ABSTRACT

A new kind of deformed calculus (the D-deformed calculus) that takes place in fractional-dimensional spaces is presented. Two simple systems, the free particle and the harmonic oscillator in fractional-dimensional spaces are reconsidered into the framework of the D-deformed quantum mechanics. D-deformed coherent states are also found.

Keywords: fractional-dimensional space, deformed calculus

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Fractional-dimensional space approaches have been shown to be useful in the study of several physical systems. Theoretical schemes dealing with non-integer space dimensionalities have frequently been considered in the study of critical phenomena (see for instance [1], [2]) and of fractal structures [3] or in modelling semiconductor heterostructure systems [4] - [8].

In the above mentioned schemes, the fractional dimensionality is not referred to the real space, but to an auxiliary effective environment used to describe the real system. Nevertheless, the idea of a real space-time having a dimension slightly different from four has also been considered by several authors [9] - [12]. Actually, the deviations of the space-time dimension from four have been found to be very small [9] - [12]. However, the question of whether the dimension of the space-time is an integer or a fractional number constitutes a basic problem not only for its conceptual significance but also because the possibility that the space-time dimension is different from four may lead to interesting consequences (e.g. it is well known that a deviation of the space-time dimension from the value four eliminates the logarithmic divergences of quantum electrodynamics, independently of how small the deviation from four may be [13]).

Recently, it has been shown the existence of some similarities between the so-called N-body Calogero models and the problem corresponding to a fractional-dimensional harmonic oscillator of a single degree of freedom [14]. This result together with the remarkable fact that fractional-dimensional bosons can be considered as generalized parabosons [14] suggests new potential applications of the non-integer-dimensional space approaches.

It was shown in [14] that the fractional-dimensional Bose operators together with the reflection operator form an R-deformed Heisenberg algebra with a deformation parameter depending on the dimension of the space. Deformations of the Heisenberg algebra leading to the so-called q-deformed quantum mechanics have been extensively investigated (see for instance [15] - [18]). Taking into account the results obtained in [14], we develop in the present paper a new deformed calculus (the D-deformed calculus) in analogy to the q-deformed calculus commonly treated in the literature [18] - [21]. The paper is organized as follows. In section 2 we introduce the D-deformed calculus. The problems corresponding to the free particle and to the harmonic oscillator in fractional-dimensional space were studied in [14]. We reconsider these problems in sections 3 and 4 respectively, but now from the point of view of the D-deformed quantum mechanics. Of course, the final results in these sections coincide with the results obtained in [14]. However, in terms of the new D-deformed calculus, the above mentioned problems can be solved immediately and in an elegant way. The D-deformed coherent states are found in section 5 and conclusions are summarized in section 6.

II. D-DEFORMED CALCULUS

It is well known that the one-dimensional momentum operator is given by

\[ P = \frac{1}{i} \frac{d}{d\xi}, \]

where we have taken \( \hbar = 1 \). However, in a fractional-dimensional space, because of the inclusion of the integration weight [22]

\[ \sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \]

this operator is no longer Hermitian. Therefore a more general momentum operator has to be defined for systems in fractional-dimensional spaces. Starting with the Wigner commutation relations for the
canonical variables of a Bose-like oscillator of a single degree of freedom, the fractional-dimensional momentum operator has been found to be [14],

\[
P = \frac{1}{i} \frac{d}{d\xi} + i \frac{(D-1)}{2\xi} R - i \frac{(D-1)}{2\xi},
\]

(3)

where \( R \) is the reflection operator.

The momentum operator (Eq. (3)) suggests a deformation of quantum mechanics in fractional-dimensional spaces. Indeed, we can introduce a new D-deformed derivative operator

\[
\frac{d_D}{d_D \xi} = \frac{d}{d\xi} + \frac{(D-1)}{2\xi}(1 - R).
\]

(4)

and then the fractional-dimensional momentum operator (Eq. (3)) can be rewritten in the standard form

\[
P = \frac{1}{i} \frac{d_D}{d_D \xi}.
\]

(5)

Thus the D-deformed annihilation and creation operators can be defined in the following way

\[
a_D = \frac{1}{\sqrt{2}} \left( \xi + \frac{d_D}{d_D \xi} \right) ; \quad a_D^\dagger = \frac{1}{\sqrt{2}} \left( \xi + \frac{d_D}{d_D \xi} \right). 
\]

(6)

The action of these operators is given as follows [14]

\[
a_D |0\rangle = 0, \quad a_D |2n\rangle = \sqrt{2n}|2n - 1\rangle ; \quad a_D |2n + 1\rangle = \sqrt{2n + D}|2n\rangle, \quad \text{and} \quad a_D^\dagger |2n\rangle = \sqrt{2n + D}|2n + 1\rangle ; \quad a_D^\dagger |2n + 1\rangle = \sqrt{2n + 2}|2n + 2\rangle,
\]

