ axis:

$$H_f \equiv \hbar \omega_g \left( f^\dagger f - \frac{1}{2} \right) \quad .$$

(24)
The four operators $H_\rho$, $H_\phi$, $H_z$ and $H_f$ form an alternative complete set of commuting operators for the single-particle Penning trap. They form a basis of the abelian center of all the degeneracy superalgebras for this system, and their associated energies are independent of each other.

5. Three distinct frequencies

For this case, $\omega_g$ must equal one of the other frequencies and the remaining two frequencies must each be distinct from this value and from each other. This can occur in numerous ways. As an example, consider the case with $\sigma = 11/6$ and $g = 18/11$ mentioned in section 3. The Hamiltonian is

$$H/\hbar \omega_z = \frac{3}{2}(a^\dagger a + f^\dagger f) - \frac{1}{3}(b^\dagger b + \frac{1}{2}) + (c^\dagger c + \frac{1}{2}).$$

(25)

Define the operators

$$J \equiv a^\dagger a + f^\dagger f,$$

$$\overline{J} \equiv a^\dagger a - f^\dagger f + 1,$$

$$F_{+1} \equiv a^\dagger f,$$

$$F_{-1} \equiv af^\dagger.$$  

(26)

Note from Eq. (8) that they depend on the value of $k$, and that for this case $k = \sqrt{\sigma^2 - 2/\sigma} = 7/11$. They commute with the Hamiltonian and generate a superalgebra. The only nonzero relations are

$$[\overline{J}, F_{\pm 1}] = \pm 2F_{\pm 1}, \quad \{F_{+1}, F_{-1}\} = J.$$  

(27)

This algebra has a nontrivial ideal spanned by $J, F_{\pm 1}$ and so is not simple. The ideal is the nilpotent superalgebra $su(1|1)$ with Lie part $u(1)$ generated by $J$. The operator $\overline{J}$ does not commute with the odd operators $F_{\pm 1}$, so we denote the superalgebra by $u(1) \otimes su(1|1)$ to indicate the absence of a direct product.

The full degeneracy algebra for the Hamiltonian Eq. (25) includes elements which complete the basis of the center. The structure is $u(1) \times u(1) \times u(1) \otimes su(1|1)$, generated by $b^\dagger b, c^\dagger c, \overline{J}$, and $\{J, F_{\pm 1}\}$.

Given a pair $F_{\pm 1}$ of mutually hermitian-conjugate generators, (self-)hermitian generators are obtained by the combinations $T_1 = (F_{+1} + F_{-1})/2$ and $T_2 = i(F_{+1} - F_{-1})/2$. 

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We define nonhermitian ladder generators because they are useful for calculations. The actual hermitian generators within the superalgebras can always be constructed by this method.

Another way to obtain a supersymmetry with three distinct frequencies in the system is to set $\omega_g = \omega_-$. Consider the example mentioned in section 3 with $g = 2/9$ and $\sigma = 9/4$, which corresponds to $k = 7/9$. The hamiltonian is

$$H/\hbar \omega_z = 2(a^\dagger a + \frac{j}{2}) - \frac{1}{4}(b^\dagger b - f^\dagger f + 1) + (c^\dagger c + \frac{1}{2}) .$$ \hspace{1cm} (28)

We define four operators that commute with the hamiltonian:

$$K \equiv b^\dagger b + f^\dagger f , \quad \overline{K} \equiv b^\dagger b - f^\dagger f + 1 , \quad F_{+2} \equiv b^\dagger f^\dagger , \quad F_{-2} \equiv bf .$$ \hspace{1cm} (29)

They generate a superalgebra with nonzero relations

$$[K, F_{\pm 2}] = \pm 2F_{\pm 2} , \quad \{F_{+2}, F_{-2}\} = \overline{K} .$$ \hspace{1cm} (30)

Comparison of these relations with those in Eq. (27) shows that the two algebras are isomorphic. The full superalgebra for this example is $u(1) \times u(1) \times u(1) \otimes su(1|1)$, generated by $a^\dagger a, c^\dagger c, K$, and $\{K, F_{\pm 2}\}$.

