THEORY OF COLOR CONFINEMENT: STATE OF THE ART.

A. DI GIACOMO AND G. PAFFUTI

Dip. Fisica Università and I.N.F.N., Sezione di Pisa,
Via Buonarroti 2, Pisa

The existing evidence for dual superconductivity as mechanism of color confinement is reviewed. We also discuss what is known on the dual excitations, which produce confinement by condensation, and what are the open problems.

1 Established results: confinement and symmetry.

An almost general consensus exists on the idea that deconfinement is an order disorder transition\(^1\). The present experimental upper limits on the observation of free quarks are 15 orders of magnitude smaller than what is expected in the absence of confinement. The only natural interpretation is that confinement is an absolute property relying on symmetry.

The confined phase of QCD is disordered (strong coupling). The symmetry of a disordered phase can be understood in terms of duality\(^2,3\). The system should admit non local topological excitations, and, besides the usual description in terms of local fields a complementary dual description exists in which the topological excitations becomes local and their v.e.v. are the order parameters. The dual effective coupling constant \(g_D\) is related to \(g\) as \(g_D \sim 1/g\). The strong coupling regime is mapped into the weak coupling of the dual description.

If this is correct, the ultimate goal is to identify the dual excitations which condense in the confined phase, or at least to identify their quantum numbers, i.e. the symmetry of the confining vacuum.

On the basis of the above argument we shall disregard all the approaches to the problem which are not based on symmetry. Two possibilities are being considered for the dual symmetry, which in principle are not mutually exclusive

1) Dual excitations carry magnetic charge, i.e. they are monopoles: their condensation produces dual superconductivity of the vacuum. Abrikosov electric flux tubes between colored particles are at the origin of confinement\(^4,5,6\).

2) Dual excitations are \(Z_N\) vortices, and their condensation is qualitatively described as “spaghetti vacuum”\(^1\).
We remark that for many years both ideas have been analyzed with a zo-
ologist’s attitude. Investigations were a more or less ingenious counting of
monopoles and vortices in connection with the deconfining transition. If use-
ful in the pioneering stages to establish the existence of the excitations, such
kinds of counting give no information on symmetry.

In a sense also the concept of dominance is similar. Vortices\(^7\) or maxi-
mal abelian monopoles\(^8\) are assumed to be the relevant dual excitations, their
contribution to observables is extracted, one way or the other, from numeri-
cally generated configurations, and shown to give a good approximation to the
full determination. Dominance is a necessary condition, but in principle not
sufficient to identify the dual excitations, without further theoretical input.
Symmetry is an exact property and has to be investigated as such.

2 Monopoles.

A conserved magnetic charge can be associated to any operator \(\vec{\varphi}(x)\) in the
adjoint representation: in what follows we shall consider \(SU(2)\) gauge group
for the sake of simplicity. Extension to higher groups will only be sketched.

Define \(\hat{\varphi} = \varphi(x)/|\varphi(x)|\), a direction in color space. \(\hat{\varphi}\) is defined everywhere
except at zeros of \(\varphi\), and\(^7\)

$$F_{\mu\nu} = \hat{\varphi} \cdot \vec{G}_{\mu\nu} - \frac{1}{g} \hat{\varphi} \cdot (D_\mu \hat{\varphi} \wedge D_\nu \hat{\varphi}) \tag{1}$$

with \(\vec{G}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g \vec{A}_\mu \wedge \vec{A}_\nu\) the field strength tensor, and \(D_\mu = \partial_\mu - ig \vec{A}_\mu \wedge\) the covariant derivative.

