Construction Formulae for Singular Vectors of the Topological
and of the Ramond N=2 Superconformal Algebras

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ABSTRACT

We write down one-to-one mappings between the singular vectors of the
Neveu-Schwarz N=2 superconformal algebra and 16 + 16 types of singular
vectors of the Topological and of the Ramond N=2 superconformal algebras. As a result one obtains construction formulae for the latter using the con-
struction formulae for the Neveu-Schwarz singular vectors due to Dörrzapf. The indecomposable singular vectors of the Topological and of the Ramond
N=2 algebras (‘no-label’ and ‘no-helicity’ singular vectors) cannot be mapped
to singular vectors of the Neveu-Schwarz N=2 algebra, but to subsingular vec-
tors, for which no construction formulae exist.

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Singular vectors of infinite dimensional algebras corresponding to conformal field theories contain an amazing amount of useful information. They are therefore far from being empty objects that one simply would like to get rid of. For example, as a general feature, their decoupling from all other states in the corresponding Verma module gives rise to differential equations satisfied by correlators of conformal fields, which can be solved as a result [1]. Also, their possible vanishing in the Fock space is directly connected with the existence of extra states in the Hilbert space that are not primary and not secondary (not included in any Verma modules) [2]. It can even happen that singular vectors of one theory are directly related to some mathematical structures of another theory (see for example ref. [3]). For these reasons it is very convenient to obtain explicit expressions for the singular vectors of a given algebra. Suitable construction formulae for these are therefore most helpful.

Regarding the construction of singular vectors, using either the “fusion” method or the “analytic continuation” method, explicit general expressions have been obtained for the singular vectors of the Virasoro algebra [4], the Sl(2) Kac-Moody algebra [5], the Affine algebra \(A_{1}^{(1)}\) [6], the N=1 superconformal algebra [7], the Neveu-Schwarz N=2 superconformal algebra [8][9], and some W algebras [10]. (There is also the method of construction of singular vertex operators, which produce singular vectors when acting on the vacuum [11]). In some cases it is even possible to transform singular vectors of an algebra into singular vectors of the same or a different algebra, simplifying notably the computation of the latter ones. For example, Kac-Moody singular vectors have been transformed into Virasoro ones, by using the Knizhnik-Zamolodchikov equation [12], and singular vectors of W algebras have been obtained out of \(A_{2}^{(1)}\) singular vectors via a quantum version of the highest weight Drinfeld-Sokolov gauge transformations [13].

The N=2 superconformal algebras have appeared in String Theory in several occasions playing quite different roles. First of all they provide the symmetries underlying the N=2 strings [14][15]. These strings seem to be related to M-theory since many of the basic objects of M-theory are realized in the heterotic (2,1) N=2 strings [16]. Second, the N=2 superconformal algebras are encountered also as the (global) symmetries of the world-sheet of the heterotic string after compactification from ten to four dimensions preserving N=1 (local) space-time supersymmetry [17]. Last but not least, the Topological N=2 superconformal algebra is realized in the world-sheet of the bosonic string [3], as well as in the world-sheet of the superstrings [18].

Four years ago the singular vectors of the Topological N=2 algebra were classified [19] taking into account the relative U(1) charge and the possible annihilation by the fermionic zero modes of the vector itself and of the primary on which it is built (BRST-
invariance properties). In generic Verma modules 20 different types of possible singular vectors were found whereas in ‘no-label’ Verma modules, built on indecomposable ‘no-label’ primaries, 9 types were found (one of them existing only at level zero). In chiral Verma modules, which are incomplete quotient modules, the number of different types of singular vectors was found to reduce to just 4. In ref. [19] the whole set of singular vectors was explicitly constructed at level 1, whereas the rigorous proofs that these types are the only existing ones were given later in ref. [20]. In the latter reference it was also proved that the Verma modules of the Topological N=2 algebra are isomorphic to the Verma modules of the Ramond N=2 algebra and therefore a similar classification of singular vectors hold. The same is not true for the Verma modules of the Neveu-Schwarz N=2 algebra, the corresponding singular vectors being in fact less than half in number than for the other two algebras.

In this paper we write down one-to-one mappings between the singular vectors of the Neveu-Schwarz N=2 algebra and 16+16 types of singular vectors of the Topological and of the Ramond N=2 algebras. As a result we obtain construction formulae for the latter (which are absent in the literature, except for some simple cases) using the construction formulae for the Neveu-Schwarz singular vectors due to Dörözapf [8][9]. As we will discuss, the indecomposable singular vectors of the Topological and of the Ramond N=2 algebras (‘no-label’ and ‘no-helicity’ singular vectors) cannot be mapped to singular vectors of the Neveu-Schwarz N=2 algebra, but only to subsingular vectors, for which no construction formulae exist.

The work is organized as follows. In section 2 we review the different types of singular vectors of the N=2 superconformal algebras and we discuss the basic ingredients to derive the mappings between the singular vectors of the Neveu-Schwarz N=2 algebra and the singular vectors of the Topological and of the Ramond N=2 algebras. In section 3 we write down these mappings, which turn into construction formulae for the singular vectors of the Topological and of the Ramond N=2 algebras once the singular vectors of the Neveu-Schwarz N=2 algebra are expressed in terms of their construction formulae themselves. Some final remarks are made in section 4.

Notation

Highest weight (h.w.) vectors denote states annihilated by all the positive modes of the generators of the algebra: \( \mathcal{L}_{n>0}|\chi\rangle = \mathcal{H}_{n>0}|\chi\rangle = \mathcal{G}_{n>0}|\chi\rangle = \mathcal{Q}_{n>0}|\chi\rangle = 0 \) for the

The Neveu-Schwarz, the Ramond and the Topological N=2 superconformal algebras are related to each other through the spectral flows and/or the topological twists. However the Verma modules of the Neveu-Schwarz N=2 algebra are not isomorphic to the Verma modules of the other two N=2 algebras and as a consequence its singular vector structure differs notably from the ones corresponding to the Ramond and to the Topological N=2 algebras.
Topological \( N=2 \) algebra and \( L_{n>0} |\chi\rangle = H_{n>0} |\chi\rangle = G^+_{n>0} |\chi\rangle = G^-_{n>0} |\chi\rangle = 0 \) for the Neveu-Schwarz and for the Ramond \( N=2 \) algebras. These annihilation conditions will be referred to as the h.w. conditions.

Primary states denote non-singular h.w. vectors.