One might expect different superalgebras to arise for the hamiltonians (25) and (28) because of the opposite signs of $a^\dagger a$ and $b^\dagger b$ relative to $f^\dagger f$. However, this is not the case, and the isomorphism relating the operators in Eq. (26) and Eq. (29) is given explicitly by

$$a \leftrightarrow b , \quad a^\dagger \leftrightarrow b^\dagger , \quad f \leftrightarrow f^\dagger .$$ \hspace{1cm} (31)

It follows from this observation that the only superalgebra that can arise for three distinct frequencies is $u(1) \otimes su(1|1)$. In all cases of this type, the full symmetry is $u(1) \times u(1) \times u(1) \otimes su(1|1)$.

This supersymmetry is relevant to experiments with electrons or positrons, where $\sigma \gg 1$ and $g \simeq 2$. Taking $g = 2$, the hamiltonian for $\sigma \gg 1$ is

$$H/\hbar \approx \omega_z(a^\dagger a + f^\dagger f) + \omega_z(c^\dagger c + \frac{j}{2}) - \omega_-(b^\dagger b + \frac{j}{2}) .$$ \hspace{1cm} (32)
with $\omega_c \gg \omega_z \gg \omega_-$. However, the supersymmetry is broken because in the physical situation $g$ is slightly larger than two, so $\omega_+$ is always slightly less than $\omega_g$ no matter how strong the magnetic field. The value of the $g$ factor determines the degree to which this supersymmetry is broken in the strong-$B$ limit. In this regime, the particle experiences a uniform magnetic field and has associated supercoherent states [22]. If $g$ were exactly equal to 2, the anomaly $\alpha_e = (g - 2)/2 \simeq 10^{-3}$ would be zero, and the spin-up and spin-down ladders would have no relative energy shift [23].

6. Two pairs of equal frequencies: $\omega_z = 2\omega_z = 2\omega_g$

For $g = 2/3$ and $\sigma = 3/2$, the Penning-trap hamiltonian is given in Eq. (17). Four linearly independent generators constructed from $a, a^\dagger, c$ and $c^\dagger$ that commute with this hamiltonian are

$$\mathcal{L} \equiv a^\dagger a + c^\dagger c + 1,$$
$$L \equiv \frac{1}{2}(a^\dagger a - c^\dagger c), \quad E_{+2} \equiv a^\dagger c, \quad E_{-2} \equiv ac^\dagger. \quad (33)$$

The generator $\mathcal{L}$ commutes with the other three, forming a $u(1)$ subalgebra. The generators $E_{+2}$ and $E_{-2}$ are hermitian conjugates and are themselves non-hermitian. They are ladder operators, which together with $L$ give the Lie algebra so(3):

$$[L, E_{\pm 2}] = \pm E_{\pm 2}, \quad [E_{+2}, E_{-2}] = 2L. \quad (34)$$

The remaining generators of the superalgebra are $K, \overline{K},$ and $F_{\pm 2}$ defined in Eq. (29), but with $k = 1/3$. They span the superalgebra $u(1) \otimes su(1|1)$ with nonzero relations given in Eq. (30).

Combining the relations of Eq. (34) and Eq. (30), the full degeneracy superalgebra for the hamiltonian in (17) is $u(1) \times so(3) \times u(1) \otimes su(1|1)$, generated by $\mathcal{T}, \{L, E_{\pm 2}\}$, $\overline{K}$, and $\{K, F_{\pm 2}\}$. It is implicit that for this case $k = 7/9$ in the definitions (29).

For $gq < 0$, the second term of the hamiltonian (17) becomes $-(b^\dagger b + f^\dagger f)/2$. A full set of generators that commute with the hamiltonian is obtained from Eq. (33) and by making the replacements $f \rightarrow f^\dagger$ and $f^\dagger \rightarrow f$ in Eq. (29). This operation is an automorphism, leaving the relations (30) and (34) unchanged.