(7, 8, 9)

where \( n = 0, 1, 2, 3, \ldots \)

By now introducing the corresponding D-factor (analogue to the q-factor) as

\[
[n]_D = n + \frac{(D-1)}{2}(1 - (-1)^n) \quad ,
\]

(10)

the Eqs. (8) and (9) can be rewritten in the usual form

\[
a_D |n\rangle = \sqrt{[n]_D} |n - 1\rangle, \quad \text{and} \quad a_D^\dagger |n\rangle = \sqrt{[n+1]_D} |n + 1\rangle,
\]

(11, 12)

respectively.
Taking into account Eq. (10) and in analogy to the q-deformed standard procedures we can define a D-deformed factorial function as follows

\[ [n]_D! = [n]_D[n-1]_D...[1]_D[0]_D! = \begin{cases} 
  2^n \left(\frac{D}{2}\right)! & \text{for } n \text{ even} \\
  \frac{2^n \Gamma\left(\frac{n+D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} & \text{for } n \text{ odd}
\end{cases} \]  

(13)

This D-deformed factorial function is a particular case of the generalized factorial function [23].

The eigenstates \(|n\rangle\) of the operator

\[ N_D|n\rangle = n|n\rangle \quad ; \quad N_D = \frac{1}{2}\{a_D^\dagger, a_D\} - D/2 \]  

(14)

may be obtained by repeated applications of \(a_D^\dagger\) on the vacuum state \(|0\rangle\)

\[ |n\rangle = \left(a_D^\dagger\right)^n |0\rangle \quad . \]  

(15)

It is easy to prove that in this Fock space, the relations

\[ a_D^\dagger a_D = [n]_D \quad ; \quad a_D a_D^\dagger = [n + 1]_D \]  

(16)

take place.

From the definition of the D-deformed derivative (Eq. (4)) we can introduce a D-deformed integration, so that if

\[ \frac{d_D f(\xi)}{d_D \xi} = F(\xi) \quad , \]  

(17)

then

\[ f(\xi) = \int F(\xi) d_D \xi + \text{const.} \]  

(18)

With the aim to find the appropriate expression for the D-deformed integration, we observe that

\[ \frac{d_D f(\xi)}{d_D \xi} = \left[ 1 + \frac{(D-1)}{2\xi} (1 - R) \right] \int d\xi \frac{df(\xi)}{d\xi} = F(\xi) \quad , \]  

(19)

and hence

\[ \frac{df(\xi)}{d\xi} = \left[ 1 + \frac{(D-1)}{2\xi} (1 - R) \int d\xi \right]^{-1} F(\xi) \quad . \]  

(20)

From the equation above it follows

\[ f(\xi) = \int F(\xi) d_D \xi = \sum_{n=0}^{\infty} \left[ - \int d\xi \frac{(D-1)}{2\xi} (1 - R) \right]^n \int d\xi F(\xi) \quad . \]  

(21)
This expression may be rewritten as
\[
\int F(\xi) d_{\xi} = \sum_{n=0}^{\infty} (-1)^n I_n,
\] (22)

where the terms \( I_n \) satisfy the following recurrence formula
\[
I_{n+1} = \int \frac{(D-1)}{2\xi} (1 - R) I_n d\xi \quad ; \quad I_0 = \int F(\xi) d\xi.
\] (23)

With respect to the D-deformed calculus induced by the fractional-dimensional integration weight (Eq. (1)), the following identities can be easily demonstrated
\[
\frac{d_D [f(\xi) g(\xi)]}{d_D \xi} = g(\xi) \frac{d_D f(\xi)}{d_D \xi} + \frac{d_D g(\xi)}{d_D \xi} R f(\xi) + \frac{d g(\xi)}{d \xi} (1 - R) f(\xi)
\] (24)

and after integrating the equation above
\[
\int g(\xi) \frac{d_D f(\xi)}{d_D \xi} d_D \xi = f(\xi) g(\xi) - \int \frac{d_D g(\xi)}{d_D \xi} R f(\xi) d_D \xi - \int \frac{d g(\xi)}{d \xi} (1 - R) f(\xi) d_D \xi.
\] (25)

One should notice that if either \( f(\xi) \) or \( g(\xi) \) is an even function of \( \xi \), Eqs. (24) and (25) reduce to a D-deformed Leibnitz rule and to a D-deformed formula of integration by parts, respectively. This is a consequence of the fact that the D-deformed derivative acts on even functions as the ordinary derivative.