Secondary or descendant states are states obtained by acting on the h.w. vectors with the negative modes of the generators of the algebra and with the fermionic zero modes (\( Q_0 \) and \( G_0 \) for the Topological \( N=2 \) algebra and \( G^+_0 \) and \( G^-_0 \) for the Ramond \( N=2 \) algebra). The fermionic zero modes can also interpolate between two h.w. vectors at the same footing (two primary states or two singular vectors).

The Verma module associated to a h.w. vector consists of the h.w. vector plus the set of secondary states built on it. For most Verma modules of the Topological and of the Ramond \( N=2 \) algebras the h.w. vector is degenerate, the fermionic zero modes interpolating between the two h.w. vectors.

Singular vectors are secondary states that satisfy the h.w. conditions. As a result they and their descendants decouple from all other states in the Verma modules and they have zero norm.

Secondary singular vectors are singular vectors built on singular vectors. Therefore they cannot ‘reach back’ the singular vectors on which they are built by acting with the algebra generators.

Subsingular vectors are secondary states that satisfy the h.w. conditions only after setting a singular vector to zero. Therefore they become singular vectors in quotient modules.

Topological states are the states in the Verma modules of the Topological \( N=2 \) algebra.

R states are the states in the Verma modules of the Ramond \( N=2 \) algebra.

NS states are the states in the Verma modules of the Neveu-Schwarz \( N=2 \) algebra.

Chiral topological states \( |\chi_T^{G,Q}\rangle \) are topological states annihilated by the two fermionic zero modes \( G_0 \) and \( Q_0 \).

Chiral R states \( |\chi_R^{+,-}\rangle \) are R states annihilated by the two fermionic zero modes \( G^+_0 \) and \( G^-_0 \).

Chiral (Antichiral) NS states \( |\chi_{NS}^{ch}\rangle (|\chi_{NS}^a\rangle) \) are NS states annihilated by \( G^+_{-1/2} (G^-_{-1/2}) \).

\( G_0 \)-closed topological states \( |\chi_T^G\rangle \) are non-chiral topological states annihilated by \( G_0 \) (they are BRST-invariant since \( Q_0 \) is the BRST charge).

Helicity (+) R states \( |\chi_R^+\rangle \) are non-chiral R states annihilated by \( G^+_0 \).

Helicity (−) R states \( |\chi_R^-\rangle \) are non-chiral R states annihilated by \( G^-_0 \).

No-label topological states \( |\chi_T\rangle \) are indecomposable topological states that cannot be ex-
pressed as linear combinations of \( G_0 \)-closed states, \( Q_0 \)-closed states and chiral states.

No-helicity \( R \) states \( |\chi_R \rangle \) are indecomposable \( R \) states that cannot be expressed as linear combinations of helicity (+) states, helicity (−) states and chiral states.

Complete Verma modules are the Verma modules which cannot be realized as quotient modules (by setting a singular vector to zero).

Generic Verma modules are complete Verma modules built on generic h.w. vectors. These are the standard h.w. vectors annihilated only by the positive modes of the generators of the algebra and by one (and just one) fermionic zero mode in the case of the Topological and of the Ramond N=2 algebras.

No-label (No-helicity) Verma modules are complete Verma modules built on no-label topological (no-helicity Ramond) h.w. vectors.

Chiral Verma modules are incomplete Verma modules built on chiral h.w. vectors.

2 Preliminaries

2.1 \( N=2 \) Superconformal algebras

The Neveu-Schwarz and the Ramond \( N=2 \) superconformal algebras \cite{14}\cite{22}\cite{25}\cite{26} can be expressed as

\[
\begin{align*}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[H_m, H_n] &= \frac{c}{3}m\delta_{m+n,0}, \\
[L_m, G^+_r] &= \left(\frac{m}{2} - r\right)G^+_m, \\
[H_m, G^+_r] &= \pm G^+_m, \\
[L_m, H_n] &= -nH_{m+n} \\
\{G^-_r, G^+_s\} &= 2L_{r+s} - (r-s)H_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \\
\end{align*}
\]

(2.1)

where \( L_m \) and \( H_m \) are the spin-2 and spin-1 bosonic generators corresponding to the stress-energy momentum tensor and the U(1) current, respectively, and \( G^+_r \) and \( G^-_r \) are the spin-3/2 fermionic generators. These are half-integer moded for the case of the Neveu-Schwarz algebra, and integer moded for the case of the Ramond algebra. The eigenvalues of the bosonic zero modes \( (L_0, H_0) \) are the conformal weight and the U(1) charge of the states. These are split conveniently as \( (\Delta + l, h + q) \) for secondary states, where \( l \) and \( q \) are the level and the relative charge of the state and \( (\Delta, h) \) are the conformal weight and U(1) charge of the primary on which the secondary is built.
Observe that we unify the notation for the $U(1)$ charge of the states of the Neveu-Schwarz algebra and the states of the Ramond algebra since the $U(1)$ charges of the $R$ states will be denoted by $h$, instead of $h \pm \frac{1}{2}$ which was commonly used in the past, and their relative charges $q$ are defined to be integer, like for the NS states.

The Topological N=2 algebra is the symmetry algebra of Topological Conformal Field Theory in two dimensions. It reads [28]

\[
\begin{align*}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, \\
[\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}, \\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, \\
[\mathcal{H}_m, \mathcal{H}_n] &= \frac{\xi}{3}m\delta_{m+n,0}, \\
[\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\
[\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{\xi}{3}(m^2 + m)\delta_{m+n,0}, \\
\end{align*}
\]

where the fermionic generators $\mathcal{Q}_m$ and $\mathcal{G}_m$ correspond to the spin-1 BRST current and the spin-2 fermionic current, respectively, $\mathcal{Q}_0$ being the BRST-charge. The eigenvalues of $(\mathcal{L}_0, \mathcal{H}_0)$ are split, as before, as $(\Delta + l, \hbar + q)$. This algebra is topological because the Virasoro generators are BRST-exact: $\mathcal{L}_m = \frac{1}{2}\{\mathcal{G}_m, \mathcal{Q}_0\}$. This implies, as is well known, that the correlators of the fields do not depend on the two-dimensional metric.

There is also the twisted N=2 algebra [26][29] for which the generators of the U(1) current are half-integer moded. As a consequence there is no U(1) charge for this algebra.