From a given state $|N_a, N_b, N_c, N_f\rangle$, the elements defined in Eqs. (29) and (33) generate all the states in the degenerate subspace. The Lie algebra so(3) generates
states differing in the $N_a$ and $N_c$ eigenvalues. For example,

$$E_{+2}|N_a, N_b, N_c, N_f⟩ \sim |N_a + 1, N_b, N_c - 1, N_f⟩ . \quad (35)$$

In contrast, the subsuperalgebra $su(1|1)$ acts to give states differing only in $N_b$ and $N_f$. For example,

$$F_{+2}|N_a, N_b, N_c, N_f⟩ \sim ((N_f + 1) \mod 2)|N_a, N_b + 1, N_c, (N_f + 1) \mod 2⟩ . \quad (36)$$

Insight into the physical implications of the superalgebra can be gained from Figure 2. It plots the energy levels of the Penning trap versus $\sigma$ for the states with quantum numbers $N_a = 0, 1, 2$, $N_b = 0 \ldots 3$, $N_c = 0, 1$, and $N_f = 0, 1$. At $\sigma = 3/2$, the Hamiltonian has the form of Eq. (17). The coefficients of the two terms show that the frequencies are in the ratio 2:1. This gives the uniform spacing of the energy levels and creates the sharply defined crossing features at this supersymmetry point on the plot.

Figure 2 also reveals the set of evenly spaced degenerate levels at $\sigma = 2.25$, for which the Hamiltonian has the form in Eq. (18).

The operators $L, \overline{L}, K,$ and $\overline{K}$ form a complete set of commuting operators for the system. They can be expressed in terms of the more physical operators defined in section 4:

$$\hbar \omega_z \overline{L} = 2H_\rho + \frac{2}{3}H_\phi + H_z ,$$

$$\hbar \omega_z L = H_\rho + \frac{1}{3}H_\phi - \frac{1}{2}H_z ,$$

$$\hbar \omega_z \overline{K} = 2H_\rho - \frac{2}{3}H_\phi - 2H_f ,$$

$$\hbar \omega_z K = 2H_\rho + \frac{2}{3}H_\phi + 2H_f .$$

7. Three equal frequencies: $\omega_+ = \omega_z = \omega_g = 2\omega_-$

Three frequencies can be equated by setting $g = 4/3$ and $\sigma = 3/2$, giving the Hamiltonian

$$H/\hbar \omega_z = (a^\dagger a + c^\dagger c + f^\dagger f + \frac{1}{2}) - \frac{1}{2}(b^\dagger b + \frac{1}{2}) \equiv M - \frac{1}{2}M .$$

$$H/\hbar \omega_z = (a^\dagger a + c^\dagger c + f^\dagger f + \frac{1}{2}) - \frac{1}{2}(b^\dagger b + \frac{1}{2}) \equiv M - \frac{1}{2}M .$$

(41)
The generators $M$ and $\overline{M}$, defined by the expressions in parentheses, commute with each other and with $H$. They therefore form an independent $u(1) \times u(1)$ subalgebra of the full degeneracy superalgebra.

In addition to $M$ and $\overline{M}$, there are four independent even elements given by

$$\tilde{L} \equiv \frac{1}{2}(a^\dagger a + c^\dagger c) + f^\dagger f$$

(42)

and by $L, E_{\pm 2}$ defined in Eq. (33). The generator $\tilde{L}$ commutes with the even elements $L, E_{\pm 2}$, which in turn satisfy the commutation relations (34) for the compact Lie algebra $su(2)$.

There are four odd elements that commute with the hamiltonian: $F_{\pm 1}$ as defined in Eq. (26) but with $k = 1/3$, and

$$F_{+3} \equiv c^\dagger f , \quad F_{-3} \equiv cf^\dagger .$$

(43)

Their nonzero anticommutation relations are

$$\{F_{+1}, F_{-1}\} = \tilde{L} + L , \quad \{F_{+3}, F_{-3}\} = \tilde{L} - L , \quad \{F_{\pm 1}, F_{\mp 3}\} = E_{\pm 2} ,$$

(44)

and their nonzero commutation relations with the even elements are

$$[\tilde{L}, F_{\pm 1}] = \mp \frac{1}{2} F_{\pm 1} , \quad [\tilde{L}, F_{\pm 3}] = \mp \frac{1}{2} F_{\pm 3} ,$$

$$[L, F_{\pm 1}] = \pm \frac{1}{2} F_{\pm 1} , \quad [L, F_{\pm 3}] = \mp \frac{1}{2} F_{\pm 3} ,$$

$$[E_{\pm 2}, F_{\pm 3}] = \pm F_{\pm 1} , \quad [E_{\pm 2}, F_{\mp 1}] = \mp F_{\mp 3} .$$

(45)

The superalgebra with generators given in Eqs. (33), (42), and (43) is $su(2|1)$, with Lie part $u(1) \times su(2)$. The first component is generated by $\tilde{L}$ and the second by $\{L, E_{\pm 2}\}$. The action of these generators on the eigenstates of the hamiltonian is similar to that displayed in Eqs. (35) and (36), except that here the values of $N_a, N_c$ and $N_f$ are affected.