III. D-DEFORMED FREE PARTICLE

The eigenstates of the momentum operator corresponding to a free particle of a single degree of freedom in a fractional-dimensional space can be found now in terms of the D-deformed calculus introduced in the previous section. Thus, the eigenstates of the fractional-dimensional momentum operator are determined by the following equation
\[
P \Psi_p = i \frac{d_D \Psi_p}{d_D \xi} = p \Psi_p.
\] (26)

The corresponding eigenfunctions are immediately found to be
\[
\Psi_p = A_p E_D(-ip\xi),
\] (27)

where \( E_D(x) \) represents the D-deformed exponential function (see Appendix A). The normalization factor \( A_p \) can be found from the orthonormalization condition
\[
\langle \Psi_p | \Psi_{p'} \rangle = \frac{\sigma(D)}{2} \lim_{\gamma \to 0} \int_{-\infty}^{\infty} e^{-\gamma \xi^2} \Psi_p(\xi) \Psi_{p'}(\xi) |D-1 d\xi = \delta(p - p') \quad (\gamma > 0).
\] (28)
in a similar way as in [14]. After the corresponding calculations we arrive to the following expression
\[ A_p = \frac{1}{2^{D/2-1} \Gamma(D/2)} \sqrt{\frac{p^{D-1}}{2\pi^D}} . \]  

One should notice that the eigenfunctions \( \Psi_p \) describing the motion of a free particle of a single degree of freedom in a fractional-dimensional space can be considered as D-deformed plane waves and they reduce to the ordinary plane de Broglie waves when \( D = 1 \).

### IV. D-DEFORMED HARMONIC OSCILLATOR

The eigenfunctions in coordinate representation corresponding to the D-deformed harmonic oscillator can be derived from Eq. (15) without much difficulty. First we consider the vacuum state \( |0\rangle \) which satisfies Eq. (7). Then, using the expression of \( a_D \) in coordinate representation (Eq. (7)) we have the following D-deformed differential equation

\[
\left( \frac{d_D}{d_D \xi} + \xi \right) \chi_0 = 0 ,
\]

where \( \chi_0 = \langle \xi | 0 \rangle \) represents the eigenfunction of the ground state of the D-deformed harmonic oscillator. By now solving Eq. (30) we found that

\[
\chi_0 = C_0 \exp[-\xi^2/2] ,
\]

where \( C_0 \) is a normalization factor. From the normalization condition

\[
\frac{\sigma(D)}{2} \int_{-\infty}^{\infty} |\chi_0|^2 |\xi|^{D-1} d\xi = 0 ,
\]

the normalization constant is found to be

\[
C_0 = \frac{1}{\pi^{D/4}} .
\]

Once the ground state has been found, the excited states may be calculated from Eq. (15). Thus the excited states are determined by

\[
\chi_n = \langle \xi | n \rangle = \frac{C_0}{\sqrt{|n|}_D!} \left[ \frac{1}{\sqrt{2}} \left( \xi - \frac{d_D}{d_D \xi} \right) \right]^n \exp[-\xi^2/2] .
\]

If we now take into account that

\[
(-1)^n \exp[\xi^2/2] \left( \frac{d_D}{d_D \xi} \right)^n \exp[-\xi^2] = \left( \xi - \frac{d_D}{d_D \xi} \right)^n \exp[-\xi^2/2] ,
\]

a relation that can be demonstrated by induction, the excited states in coordinate representation can be written as
\[ \chi_n = \langle \xi | n \rangle = \frac{[n]_D! \exp[-\xi^2/2]}{n! \sqrt{\pi D/2}} [n]_D! H_n^D(\xi) , \quad (36) \]

where

\[ H_n^D(\xi) = \frac{n!}{[n]_D!} (-1)^n \exp[\xi^2] \left( \frac{d_D}{d_D^*} \right)^n \exp[-\xi^2] , \quad (37) \]

can be understood as D-deformed Hermite polynomials. In fact the polynomials \( H_n^D(\xi) \) above defined are a particular case of the generalized Hermite polynomials studied in [23].

It is worth remarking that if \( D = 1 \) the results obtained in the present section reduce to the well known results corresponding to the undeformed one-dimensional case.