The two terms in eq.(1) are separately gauge invariant and color singlets:
the specific combination is chosen in such a way that bilinears in \(A_\mu A_\nu\) and
\(A_\mu \partial_\nu \varphi\) cancel, so that one has identically

$$F_{\mu\nu} = \partial_\mu (\hat{\varphi} \vec{A}_\nu) - \partial_\nu (\hat{\varphi} \vec{A}_\mu) - \frac{1}{g} \hat{\varphi} (\partial_\mu \hat{\varphi} \wedge \partial_\nu \hat{\varphi}) \tag{2}$$

A gauge transformation bringing \(\hat{\varphi}\) to a fixed direction, say 3 axis (abelian
projection) eliminates the second term of eq.(2) and makes \(F_{\mu\nu}\) an abelian
field\(^6\)

$$F_{\mu\nu} = \partial_\mu A_3^\nu - \partial_\nu A_3^\mu$$

The dual field \(F^\ast_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu}\) defines a magnetic current

$$j_\alpha = \partial^\beta F^\ast_{\alpha\beta} \tag{3}$$
which is identically conserved $\partial^\alpha j_\alpha = 0$. In a non compact formulation of the theory $j_\alpha$ of eq.(3) is identically zero (Bianchi identity): in a compact formulation it can be different from zero.

After abelian projection $U(1)$ Dirac monopoles coupled to $F_{\mu\nu}$ show up at zeros of $\vec{\varphi}$.

For generic $SU(N)$ gauge group cancellation of bilinears identifies a set of $N - 1$ directions in the algebra, $\Phi_i = \varphi_i^a T^a$, and each of them identifies a symmetric subspace of the algebra. The field $\Phi$ can then be written as a superposition of $\Phi_i$

$$\Phi = \sum_{i=1}^{N-1} c_i(x) \Phi_i$$

Monopoles appear at the zeros of $c_i(x)$, as magnetic $U(1)$ charges.

Dual superconductivity will occur when any of the above magnetic charges condense in the vacuum, so that magnetic $U(1)$ is broken à la Higgs.

A disorder parameter (order parameter of the dual system) $\langle \mu \rangle$ can be defined, which is the vacuum expectation value of an operator $\mu$ carrying magnetic charge$^{7,8}$.

The operator $\mu$ is magnetic $U(1)$ charged (and gauge invariant). It is non local but localized enough to obey cluster property.

$\langle \mu \rangle$ is measured around the deconfining phase transition on lattices of different spatial size, and then the thermodynamic limit $V \to \infty$ is performed by use of finite size scaling techniques$^{11}$.

The result is

$$\langle \mu \rangle \neq 0 \quad T < T_C$$

$$\langle \mu \rangle = 0 \quad T > T_C$$

$$\langle \mu \rangle \approx \tau^\delta, \quad \tau \equiv 1 - \frac{T}{T_C} \quad (4)$$

The other important result is that$^{12,13}$ the behaviour of $\langle \mu \rangle$ is independent of the abelian projection used to define the magnetic charge.

In fact the computation of $\langle \mu \rangle$ is the computation of a partition function, which is very noisy. All the relevant information is, however, contained in the quantity

$$\rho = \frac{d}{d\beta} \ln \langle \mu \rangle \quad (5)$$

$\langle \mu \rangle$ in dimensionless, and depends on the ratios of the lengths involved, the lattice spacing $a$, the spatial lattice size $L$ and the correlation length $\Lambda$. The
time extension of the lattice identifies the temperature

\[ \langle \mu \rangle = \Phi \left( \frac{a}{\Lambda}, \frac{L}{\Lambda} \right) \]

At \( T_C \) the correlation length \( \Lambda \) goes large, the transition being 2nd order for \( SU(2) \), weak first order for \( SU(3) \)

\[ \Lambda \sim \tau^{-\nu} \]

so that \( a/\Lambda \approx 0 \) and \( \langle \mu \rangle = \Phi \left( 0, \frac{L}{\Lambda} \right) \). The scaling law follows

\[ \rho/L^{1/\nu} = f(\tau L^{1/\nu}) \]

Using data from different sizes of the lattice \( \nu \), \( T_C \) and \( \delta \) can be determined. The result is

\[ SU(2) \quad \nu = 0.62(1) \quad \delta = 0.7(2) \]
\[ SU(3) \quad \nu = 0.33 \quad \delta = 0.5(1) \]

\( \nu \) and \( T_C \) agree with other determinations. In particular it is confirmed that the transition is weak first order for \( SU(3) \), second order for \( SU(2) \), in the universality class of the 3d Ising model.