The full degeneracy structure of the hamiltonian (41) is $u(1) \times u(1) \times su(2|1)$. It has three subalgebras, generated by the sets $\{M\}$, $\{\overline{M}\}$, and $\{L, \tilde{L}, E_{\pm 2}, F_{\pm 1}, F_{\pm 3}\}$.

The hamiltonian for $gq < 0$ is found by replacing $(f^\dagger f - 1/2) \rightarrow -(f^\dagger f - 1/2)$ in (41). To obtain the generators commuting with this hamiltonian, the replacements
$f \to f^\dagger$ and $f^\dagger \to f$ are made in the definitions for all the operators. This automorphism leaves unchanged the superalgebra relations. Thus, the algebraic structure is again independent of the sign of $gq$ for the trapped particle.

Figure 3 plots the energy levels versus $\sigma$ for the states with quantum numbers $N_a = 0, 1, 2$, $N_b = 0, 1, 2$, $N_c = 0, 1$, and $N_f = 0, 1$. Because the frequencies are in a rational ratio, the supersymmetry point has uniformly spaced crossings at $\sigma = 3/2$.

The operators $M, M, L$, and $L$ form a complete set of commuting operators for the system. They can be expressed in terms of the alternative basis of section 4:

\begin{align*}
\hbar \omega_z M &= 2 H_\rho - \frac{2}{3} H_\phi, \\
\hbar \omega_z M &= 2 H_\rho + \frac{2}{3} H_\phi + H_z + H_f, \\
\hbar \omega_z \bar{L} &= H_\rho + \frac{1}{3} H_\phi + \frac{1}{2} H_z + H_f, \\
\hbar \omega_z L &= H_\rho + \frac{1}{3} H_\phi - \frac{1}{2} H_z.
\end{align*}

These expressions can be inverted. For example, the spin-splitting operator $H_f$ can be shown to be $H_f = \hbar \omega_z (2 \bar{L} - M)$.

8. Hypothetical case of four equal frequencies

The largest possible degeneracy superalgebra in a system of the form of (13) would arise if all the frequencies could be set equal. No choices of $g$ and $\sigma$ allow this in the Penning trap, as can be seen from Figure 1. Nonetheless, it is of interest to consider the degeneracy superalgebra that would arise from a hamiltonian of the form

$$H_0 \equiv a^\dagger a - b^\dagger b + c^\dagger c + f^\dagger f,$$  

where $f$ and $f^\dagger$ are fermionic and the other operators are bosonic, because this superalgebra contains all the superalgebras discussed in sections 5, 6, and 7 as subsuperalgebras. This superalgebra is $u(1) \times su(2, 1|1)$, as shown below.

The hamiltonian $H_0$ forms an independent $u(1)$ subalgebra by definition. There are eight other independent generators commuting with this hamiltonian that are constructed only from bosonic operators. Expressed in the Cartan-Weyl basis, they
are $E_{\pm 2}$ already defined in Eq. (33), and

$$
H_1 \equiv b^\dagger b + c^\dagger c + 1 , \quad H_2 \equiv a^\dagger a + b^\dagger b + 1 ,
$$

$$
E_{+1} \equiv b^\dagger c^\dagger , \quad E_{-1} \equiv bc ,
$$

$$
E_{+3} \equiv a^\dagger b^\dagger , \quad E_{-3} \equiv ab ,
$$

and they satisfy the nonzero commutation relations

$$
[H_1, E_{\pm 1}] = \pm 2E_{\pm 1} , \quad [H_1, E_{\pm 2}] = \mp E_{\pm 2} , \quad [H_1, E_{\pm 3}] = \pm E_{\pm 3} ,
$$

$$
[H_2, E_{\pm 1}] = \pm E_{\pm 1} , \quad [H_2, E_{\pm 2}] = \mp E_{\pm 2} , \quad [H_2, E_{\pm 3}] = \pm 2E_{\pm 3} ,
$$

$$
[E_{\pm 2}, E_{\mp 3}] = \mp E_{\pm 1} , \quad [E_{\pm 3}, E_{\mp 1}] = \mp E_{\pm 2} , \quad [E_{\pm 1}, E_{\pm 2}] = \mp E_{\pm 3} ,
$$

$$
[E_{+1}, E_{-1}] = -H_1 , \quad [E_{+2}, E_{-2}] = -H_1 + H_2 , \quad [E_{+3}, E_{-3}] = -H_2 .
$$

These generators provide a description of the Lie algebra $\text{su}(2, 1)$.