**V. D-DEFORMED COHERENT STATES**

We now observe the spectrum problem corresponding to a D-deformed annihilation operator \( a_D \). The eigenstates of \( a_D \):

\[ a_D |\alpha\rangle = \alpha |\alpha\rangle \quad (38) \]

are a D-deformation of usual coherent states. The solution of Eq.(38) is given by

\[ |\alpha\rangle = A_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_D!}} |n\rangle . \quad (39) \]

From the normalization condition \( \langle \alpha | \alpha \rangle \) the normalization constant is found to be

\[ A_\alpha = \frac{1}{\sqrt{E_D(|\alpha|^2)}} \quad (40) \]

As usually, the Eq. (39) can be written in term of the vacuum state as follows

\[ |\alpha\rangle = \frac{1}{\sqrt{E_D(|\alpha|^2)}} E_D(\alpha a_D^\dagger) |0\rangle . \quad (41) \]

Once we have found the expression of the D-deformed coherent states in the Fock representation, we can easily obtain its expression in coordinate representation by using the relation

\[ \Phi_\alpha(\xi) = \langle \xi | \alpha \rangle = \sum_{n=0}^{\infty} \langle \xi | n \rangle \langle n | \alpha \rangle . \quad (42) \]

By now considering Eqs. (36) and (39) we arrive to the following result

\[ \Phi_\alpha(\xi) = \frac{\exp[-\xi^2/2]}{\sqrt{E_D(|\alpha|^2)}} \frac{1}{\pi^{D/4}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{2^n n!}} H_n^D(\xi) . \quad (43) \]
From Eq. (39), the probability distribution of a D-deformed coherent state in Fock representation is found to be

$$|\langle n|\alpha\rangle|^2 = \frac{1}{E_D(|\alpha|^2)} \frac{(|\alpha|^2)^n}{[n]_D!}$$  \hspace{1cm} (44)

i. e. a D-deformation of the Poisson distribution.

VI. CONCLUSIONS

Summing up, taking into account recent developments in the mathematical physics of the fractional-dimensional space and in analogy to the q-deformed calculus we have developed a new deformed calculus that we have called D-deformed calculus. Two systems, the free particle and the harmonic oscillator in a fractional-dimensional space have been reconsidered into the frame work of the D-deformed quantum mechanics. Finally, the D-deformed coherent states are found.

APPENDIX

Here we will study the properties of some D-deformed functions. From the definition of the D-deformed derivative (Eq. (4)) it is easy to find that

$$\frac{d_D^n}{d_D \xi} = [n]_D \xi^{n-1}. \hspace{1cm} (A1)$$

On the other hand, from the definition of the D-deformed integration (Eq. (22)), we also found

$$\int \xi^n d_D \xi = \frac{\xi^{n+1}}{[n+1]_D} + \text{const.} \hspace{1cm} (A2)$$

In this way, one can introduce the D-deformed exponential function as follows

$$E_D(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{[n]_D!} \hspace{1cm} (A3)$$

From Eq. (A3) and making use of the Eqs. (A1) and (A2) one can straightforwardly demonstrate that

$$\frac{d_D \xi^n}{d_D \xi} = \lambda E_D(\lambda \xi) \hspace{1cm} \lambda = \text{const.} \hspace{1cm} (A4)$$

and consequently

$$\int E_D(\lambda \xi) d_D \xi = \frac{E_D(\lambda \xi)}{\lambda} + \text{const.} \hspace{1cm} (A5)$$

Actually, the D-deformed exponential function is a particular case of the generalized exponential function defined in [24] and can be represented as follows
\[ E_D(\xi) = \exp[\xi \Phi\left(\frac{D - 1}{2}, D, -2\xi\right)] , \]  
(A6)

or

\[ E_D(\xi) = \Gamma(D/2) \left(\frac{\xi}{2}\right)^{1-D/2} [I_{D/2-1}(\xi) + I_{D/2}(\xi)] , \]  
(A7)

where \( \Phi \) and \( I_\nu \) are the confluent hypergeometric function and the modified Bessel function respectively.

We can also introduce D-deformed cosine and sine functions through the following definitions

\[ \text{COS}_D \xi = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n}}{[2n]_D!} = \Gamma(D/2) \left(\frac{\xi}{2}\right)^{1-D/2} J_{D/2-1}(\xi) , \]  
(A8)

and

\[ \text{SIN}_D \xi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \xi^{2n-1}}{[2n-1]_D!} = \Gamma(D/2) \left(\frac{\xi}{2}\right)^{1-D/2} J_{D/2}(\xi) , \]  
(A9)

where \( J_\nu \) represents the Bessel function. Thus, the following identities can be easily verified

\[ E_D(\pm i\xi) = \text{COS}_D \xi \pm i \text{SIN}_D \xi , \]  
(A10)

and

\[ \frac{d_D \text{COS}_D \xi}{d_D \xi} = -\text{SIN}_D \xi ; \quad \frac{d_D \text{SIN}_D \xi}{d_D \xi} = \text{COS}_D \xi \]  
(A11)

One should notice that all the definitions and equations given in this appendix reduce to the corresponding undeformed expressions when \( D = 1 \), as it must.

References

1. S. Ma, Rev. Mod. Phys. 45 (1973) 589.