### 2.2 Relations between the N=2 Superconformal algebras

The Neveu-Schwarz, the Ramond and the Topological N=2 superconformal algebras are connected to each other by the spectral flows and/or the topological twists. To be precise, the Neveu-Schwarz and the Topological N=2 algebras are related by the topological twists whereas the Neveu-Schwarz and the Ramond N=2 algebras are related by the spectral flows. The twisted N=2 algebra, however, is not connected to these three N=2 algebras and therefore will not be considered in what follows.

#### 2.2.1 Topological twists

The Topological N=2 algebra (2.2) can be obtained from the Neveu-Schwarz N=2 algebra (2.1) by using one of the two topological twists:
\[
\mathcal{L}_{m}^{(\pm)} = L_{m} \pm \frac{1}{2}(m + 1)H_{m},
\]
\[
\mathcal{H}_{m}^{(\pm)} = \pm H_{m},
\]
\[
G_{m}^{(\pm)} = G_{m+\frac{1}{2}}, \quad Q_{m}^{(\pm)} = G_{m-\frac{1}{2}}^{\mp}, \tag{2.3}
\]

These twists, which we denote as \(T_{W}^{\pm}\), are mirrored under the interchange \(H_{m} \leftrightarrow -H_{m}\), \(G_{r}^{+} \leftrightarrow G_{r}^{-}\). Observe that the h.w. conditions \(G_{1/2}^{\pm} |\chi_{NS}\rangle = 0\) of the Neveu-Schwarz algebra read \(G_{0} |\chi_{T}\rangle = 0\) under \(T_{W}^{\pm}\), respectively. Therefore, any NS h.w. vector results in a \(G_{0}\)-closed or chiral topological state, which is also a h.w. vector as one can easily verify by inspecting the twists (2.3). Conversely, any \(G_{0}\)-closed or chiral topological h.w. vector (and only these) transforms under the twists \(T_{W}^{\pm}\) into a NS h.w. vector. The zero mode \(Q_{0}\), in turn, corresponds to the negative modes \(G_{1/2}^{\mp}\) of the Neveu-Schwarz algebra. Therefore the topological states annihilated by \(Q_{0}\) become antichiral and chiral NS states under the twists \(T_{W}^{\pm}\), respectively.

### 2.2.2 Spectral flows

The even and the odd spectral flows \(U_{\theta}\) and \(A_{\theta}\) are one-parameter families of transformations providing a continuum of \(N=2\) Superconformal algebras [21][22][23]. The even spectral flow \(U_{\theta}\) acting on the generators of the Neveu-Schwarz and of the Ramond \(N=2\) algebras reads

\[
U_{\theta}L_{m}U_{\theta}^{-1} = L_{m} + \theta H_{m} + \frac{\xi}{6}\theta^{2}\delta_{m,0},
\]
\[
U_{\theta}H_{m}U_{\theta}^{-1} = H_{m} + \frac{\xi}{3}\theta\delta_{m,0},
\]
\[
U_{\theta}G_{r}^{+}U_{\theta}^{-1} = G_{r+\theta}^{+},
\]
\[
U_{\theta}G_{r}^{-}U_{\theta}^{-1} = G_{r-\theta}^{-}, \tag{2.4}
\]
satisfying \(U_{\theta}^{-1} = U(-\theta)\). For \(\theta = 0\) it is just the identity operator, i.e. \(U_{0} = 1\). It transforms the \((L_{0}, H_{0})\) eigenvalues, i.e. the conformal weight and the \(U(1)\) charge, \((\Delta, h)\) of a given state as \((\Delta - \theta h + \frac{\xi}{6}\theta^{2}, h - \frac{\xi}{3}\theta)\). From this one gets straightforwardly that the level \(l\) of any secondary state changes to \(l - \theta q\), while the relative charge \(q\) remains equal.

The odd spectral flow \(A_{\theta}\) [23][24] is given by

\[
A_{\theta}L_{m}A_{\theta}^{-1} = L_{m} + \theta H_{m} + \frac{\xi}{6}\theta^{2}\delta_{m,0},
\]
\[
A_{\theta}H_{m}A_{\theta}^{-1} = -H_{m} - \frac{\xi}{3}\theta\delta_{m,0},
\]
\[
A_{\theta}G_{r}^{+}A_{\theta}^{-1} = G_{r-\theta}^{-},
\]
\[
A_{\theta}G_{r}^{-}A_{\theta}^{-1} = G_{r+\theta}^{+}, \tag{2.5}
\]
satisfying $A_\theta^{-1} = A_\theta$. It is therefore an involution. $A_\theta$ is ‘quasi’ mirror symmetric to $U_\theta$, under the exchange $H_m \rightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$ and $\theta \rightarrow -\theta$, and it is in fact the only fundamental spectral flow since it generates the latter [23]. For $\theta = 0$ it is the mirror map, i.e. $A_0 = M$. It transforms the $(L_0, H_0)$ eigenvalues of the states as $(\Delta + \theta h + \frac{c}{6} \theta^2, -h - \frac{c}{3} \theta)$. The level $l$ of the secondary states changes to $l + \theta q$, while the relative charge $q$ reverses its sign.

For half-integer values of $\theta$ the two spectral flows interpolate between the Neveu-Schwarz algebra and the Ramond algebra. In particular, for $\theta = \pm 1/2$ the NS h.w. vectors are transformed into R h.w. vectors algebra with helicities $(\mp)$. As a result the NS singular vectors are transformed into R singular vectors with helicities $(\mp)$ built on R primaries with the same helicities.

By performing the topological twists $T_W^\pm$ (2.3) on the spectral flows (2.4) and (2.5) one obtains the topological spectral flows [30][23]. Of special importance is the topological odd spectral flow $A_1$, denoted simply as $A$, since it transforms all kinds of topological primary states and singular vectors back to topological primary states and singular vectors. It is given by

$$
\begin{align*}
A \mathcal{L}_m A^{-1} &= \mathcal{L}_m - m \mathcal{H}_m, \\
A \mathcal{H}_m A^{-1} &= -\mathcal{H}_m - \frac{c}{3} \delta_{m,0}, \\
A \mathcal{Q}_m A^{-1} &= \mathcal{Q}_m, \\
A \mathcal{G}_m A^{-1} &= \mathcal{G}_m.
\end{align*}
$$

with $A^{-1} = A$. It is therefore an involutive automorphism. It transforms the $(\mathcal{L}_0, \mathcal{H}_0)$ eigenvalues $(\Delta, h)$ of the states as $(\Delta, -h - \frac{c}{3})$, reversing the relative charge of the secondary states and leaving the level invariant, as a consequence. In addition, $A$ also reverses the BRST-invariance properties of the states (primary as well as secondary) mapping $\mathcal{G}_0$-closed $(\mathcal{Q}_0$-closed) states into $\mathcal{Q}_0$-closed $(\mathcal{G}_0$-closed) states, chiral states into chiral states and no-label states into no-label states. Hence the action of $A$ results in the following mappings between topological singular vectors in different Verma modules:

$$
A |\chi_T^{(q)G}_{l,|\Delta, h\rangle}G \rightarrow |\chi_T^{(-q)G}_{l,|\Delta, -h - \frac{c}{3}\rangle}G, \\
A |\chi_T^{(q)G,Q}_{l,|\Delta, h\rangle}G \rightarrow |\chi_T^{(-q)G,Q}_{l,|\Delta, -h - \frac{c}{3}\rangle}G, \\
A |\chi_T^{(q)G,Q}_{l,|\Delta, h\rangle}G \rightarrow |\chi_T^{(-q)G,Q}_{l,|\Delta, -h - \frac{c}{3}\rangle}G, \\
A |\chi_T^{(q)\mathcal{Q}}_{l,|0, h\rangle} \rightarrow |\chi_T^{(-q)\mathcal{Q}}_{l,|0, -h - \frac{c}{3}}\rangle \mathcal{Q},
\$$

and their inverses.
In what follows the singular vectors of the Topological, the Neveu-Schwarz and the Ramond N=2 superconformal algebras will be denoted as $|\chi_T\rangle$, $|\chi_{NS}\rangle$ and $|\chi_R\rangle$, respectively.

### 2.3.1 Singular Vectors of the Topological N=2 Algebra

From the anticommutator $\{Q_0, G_0\} = 2L_0$ one deduces [19] that a topological state (primary or secondary) with non-zero conformal weight can be either $G_0$-closed, or $Q_0$-closed, or a linear combination of both types. The topological states with zero conformal weight, however, can be $Q_0$-closed, or $G_0$-closed, or chiral, or no-label (indecomposable).

As a first classification of the topological secondary states one considers their level $l$, their relative $U(1)$ charge $q$ and their transformation properties under $Q_0$ and $G_0$ (BRST-invariance properties). Hence the topological secondary states will be denoted as $|\chi_T\rangle_i^{(q)G}$ ($G_0$-closed), $|\chi_T\rangle_i^{(q)Q}$ ($Q_0$-closed), $|\chi_T\rangle_i^{(q)G,Q}$ (chiral), and $|\chi_T\rangle_i^{(q)}$ (no-label). The conformal weight and the total $U(1)$ charge of the secondary states are given by $\Delta + l$ and $h + q$, respectively, where $\Delta$ and $h$ are the conformal weight and the $U(1)$ charge of the primary state on which the secondary is built. Therefore only for $\Delta + l = 0$ the secondary states can be chiral or no-label.

There are three different types of topological primaries giving rise to complete Verma modules: $G_0$-closed primaries $|\Delta, h\rangle^G$, $Q_0$-closed primaries $|\Delta, h\rangle^Q$, and no-label primaries $|0, h\rangle$. The singular vectors in generic Verma modules, that is built on $G_0$-closed primaries and/or on $Q_0$-closed primaries are distributed in the following way [19][20]:

- Ten types built on $G_0$-closed primaries $|\Delta, h\rangle^G$:

\[
\begin{array}{c|cccc}
\hline
& q = -2 & q = -1 & q = 0 & q = 1 \\
\hline
G_0\text{-closed} & - & |\chi_T\rangle_i^{(-1)G} & |\chi_T\rangle_i^{(0)G} & |\chi_T\rangle_i^{(1)G} \\
Q_0\text{-closed} & |\chi_T\rangle_i^{(-2)Q} & |\chi_T\rangle_i^{(-1)Q} & |\chi_T\rangle_i^{(0)Q} & - \\
\text{chiral} & - & |\chi_T\rangle_i^{(-1)G,Q} & |\chi_T\rangle_i^{(0)G,Q} & - \\
\text{no-label} & - & |\chi_T\rangle_i^{(-1)} & |\chi_T\rangle_i^{(0)} & - \\
\hline
\end{array}
\]

- Ten types built on $Q_0$-closed primaries $|\Delta, h\rangle^Q$:  

For $\Delta \neq 0$ the h.w. vector of the Verma module is degenerate: there is one $G_0$-closed primary state as well as one $Q_0$-closed primary state, $Q_0$ and $G_0$ interpolating between them. As a result, for $\Delta \neq 0$ the singular vectors of table (2.8) are equivalent to singular vectors of table (2.9) with a shift on the U(1) charges. That is, the singular vectors can be expressed as built on the $G_0$-closed primary or as built on the $Q_0$-closed primary (for the details see ref. [19]). In particular, the uncharged chiral singular vectors are equivalent to charged chiral singular vectors, provided the level $l > 0$, since for them $\Delta = -l \neq 0$. The spectrum of $(\Delta, h)$ corresponding to the $G_0$-closed primaries $|\Delta, h\rangle^G$ and to the $Q_0$-closed primaries $|\Delta, h\rangle^Q$ which contain singular vectors in their Verma modules was derived partially in ref. [19] and completely in ref. [33].

An important observation is that chiral singular vectors $|\chi_T\rangle^{(q)G,Q}_l$ can be regarded as particular cases of $G_0$-closed singular vectors $|\chi_T\rangle^{(q)G}_l$ and/or as particular cases of $Q_0$-closed singular vectors $|\chi_T\rangle^{(q)Q}_l$. That is, $G_0$-closed and $Q_0$-closed singular vectors may ‘become’ chiral (although not necessarily) when the conformal weight of the singular vector turns out to be zero, i.e. $\Delta + l = 0$. In fact the singular vectors of types $|\chi_T\rangle^{(0)Q}_l$ and $|\chi_T\rangle^{(-1)Q}_l$ in table (2.8) always become chiral when the conformal weight is zero and the same is true for the singular vectors of types $|\chi_T\rangle^{(0)G}_l$ and $|\chi_T\rangle^{(1)G}_l$ in table (2.9) [20]. In other words, for zero conformal weight $\Delta + l = 0$ these types of singular vectors are absent from tables (2.8) and (2.9).