Including the two fermionic operators $f$ and $f^\dagger$ allows the introduction of seven
more generators that commute with the hamiltonian (50), of which one,

$$
H_3 \equiv a^\dagger a - b^\dagger b + c^\dagger c + 3f^\dagger f - 1 ,
$$

is even and commutes with the eight other even generators. The six others are odd generators defined earlier: $F_{\pm 1}$, $F_{\pm 2}$, and $F_{\pm 3}$. They satisfy anticommutation relations, of which the only nonzero ones are

$$
\{F_{\pm 2}, F_{\mp 3}\} = E_{\pm 1} , \quad \{F_{\pm 1}, F_{\mp 3}\} = E_{\pm 2} , \quad \{F_{\pm 1}, F_{\pm 2}\} = E_{\pm 3} ,
$$

$$
\{F_{+1}, F_{-1}\} = -\frac{1}{3}H_1 + \frac{2}{3}H_2 + \frac{1}{3}H_3 ,
$$

$$
\{F_{+2}, F_{-2}\} = \frac{5}{3}H_1 - \frac{2}{3}H_2 - \frac{1}{3}H_3 ,
$$

$$
\{F_{+3}, F_{-3}\} = \frac{2}{3}H_1 - \frac{1}{3}H_2 + \frac{1}{3}H_3 .
$$

Note that these anticommutators yield elements within the even part of the superalgebra, as expected. Commutation relations between even and odd generators produce generators in the odd part of the superalgebra. The nonzero cases are

$$
[H_3, F_{\pm 1}] = \mp 2F_{\pm 1} , \quad [H_3, F_{\pm 2}] = \pm 2F_{\pm 2} , \quad [H_3, F_{\pm 3}] = \mp 2F_{\pm 3} ,
$$

$$
[H_1, F_{\pm 2}] = \pm F_{\pm 2} , \quad [H_1, F_{\pm 3}] = \pm F_{\pm 3} ,
$$

$$
[H_2, F_{\pm 1}] = \pm F_{\pm 1} , \quad [H_2, F_{\pm 2}] = \pm F_{\pm 2} ,
$$

$$
[E_{\pm 1}, F_{\mp 2}] = \mp F_{\pm 3} , \quad [E_{\pm 1}, F_{\mp 3}] = \mp F_{\pm 2} ,
$$

$$
[E_{\pm 3}, F_{\mp 1}] = \mp F_{\pm 2} , \quad [E_{\pm 3}, F_{\mp 2}] = \mp F_{\pm 1} ,
$$

(55)
and the last two relations of (45). The fifteen-dimensional superalgebra $su(2,1|1)$ considered here has Lie subalgebra $u(1) \times su(2,1)$, with the first component generated by $H_3$. The $su(2,1)$ subalgebra has eight dimensions, with basis given in Eq. (51).

Including the $u(1)$ algebra generated by $H_0$, the full degeneracy superalgebra for the hamiltonian (50) is $u(1) \times su(2,1|1)$.

9. Phase-space superalgebra

The degeneracy superalgebras considered above are subsuperalgebras of a still larger superalgebra $\mathcal{A}$, where the generators are formed from all possible independent quadratic combinations of creation or annihilation operators. This algebra is not a degeneracy superalgebra, although it contains the degeneracy superalgebras mentioned in the previous sections. In the superalgebra $\mathcal{A}$, there are 12 odd generators formed by pairing each of the six bosonic operators $a, a^\dagger, b, b^\dagger, c, c^\dagger$, with each of the fermionic operators $f, f^\dagger$. There are 21 even generators formed from pairs of bosonic operators including, for example, $a^\dagger a^\dagger, a^\dagger b, bb, bc^\dagger$. These generate an $sp(6)$ subalgebra. A further even generator, $f^\dagger f$, is formed from the fermionic operators. Taken together, the 34 generators define the superalgebra $osp(2|6)$, which has even part $sp(6) \times so(2)$.