This result implies that vacuum is superconducting in all the abelian projected \( U(1) \)'s.

A typical behaviour of \( \mu \) and \( \rho \) are shown in fig.1,2.

![Figure 1. \( \langle \mu \rangle \) vs. \( \beta \) for \( SU(2) \) gauge theory. Plaquette projection, lattice \( 16^3 \times 4 \).](image-url)
3 Vortices.

In 2+1 dimension vortices are local operators carrying a conserved quantum number\(^1\). In 3+1 dimension they are closed defect lines associated to a path \(c\). If \(B(C)\) is the operator which creates a vortex on the spatial contour \(C\) at time \(t\) and \(W(C')\) creates a Wilson loop on the contour \(C'\) at time \(t\), then\(^1\)

\[
W(C')B(C) = B(C)W(C')\exp\left(\frac{2\pi}{N}in_{CC'}\right)
\]

where \(n_{CC'}\) is the winding number of the two paths. Eq.(6) implies that if \(\langle W(C') \rangle\) obeys the area law, i.e. if it behaves as \(\exp(-A_{C'})\) when the size of the path goes large with respect the correlation length, then \(\langle B(C) \rangle\) obeys the perimeter law, and vice versa if \(\langle B(C) \rangle\) obeys the area law \(\langle W(C') \rangle\) will obey the perimeter law. The fact that \(\langle B(C) \rangle \neq 0\) per se does not imply any symmetry breaking. Some groups\(^13\) have checked this behaviour by looking at a series of loops. A systematic procedure is to look at the “dual Polyakov line” \(\langle \bar{L} \rangle\), i.e. the straight path wrapping around the lattice by periodic boundary conditions\(^14,15\). If \(\langle B \rangle\) obeys the area law then \(\langle \bar{L} \rangle = 0\), \(\langle \bar{L} \rangle \neq 0\) signal perimeter law.

Numerical simulations show that \(\langle \bar{L} \rangle\) is a good order parameter for confinement, being nonzero in the confined phase, and zero in the deconfined one.
Moreover the critical index by which it vanishes with \( \tau = 1 - t/T_C \), \( \langle \bar{L} \rangle \sim \tau^\delta \) is \( \delta = 0.5 \pm .15 \), equal within errors to that of \( \langle \mu \rangle \) the disorder parameter for dual superconductivity\(^{14,15}\).

4 Discussion

The question: who is \( \vec{\Phi} \)? was asked already in ref.\(^6\). For long time a zoologically minded answer was given to it. The relevant abelian projection was taken to be maximal abelian gauge because of monopole dominance in that projection. Investigation of symmetry, however, shows that all abelian projections are physically equivalent\(^{11,12}\), a possibility already suggested in ref.\(^5\).

The operator \( \mu \) detecting dual superconductivity is defined at the level of rigor of constructive field theory in compact \( U(1) \). In that case everything is a theorem: numerical simulations were made only to explore the numerical viability of the method\(^{16}\). Incidentally \( \mu \) is constructed as a gauge invariant charged operator à la Dirac\(^7\), non local, but local enough to obey the cluster property, so that \( \langle \mu \rangle \neq 0 \) does not violate by any means Elitzur’s theorem\(^{18}\).

In non abelian theories charged fields are present, and in principle the abelian projection is defined up to terms \( O(a^2) \), with \( a \) the lattice spacing. This can create a problem at short distances with the Dirac quantization condition. Looking for a way out can probably suggest good ideas to identify the dual excitations\(^{19}\).

Our conclusion is that dual superconductivity is the mechanism of confinement. The disorder parameter \( \langle \mu \rangle \) detects it, both in pure gauge and in full QCD\(^{20}\), as expected in the philosophy of \( N_C \to \infty \): indeed quark loops are non leading in that expansion, and the mechanism of confinement is expected to be the same with and without quarks. A careful analysis of the relation between \( \langle \mu \rangle \) and the chiral order parameter is on the way.

More investigation is needed to fully understand the dual description of QCD.

References