Inside a given Verma module the topological singular vectors come in pairs at the same level, one of them $G_0$-closed or chiral (annihilated necessarily by $G_0$) and the other $Q_0$-closed or chiral (annihilated necessarily by $Q_0$). The only exception to this rule occurs in the presence of no-label singular vectors. In this ‘degenerate’ case there are four singular vectors at the same level: one $G_0$-closed, one $Q_0$-closed, one chiral and one no-label. In the general case there are only two singular vectors at the same level, the fermionic zero modes $G_0$ and $Q_0$ interpolating between them, although it can also happen that both are chiral and then disconnected from each other [33]. To be precise, $G_0$ and $Q_0$ interpolate between two singular vectors with non-zero conformal weight in both directions, whereas they produce secondary singular vectors when acting on singular vectors with zero conformal weight. That is, inside a given Verma module $V(\Delta, h)$ and for a given level $l$ the topological singular vectors with non-zero conformal weight are

<table>
<thead>
<tr>
<th>$q = -1$</th>
<th>$q = 0$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$-closed</td>
<td>$\chi_T^{(0)G}_l$</td>
<td>$\chi_T^{(1)G}_l$</td>
<td>$\chi_T^{(2)G}_l$</td>
</tr>
<tr>
<td>$Q_0$-closed</td>
<td>$\chi_T^{(-1)Q}_l$</td>
<td>$\chi_T^{(0)Q}_l$</td>
<td>$\chi_T^{(1)Q}_l$</td>
</tr>
<tr>
<td>chiral</td>
<td>$\chi_T^{(0)G,Q}_l$</td>
<td>$\chi_T^{(1)G,Q}_l$</td>
<td>$\chi_T^{(2)G,Q}_l$</td>
</tr>
<tr>
<td>no-label</td>
<td>$\chi_T^{(0)}_l$</td>
<td>$\chi_T^{(1)}_l$</td>
<td>$\chi_T^{(2)}_l$</td>
</tr>
</tbody>
</table>
connected by the action of $Q_0$ and $G_0$ as:
\[ Q_0 \vert \chi_T \rangle_i^{(q)G} \rightarrow \vert \chi_T \rangle_i^{(q-1)Q}, \quad G_0 \vert \chi_T \rangle_i^{(q)Q} \rightarrow \vert \chi_T \rangle_i^{(q+1)G}, \quad (2.10) \]
where the arrows can be reversed (up to constants), using $G_0$ and $Q_0$ respectively, since the conformal weight of the singular vectors is different from zero, \textit{i.e.} $\Delta + l \neq 0$. Otherwise, on the right-hand side of the arrows one obtains \textit{chiral secondary} singular vectors which cannot ‘come back’ to the singular vectors on the left-hand side:
\[ Q_0 \vert \chi_T \rangle_i^{(q)G} \rightarrow \vert \chi_T \rangle_i^{(q-1)G,Q}, \quad G_0 \vert \chi_T \rangle_i^{(q)Q} \rightarrow \vert \chi_T \rangle_i^{(q+1)G,Q}. \quad (2.11) \]

Regarding no-label singular vectors $\vert \chi_T \rangle_i^{(q)}$, they always satisfy $\Delta + l = 0$. The action of $G_0$ and $Q_0$ on a no-label singular vector produce three secondary singular vectors which cannot come back to the no-label singular vector $\vert \chi_T \rangle_i^{(q)}_{l=-\Delta}$ by acting with $G_0$ and $Q_0$:
\[ Q_0 \vert \chi_T \rangle_i^{(q)}_{l=-\Delta} \rightarrow \vert \chi_T \rangle_i^{(q-1)G,Q}_{l=-\Delta}, \quad G_0 \vert \chi_T \rangle_i^{(q)}_{l=-\Delta} \rightarrow \vert \chi_T \rangle_i^{(q+1)G,Q}_{l=-\Delta}. \quad (2.12) \]

The $Q_0$-closed and no-label h.w. vectors (primaries or singular vectors) are transformed, under the twists $T_W^\pm$ (2.3), into states of the Neveu-Schwarz algebra which are not h.w. vectors since they are not annihilated by one of the modes $G_{1/2}^\pm$ (because the topological state is not annihilated by $G_0$). The $G_0$-closed and chiral h.w. vectors, however, are transformed under $T_W^\pm$ into NS h.w. vectors. In particular the $G_0$-closed primaries $\vert \Delta, h \rangle^G$ generate the topological Verma modules which are mapped to the complete NS Verma modules, whereas the chiral primaries $\vert 0, h \rangle^G$ generate topological chiral (incomplete) Verma modules which are mapped to antichiral and chiral (incomplete) NS Verma modules under $T_W^\pm$, respectively.

We will see in next section that, with the exception of the indecomposable no-label singular vectors, all other 16 types of topological singular vectors in tables (2.8) and (2.9) can be mapped to NS singular vectors using the twists $T_W^\pm$ and the involutive automorphism $A$ (2.6). The no-label singular vectors, however, can be mapped only to NS \textit{subsingular} vectors [31], as was shown in ref. [32], for which there are no construction formulae.

Regarding no-label Verma modules, the spectrum of $h$ corresponding to the no-label primaries $\vert 0, h \rangle$ containing singular vectors in their Verma modules was derived in ref. [33]. The possible existing types of singular vectors in no-label Verma modules are the following nine types [19][20][33] built on no-label primaries $\vert 0, h \rangle$:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$-closed</td>
<td>$\rightarrow$</td>
<td>$\vert \chi_T \rangle_i^{(-1)G} \vert \chi_T \rangle_i^{(0)Q}$</td>
<td>$\vert \chi_T \rangle_i^{(1)G} \vert \chi_T \rangle_i^{(2)Q}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_0$-closed</td>
<td>$\rightarrow$</td>
<td>$\vert \chi_T \rangle_i^{(-2)Q}$</td>
<td>$\vert \chi_T \rangle_i^{(-1)Q}$</td>
<td>$\vert \chi_T \rangle_i^{(0)Q}$</td>
<td>$\vert \chi_T \rangle_i^{(1)Q}$</td>
</tr>
<tr>
<td>chiral</td>
<td>$\rightarrow$</td>
<td>$\rightarrow$</td>
<td>$\rightarrow$</td>
<td>$\rightarrow$</td>
<td>$\rightarrow$</td>
</tr>
</tbody>
</table>
Since the no-label primaries do not have a counterpart in the Neveu-Schwarz algebra, the singular vectors built on them cannot be mapped to NS singular vectors unless they are located in the submodules generated by the level-zero singular vectors $G_0|0, h\rangle = |0, h + 1\rangle^G$ or $Q_0|0, h\rangle = |0, h - 1\rangle^Q$. In these cases the singular vectors can be regarded as built on the $G_0$-closed h.w. vector $|0, h + 1\rangle^G$ or on the $Q_0$-closed h.w. vector $|0, h - 1\rangle^Q$ and they fit into the tables (2.8) and (2.9), respectively. Observe that all the singular vectors in no-label Verma modules are either $G_0$-closed or $Q_0$-closed, except a unique chiral singular vector which is at level zero: $G_0 Q_0|0, h\rangle$. There are no no-label singular vectors built on no-label primaries because their level, that is their conformal weight, must be zero and it is not possible to construct a no-label singular vector at level zero just by the action of the fermionic zero modes $G_0$ and $Q_0$.

