Ising model, pointed out that the non-relativistic limit is more straightforward if \( i \) is replaced by \(-i\) in (1). Hull and Treumann[11] also noted that paths fixed at both ends have \((R-1)\) degrees of freedom, so the \( R \) in (1) may be replaced by \((R-1)\) without interfering with the continuum limit.

The formal analytic continuation involved in the chessboard model is more subtle than in the non-relativistic case. Note that the effect of \( i \) in (1) above is to partition the sum into 4 components, i.e.:

\[
K(b, a) = \left( \sum_{R=0,3,6,...} N(R)\langle cm\rangle^R - \sum_{R=1,4,7,...} N(R)\langle cm\rangle^R \right) + i\left( \sum_{R=1,4,7,...} N(R)\langle cm\rangle^R - \sum_{R=0,3,6,...} N(R)\langle cm\rangle^R \right)
\]

\[
= \Phi_+ + i\Phi_-
\]

Each sum by itself is a weighted sum, a partition function for a random walk in which the term \( \langle cm\rangle^R \) is just a Boltzmann weight. The interference of alternative paths is a result of the two subtractions in (2). If we remove the minus signs, the resulting propagator is related to the Telegraph equation, which in turn becomes the diffusion equation in the ‘non-relativistic’ limit[12]. The underlying stochastic model has been studied by Kac[13] and its relation to the Dirac equation through analytic continuation has been discussed by Gaveau et. al.[14] and Jacobson and Schulman[10]. The \( \hat{\Phi} \) which appears in (2) just expresses \( K \) as a particularly convenient linear combination of the real amplitudes \( \Phi_\pm \). It does not represent the formal analytic continuation of quantization which is, in this case, implemented through the alternating sign of \( (\hat{\mathcal{F}})^n \) which appears in (1) and the resulting subtractions which appear in (2). Since it is the occurrence of the minus signs in the propagator which is essential for interference we look for a physical basis for the subtractions.

Regarding Fig(1) we can codify the counting and subtractions involved in (2) by colouring the trajectories with two colours, say blue and red. If the trajectories start out blue, they change to red at the second corner, return to blue at the fourth and so on. The sign of the contribution of a trajectory is then determined by its colour at the end point, + for blue, - for red. Red contributions behave like antiparticles in that they reduce the contribution of the particles, providing interference effects. The ensemble of such coloured paths between \( a \) and \( b \) provides the appropriate contribution to a quantum propagator, but is not explicitly traversed as a single path. What we would like to do is to sew together the Chessboard paths in such a way that they may be traversed by a single path which also provides the alternating colours of the trajectories through the direction in time of the traversal. If we can do this, then the subtractions in (2) become a measure of the difference between the number of particles and the number of ‘anti-particles’ in the system[15, 16, 17]. To this end, we note from Fig 2 that each Chessboard path has an orthogonal twin.

The orthogonal twin starts from the origin moving in the opposite direction with the opposite colour. It moves the same distance as the second leg of its twin’s path, reverses direction and moves the same distance as its twin’s first leg. Twins meet at every second corner where they both change colour. This is repeated until the twins meet at or just after \( t_b \). The orthogonal twin is also a chessboard path with colouring \( 180^\circ \) out of phase with the original.

Now consider the following ‘entwined’ traversal of the two paths. Follow the first twin to the first meeting, the second to the second meeting and so on. This path is blue from the origin to \( b \). From \( b \) reverse the direction in
$t$ by proceeding down the remaining red sections. This brings you back to the origin on an entirely red path. Notice that this traversal gives a meaning to the original Feynman colouring; the colouring corresponds to the direction in time of an entwined path traversal. Blue corresponds to forward in $t$, red to backwards. Note that entwined pairs conserve charge if we associate opposite charges with reversed time segments. Now each chessboard trajectory in (2) has a unique orthogonal twin. Let $P_R$ be an arbitrary $n$-step R-cornered Chessboard path. Write $P_R = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ where $\sigma_k = \pm$ accordingly as the $k$-th step of the path is in a plus or minus $x$-direction. For example, Chessboard paths with $R$ corners which start in the plus-$x$ direction are of the form $(+, \sigma_2, \ldots, \sigma_n)$. If we define a ‘leg’ as a set of contiguous steps all in the same direction and bounded by either corners or ends of a path (i.e. a domain in the Ising analogy), then we may write $P_R = (l_1, l_2, \ldots, l_{R+1})$ with the understanding that $l_1$ stands for the first leg, $l_2$ stands for the second and so on. We may then define the orthogonal twin to $P_R$ as