The $osp(2|6)$ superalgebra $\mathcal{A}$ is not unique to the Penning-trap system, since it would arise for any combination of signs for the number operators in the hamiltonian (13). The point is that $\mathcal{A}$ exists even before a potential for the physical problem is defined. The only requirement for $\mathcal{A}$ to be a relevant algebra is that the system describe a single fermion in a phase space with three space and three momentum dimensions. Thus, the superalgebra $osp(2|6)$ describes the properties of the phase space for the problem.

The hamiltonian for the Penning trap is fixed by specifying the parameters $\omega_c$, $\omega_z$ and $g$. For each of the cases in sections 5, 6, and 7, the degeneracy superalgebra is a subsuperalgebra of the phase-space superalgebra. We therefore find a hierarchy of nested superalgebras: $\mathcal{A} = osp(2|6) \supset so(2,1|1) \supset \mathcal{D}$, where $\mathcal{D}$ is any of the degeneracy superalgebras of sections 5, 6 or 7.

We have considered only structures arising from quadratic combinations of creation
or annihilation operators. The issue of the role played by higher-order combinations, such as those commuting with Eq. (18), is related to Clifford-algebra theory [25] but lies outside the scope of this paper.

10. Summary and Discussion

Several superalgebras are associated with the single-particle Penning trap. The various cases depend on the gyromagnetic ratio of the trapped particle and the relative strengths of the magnetic and electric fields. This paper considers the degeneracy superalgebras of operators that commute with the hamiltonian. The relevant superalgebras are summarized in Table 1.

In general, superalgebra descriptions might be expected for trap systems having energy separations between spin states equal to the separations between the bosonic oscillator-like levels. This guarantees the existence of odd generators that commute with the hamiltonian. Traps in which the spin cannot to be reversed, such as the TOP or Ioffe-Pritchard traps [15], are therefore unlikely to have superalgebra structures of the type described here. Superalgebras of this kind are also unlikely for traps where the spin states are independent of a magnetic field, as is the case for the Paul trap [15]. However, supersymmetries of another type do appear in these systems [15].

Some other issues beyond the scope of this paper are of potential interest. In particular, the spectrum-generating superalgebras would be relevant to a complete study of the properties of the Penning trap. Furthermore, higher-rank combinations of operators, such as those mentioned for the $\sigma = 9/4$ point in Figure 1, can be expected to arise in a study of the relevant Clifford algebras.

11. Acknowledgments

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References


25. See, for example, W.K. Clifford, Am. J. Math. 1, 350 (1887); A. Crumeyrolle, p.
Figure 1: The four Penning-trap frequencies $\omega_+, \omega_-, \omega_g$, and $\omega_z$ as functions of the parameter $\sigma = \omega_c/\omega_z$. The dashed lines show $\omega_g$ for $g = 4/3$ and $g = 2/3$. For $g = 4/3$, and there are three equal frequencies $\omega_+ = \omega_z = \omega_g = 2\omega_-$ at the supersymmetric point $\sigma = 3/2$. For $g = 3/2$, there are two pairs of distinct equal frequencies at the supersymmetric point. The frequencies $\omega_+$ and $\omega_-$ have infinite slopes where they meet at $\sigma = 2^{1/2}$. 
There are conspicuous degeneracies of the levels at the supersymmetric point $\sigma = 3/2$, arising from the superalgebra structure discussed in section 6. Another degeneracy occurs at $\sigma = 2.25$. In this plot, $\hbar = 1$. 

Figure 2: Penning-trap energies as a function of $\sigma$ for various states, with $g = 2/3$. 

Figure 3: Penning-trap energies as a function of $\sigma$ for various states, with $g = 4/3$. The evenly spaced crossings at the supersymmetric point $\sigma = 3/2$ are discussed in section 7. For this plot, $\hbar = 1$. 
Table 1: Penning trap superalgebras for the supersymmetric configuration $\sigma = 3/2$. The particle $g$ factor is given in the first column, and the four frequencies in units of $\omega_z$ are given in the next four columns. The algebraic structures found and the sections where they are discussed are given in the final two columns. The symbol $\ominus$ is defined in section 5. The bottom row represents the hypothetical case with four equal frequencies.

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