2.3.2 Singular Vectors of the Ramond N=2 Algebra

The fermionic zero modes characterize the R states as being annihilated by $G_0^+$ or by $G_0^-$ (or as a linear combination of these), as the anticommutator $\{G_0^+, G_0^-\} = 2L_0 - \frac{c}{12}$ shows. We call these states ‘helicity’ (+) and ‘helicity’ (−) states. However, for conformal weight $\Delta = \frac{c}{24}$ the R states can be annihilated by both $G_0^+$ and $G_0^-$ (we call them chiral), and there also exist indecomposable ‘no-helicity’ states which cannot be expressed as linear combinations of helicity (+), helicity (−) and chiral states. These no-helicity states, singular vectors in particular, have been reported for the first time in ref. [32], and they were completely overlooked in the early literature. The R singular vectors will be denoted therefore as $|\chi^+_{R,l}\rangle^{(q)+}$, $|\chi^-_{R,l}\rangle^{(q)-}$, $|\chi^{+-}_{R,l}\rangle^{(q)+-}$ or $|\chi^{q}_{R,l}\rangle$, where, in addition to the level $l$ and the relative charge $q$, the helicities indicate that the vector is annihilated by $G_0^+$ or $G_0^-$, or both, or none, respectively.

We already see that the case $\Delta = \frac{c}{24}$ for the Ramond N=2 algebra is very similar to the case $\Delta = 0$ for the Topological N=2 algebra, the chiral (+−) and no-helicity R states being analogous to the chiral (G, Q) and no-label topological states, respectively. This is not just a similarity but a consequence of the fact that the Verma modules of the Ramond N=2 algebra are isomorphic to the Verma modules of the Topological N=2 algebra [33]. In particular one can construct a one-to-one map between each R singular vector and a topological singular vector at the same level and with the same relative charge. As a consequence, most results concerning the topological singular vectors, in particular the classification summarized in tables (2.8), (2.9) and (2.13), can be transferred straightforwardly to the R singular vectors. The argument goes as follows [33].

Let us compose the topological twists (2.3), which transform the Topological N=2 algebra into the Neveu-Schwarz N=2 algebra, with the spectral flows (2.4) and (2.5), which transform the Neveu-Schwarz N=2 algebra into the Ramond N=2 algebra. By
analysing all possible combinations one obtains that the only map that transforms the topological states into R states, preserving the level $l$ and the relative charge $q$, is given by the exactly equivalent compositions $\mathcal{A}_{-1/2} (T_W^-)^{-1}$ and $\mathcal{U}_{-1/2} (T_W^+)^{-1}$:

$$|\chi_R|^{(q)}_l = \mathcal{A}_{-1/2} (T_W^-)^{-1} |\chi_T|^{(q)}_l = \mathcal{U}_{-1/2} (T_W^+)^{-1} |\chi_T|^{(q)}_l.$$  \hspace{1cm} (2.14)

This map is one-to-one because it transforms every topological state into a R state, and the other way around: $|\chi_T|^{(q)}_l = (T_W^-) \mathcal{A}_{-1/2} |\chi_R|^{(q)}_l = (T_W^+) \mathcal{U}_{1/2} |\chi_R|^{(q)}_l$ (from now on we will keep only the composition $\mathcal{A}_{-1/2} (T_W^-)^{-1}$). Furthermore if $|\chi_T|^{(q)}_l$ is singular, i.e. satisfies the topological h.w. conditions $L_{n\geq 1} |\chi_T|^{(q)}_l = H_{n\geq 1} |\chi_T|^{(q)}_l = G_{n\geq 1} |\chi_T|^{(q)}_l = G_{n\geq 1} |\chi_T|^{(q)}_l = 0$, then $|\chi_R|^{(q)}_l$ is also singular, satisfying in turn the R h.w. conditions $L_{n\geq 1} |\chi_R|^{(q)}_l = H_{n\geq 1} |\chi_R|^{(q)}_l = G^+_{n\geq 1} |\chi_R|^{(q)}_l = G^-_{n\geq 1} |\chi_R|^{(q)}_l = 0$. To see this one has to study first the transformation, under $\mathcal{A}_{-1/2} (T_W^-)^{-1}$, of the h.w. vectors of the topological Verma modules. There are four cases to analyse, corresponding to the topological h.w. vectors being $\mathcal{G}_0$-closed $|\Delta, h\rangle^G$, $\mathcal{Q}_0$-closed $|\Delta, h\rangle^Q$, chiral $|0, h\rangle^{G,Q}$, and no-label $|0, h\rangle$. By carefully keeping track of the transformation of the positive and zero modes of the topological generators one obtains that under $\mathcal{A}_{-1/2} (T_W^-)^{-1}$:

i) The $\mathcal{G}_0$-closed topological h.w. vectors $|\Delta, h\rangle^G$ are mapped to helicity-$(+)$ R h.w. vectors $|\Delta_R, h_R\rangle_R^+ = |\Delta + \frac{q}{24}, h + \frac{q}{6}\rangle_R^+$.  

ii) The $\mathcal{Q}_0$-closed topological h.w. vectors $|\Delta, h\rangle^Q$ are mapped to helicity-$(−)$ R h.w. vectors $|\Delta_R, h_R\rangle_R^− = |\Delta + \frac{q}{24}, h + \frac{q}{6}\rangle_R^−$.  

iii) The chiral topological h.w. vectors $|0, h\rangle^{G,Q}$ are mapped to chiral R h.w. vectors $|\Delta_R, h_R\rangle_R = |\frac{q}{24}, h + \frac{q}{6}\rangle_R$.  

iv) The no-label topological h.w. vectors $|0, h\rangle$ are mapped to no-helicity R h.w. vectors $|\Delta_R, h_R\rangle_R = |\frac{q}{24}, h + \frac{q}{6}\rangle_R$.