$$P_{R}^\dagger = -(l_2, l_1, \ldots, l_{R+1}, l_R) \quad R \text{ odd}$$

$$= -(l_2, l_1, \ldots, l_R, l_{R+1}) \quad R \text{ even} \quad (3)$$

The definition of the orthogonal twin for even $R$ is chosen to allow us to join it to its sibling by imposing a corner at the end of the last leg, and adding a leg in the new direction the same length as the last leg. Thus paths with even $R$ are extended by one leg to allow contact with the orthogonal twin. Because $P_{R}^\dagger$ is a unique permutation of $P_R$ with the signs of the $\sigma$’s flipped, the ensemble, $\mathcal{E}_R$, of all $n$-step paths $P_R$ from the origin is the same as the ensemble of all paths $P_{R}^\dagger$ from the origin. Furthermore, this is the same as the ensemble of paths of the form $(+, \sigma_2, \ldots, \sigma_n)$ combined with all orthogonal twins. Thus we may cover all paths in $\mathcal{E}_R$ with the correct Chessboard colouring, just by traversing all entwined pairs. This may be done through a single continuous (in the sense of the lattice) path since all entwined loops intersect at the origin. Furthermore, entwined pairs fixed at the origin and at time $t_0$ have the same number of degrees of freedom as their individual component Chessboard paths (i.e. $R-1$) and each pair may be given the statistical weight $(im)^R$ which correctly weights the component Chessboard trajectories. Thus the following classical stochastic process gives rise to a properly weighted chessboard ensemble of coloured paths. Start a random walk at the origin and allow the walker to choose entwined paths according to the number of free corners, either in the entwined path or one of the pairs. The walker traverses the entwined path as above so as to maintain both the Chessboard and time-sense colouring. The walker ends up at the origin at the end of the traversal and repeats the process. The space-time lattice records the net number of traversals in the $+t$ direction as the walker passes, by registering a plus one for a positive traversal and a minus one for a negative traversal, thus accumulating positive and negative integers. The traversal weighting ensures that the constituent Chessboard paths have the correct expected weight, and the ergodic nature of the walks ensures that, with enough loops, you can get as close as you like to a uniform coverage of the ensemble.

If we allow a walker to cycle through the entwined paths according to the above prescription, we can immediately write down the expected net charge accumulated on the lattice. If the walker loops over $N$ entwined pairs and $(x,t)$ is a lattice point within the light cone and $t < t_0$ then the contribution to the $+ \text{-component}$ of the net charge is proportional to $N\left(K_{++}(x,t) - K_{+-}(x,t)\right)$. This is because an entwined loop corresponds to two forward Chessboard paths, one originating from a positive right moving source at the origin and one originating from a negative left moving source. Similarly the $- \text{- component}$ $K_{-+}(x,t) - K_{--}(x,t)$ is proportional to $N\left(K_{-+}(x,t) - K_{--}(x,t)\right)$.

Unlike the predecessors of this model [15, 16, 17] which did not use bound pairs of trajectories in any way, this new model is relatively easy to simulate on a lattice. Fig. (3) shows an example of such a simulation. Plotted is the sum of the real and imaginary parts of the propagator expected from the Chessboard model at the same lattice resolution (continuous curve) as the results of a simulation with a single path which loops over the lattice $10^8$ times. At smaller values of $t$, the simulation and the Chessboard model are indistinguishable on the scale of the figure, at larger values of $t$ the single path gives sparser coverage of the chessboard ensemble and the scatter increases.

The above result has several appealing features. The first feature is that we do not have to resort to quantum mechanics or formal analytic continuation to arrive at the $\Phi_+$ of (2). The calculation shows that the Feynman
The propagator has an independent existence as an expected net charge over an ensemble of entwined paths which can be joined into a single trajectory. In this context, the propagator has an underlying classical stochastic model which is in effect self-quantizing.

A second feature is that the above observations unite two perspectives on quantum mechanics by pointing out a third. Regarding Fig. 2, there are three ways of viewing the two trajectories. We can view them as two separate chessboard trajectories, coloured according to Feynman’s corner rule. An ensemble of such trajectories would build a quantum propagator as a sum-over-histories. This is the view which allows us to calculate the propagator so easily using previous work.

A second picture is to note that the two trajectories form a chain of creation/annihilation events. An ensemble of these would provide a vacuum of virtual particles upon which an excitation could presumably propagate. This is close to a field theory perspective.

The third picture, which we favour, is the continuous loop in space-time, coloured according to direction of motion in time. This provides a bridge between the previous two pictures. In this picture, the phase of the wave function, ‘zitterbewegung’, and the presence of virtual particles are all manifestations of paths which form entwined space-time loops. Noticing that an ensemble of orthogonal twins could be traversed in many ways and still form a continuous space-time curve, an interesting next step would be to see if a traversal scheme could be found that would show the existence of a Born postulate in the system. With the traversal suggested by the path twins, there is already a form of wave function collapse present, since if the particle is prevented from revisiting its macroscopic past in space-time, the propagator has to readjust, fixing the wave function in the past light cone while forcing the propagator in the future light cone to be redrawn.

It is worth reiterating that although we have interpreted the above entwined-pair chessboard model in terms of quantum mechanics it is not, as it stands, quantum mechanics. The model is explicitly realist, we have not quantized a classical system, and there is no ambiguity as to what a wave function or propagator represents.

We do not have the interpretative problems of quantum mechanics because the underlying microscopic dynamics are fixed and transparent. However, if we are to maintain the realistic status of the model, we do not have the luxury of measurement postulates either. If the model is to provide more than just an algorithm for the calculation of wave functions, then the measurement postulates must have a direct basis in the dynamics themselves. This possibility is currently under investigation.

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[8] However, a recent result by Kröger has motivated a conjecture that for a particular class of non-relativistic path integrals, the sum-over-paths for a particular transition amplitude may be replaced by a sum over a single path with a renormalized action. Jiari et al., Phys. Rev. Lett. 86 (2001), 887; Jiari et al. Phys. Lett. A 281 (2001) 1