Now by taking into account that singular vectors are just particular cases of h.w. vectors one deduces that the topological singular vectors are transformed under $\mathcal{A}_{-1/2} (T_W^-)^{-1}$ into R singular vectors at the same level $l$, with the same relative charge $q$, and with the helicities determined by the exchange $G \rightarrow +, Q \rightarrow −$. That is, $|\chi_T|^{(q)}_l |\Delta, h\rangle^G \rightarrow |\chi_R|^{(q)}_l |\Delta_R, h_R\rangle_R^+$, $|\chi_T|^{(q)}_l |\Delta, h\rangle^Q \rightarrow |\chi_R|^{(q)}_l |\Delta_R, h_R\rangle_R^−$, $|\chi_T|^{(q)}_l |0, h\rangle^{G,Q} \rightarrow |\chi_R|^{(q)}_l |\Delta_R, h_R\rangle_R$ and $|\chi_T|^{(q)}_l \rightarrow |\chi_R|^{(q)}_l$. Therefore the classification of the topological singular vectors in tables (2.8), (2.9) and (2.13), is also valid for the R singular vectors, simply by taking into account that $\Delta_R = \Delta + \frac{q}{24}$, $h_R = h + \frac{q}{6}$, and exchanging the labels $G \rightarrow +$ and $Q \rightarrow −$. As a consequence the R singular vectors in complete Verma modules are distributed as follows:

- Ten types built on helicity $(+)$ primaries $|\Delta, h\rangle^+$:
\begin{align}
|q| = -2 & & |q| = -1 & & |q| = 0 & & |q| = 1 \\
\text{helicity (+)} & - & |\chi_{R}^{(-1)+}_l| & & |\chi_{R}^{(0)+}_l| & & |\chi_{R}^{(1)+}_l| & & (2.15) \\
\text{helicity (-)} & |\chi_{R}^{(-2)-}_l| & & |\chi_{R}^{(-1)-}_l| & & |\chi_{R}^{(0)-}_l| & & |\chi_{R}^{(1)-}_l| & & \\
\text{chiral} & - & |\chi_{R}^{(-1)+,-}_l| & & |\chi_{R}^{(0)+,-}_l| & & - & & - & & \\
\text{no-helicity} & - & |\chi_{R}^{(-1)}_l| & & |\chi_{R}^{(0)}_l| & & - & & - & & \\
\end{align}

- Ten types built on helicity (-) primaries $|\Delta, h\rangle^-$:

\begin{align}
|q| = -1 & & |q| = 0 & & |q| = 1 & & |q| = 2 \\
\text{helicity (+)} & - & |\chi_{R}^{(0)+}_l| & & |\chi_{R}^{(1)+}_l| & & |\chi_{R}^{(2)+}_l| & & (2.16) \\
\text{helicity (-)} & |\chi_{R}^{(-1)-}_l| & & |\chi_{R}^{(0)-}_l| & & |\chi_{R}^{(1)-}_l| & & - & & \\
\text{chiral} & - & |\chi_{R}^{(0)+,-}_l| & & |\chi_{R}^{(1)+,-}_l| & & - & & - & & \\
\text{no-helicity} & - & |\chi_{R}^{(0)}_l| & & |\chi_{R}^{(1)}_l| & & - & & - & & \\
\end{align}

- Nine types built on no-helicity primaries $|\frac{c}{24}, h\rangle$:

\begin{align}
|q| = -2 & & |q| = -1 & & |q| = 0 & & |q| = 1 & & |q| = 2 \\
\text{helicity (+)} & - & |\chi_{R}^{(-1)+}_l| & & |\chi_{R}^{(0)+}_l| & & |\chi_{R}^{(1)+}_l| & & |\chi_{R}^{(2)+}_l| & & (2.17) \\
\text{helicity (-)} & |\chi_{R}^{(-2)-}_l| & & |\chi_{R}^{(-1)-}_l| & & |\chi_{R}^{(0)-}_l| & & |\chi_{R}^{(1)-}_l| & & - & & \\
\text{chiral} & - & - & |\chi_{R}^{(0)+,-}_l| & & |\chi_{R}^{(1)+,-}_l| & & - & & - & & \\
\end{align}

Analogously as for the topological case, for $\Delta \neq \frac{c}{24}$ the h.w. vector of the generic Verma modules is degenerate: there is one helicity (+) primary state as well as one helicity (-) primary state, $G_0^+$ and $G_0^-$ interpolating between them. In addition, for $\Delta + l = \frac{c}{24}$ some types of singular vectors are absent from tables (2.15) and (2.16): the singular vectors of types $|\chi_{R}^{(0)-}_l|$ and $|\chi_{R}^{(-1)+}_l|$ in table (2.15) and the singular vectors of types $|\chi_{R}^{(0)+}_l|$ and $|\chi_{R}^{(1)+}_l|$ in table (2.16) ‘become’ chiral. Inside a Verma module $G_0^+$ and $G_0^-$ interpolate between two singular vectors, in both directions if $\Delta + l \neq \frac{c}{24}$, or in only one direction producing secondary singular vectors if $\Delta + l = \frac{c}{24}$. Therefore the $R$ singular vectors come in pairs at the same level, like the topological ones. Generically one of the singular vectors belong to the ‘(+’) sector’ and the other to the ‘(−) sector’, as was stated in the early literature, although there are many exceptions (see the discussion in ref. [33]). For example, the no-helicity singular vectors do not belong to any of these sectors.
The spectrum of \((\Delta, h)\) corresponding to the helicity \((+)\) primaries \(|\Delta, h\rangle^+\) and to the helicity \((-)\) primaries \(|\Delta, h\rangle^-\) which contain singular vectors in their Verma modules was written down for the first time in ref. [26]. The existence of no-helicity singular vectors was overlooked, however. The spectrum of \(h\) corresponding to the no-helicity primaries \(|\frac{c}{24}, h\rangle\) containing singular vectors in their Verma modules was written in ref. [33].

In the case of the Ramond N=2 algebra both the helicity \((+)\) and the helicity \((-)\) h.w. vectors are transformed into NS h.w. vectors by the spectral flows \(U_\theta\) and \(A_\theta\) (for suitable values of \(\theta\)). This contrasts the situation for the case of the Topological N=2 algebra where the \(Q_0\)-closed h.w. vectors are not mapped to NS h.w. vectors by the topological twists. The chiral R h.w. vectors in turn are transformed into chiral and antichiral NS h.w. vectors too. The no-helicity h.w. vectors (primaries or singular vectors) are not transformed into NS h.w. vectors under the spectral flows \(U_\theta\) and \(A_\theta\), for any value of \(\theta\). The no-helicity singular vectors can be mapped only to NS subsingular vectors [32], as was the case for the no-label topological singular vectors. All other 16 types of R singular vectors in tables (2.15) and (2.16) can be mapped to NS singular vectors using the spectral flows. The singular vectors built on no-helicity h.w. vectors can be mapped to NS singular vectors only if they are descendants of the level-zero singular vectors \(G^+_0 |\frac{c}{24}, h\rangle = |\frac{c}{24}, h + 1\rangle^+ \) or \(G^-_0 |\frac{c}{24}, h\rangle = |\frac{c}{24}, h - 1\rangle^-\) since in these cases they fit in tables (2.15) and (2.16), respectively.

### 2.3.3 Singular Vectors of the Neveu-Schwarz N=2 Algebra

The Neveu-Schwarz N=2 algebra contains no fermionic zero modes. As a consequence the classification of h.w. vectors, singular vectors and Verma modules is much simpler for this algebra than for the Topological and for the Ramond N=2 algebras. Basically one considers only generic and chiral (and antichiral) h.w. vectors and singular vectors. The chiral and antichiral h.w. vectors and singular vectors are annihilated by the negative modes \(G^+_{-1/2}\) and \(G^-_{-1/2}\), respectively. The generic h.w. vectors give rise to complete Verma modules while the chiral and antichiral h.w. vectors produce incomplete chiral Verma modules. In the generic Verma modules the possible singular vectors belong to the following types [9][20]:

\[
\begin{array}{c|ccc}
q & -1 & 0 & 1 \\
\hline
\text{generic} & |\chi_{NS}^{(-1)}\rangle & |\chi_{NS}^{(0)}\rangle & |\chi_{NS}^{(1)}\rangle \\
\text{chiral} & - & |\chi_{NS}^{(0)ch}\rangle & |\chi_{NS}^{(1)ch}\rangle \\
\text{antichiral} & |\chi_{NS}^{(-1)a}\rangle & |\chi_{NS}^{(0)a}\rangle & - \\
\end{array}
\]

The spectrum of \((\Delta, h)\) corresponding to the generic NS primaries \(|\Delta, h\rangle\) which con-
tain singular vectors in their Verma modules was obtained for the first time in refs. [26] and [27]. The fact that \(|q| \leq 1\) for all NS singular vectors, not only for the primitive singular vectors but also for the secondary ones, was only realized and proved in ref. [9], however.

All the h.w. vectors and singular vectors of the Neveu-Schwarz N=2 Algebra can be mapped to h.w. vectors and singular vectors of the Topological and of the Ramond N=2 algebras. The converse is not true, however, as we have already pointed out.

3 Construction Formulae

About six years ago construction formulae for the singular vectors of the Neveu-Schwarz N=2 algebra were computed by Dörrzapf. Using the fusion method explicit formulae were obtained [8] for all the charged singular vectors, and for a class of uncharged singular vectors. Later using the analytic continuation method explicit formulae were obtained for all the uncharged singular vectors [9]. In what follows we will show that these construction formulae for the NS singular vectors also provide construction formulae for 16 types of topological singular vectors and for 16 types of R singular vectors, as one can write maps from the NS singular vectors to these 16 + 16 types of topological and R singular vectors.

3.1 Maps from NS to topological singular vectors

To derive the maps from the NS singular vectors to the topological ones we will proceed in the following way. First we will construct the ‘box’ diagrams obtained by the actions of the fermionic zero modes \(G_0, Q_0\), eqns. (2.10)-(2.11), and the action of the spectral flow automorphism \(A\), eqns. (2.7). We will be interested only in the box diagrams which contain \(G_0\)-closed topological singular vectors built on \(G_0\)-closed primaries; \textit{i.e.} the singular vectors of type \(|\chi_{T/l}|_{\phi}^G\) in table (2.8), since only these types have a direct relation with the generic NS singular vectors via the topological twists. From the box diagrams one can deduce straightforwardly two different maps from the NS singular vectors to each topological singular vector, taking into account that in every box diagram the topological singular vector of type \(|\chi_{T/l}|_{\phi}^G\) can be transformed into two NS singular vectors using the two topological twists \(T_{\pm}W\) (2.3). These two NS singular vectors are mirror-symmetric under the exchange \(H_m \rightarrow -H_m\) and \(G_r^+ \leftrightarrow G_r^-\); therefore they have opposite U(1) charges and are located in mirror-symmetric Verma modules:

\[
|\chi_{T/l}|_{\phi}^G = T_+^W |\chi_{NS}^{(q)}\rangle_{l-q/2,\Delta-h/2, h} = T_-^W |\chi_{NS}^{(-q)}\rangle_{l-q/2,\Delta-h/2, -h}. \tag{3.1}
\]
Let us start with the box diagrams which contain an uncharged singular vector of type $|\chi_T^{(0)}G_l,|\phi\rangle_G^{(0)}$ in the Verma module $V(|\Delta, h)^G_l$. For non-zero conformal weight, $\Delta+l \neq 0$, the box diagram, shown in (3.2), consists of singular vectors of types $|\chi_T^{(0)}G_l,|\phi\rangle_G^{(0)}$ and $|\chi_T^{(-1)}G_l,|\phi\rangle_G^{(0)}$, at level $l$ in the Verma module $V(|\Delta, h)^G_l$, and singular vectors of types $|\chi_T^{(0)}Q_l,|\phi\rangle_Q^{(0)}$ and $|\chi_T^{(1)}Q_l,|\phi\rangle_Q^{(0)}$, also at level $l$ in the Verma module $V(|\Delta, -h - c/3)^Q_l)$.

The arrows $G_0$ and $Q_0$ can be reversed (up to constants) using $Q_0$ and $G_0$ respectively; that is, the fermionic zero modes interpolate between two singular vectors, one charged and one uncharged, at the same level in the same Verma module.

For $\Delta + l = 0$, the conformal weight of the singular vectors is zero, so that the corresponding arrows $Q_0$, $G_0$ cannot be reversed, producing secondary chiral singular vectors $|\chi_T^{(-1)}G_l,|\phi\rangle_G^{(0)}$ and $|\chi_T^{(1)}G_l,|\phi\rangle_G^{(0)}$ on the right-hand side, at level zero with respect to the singular vectors on the left-hand side. The corresponding box diagram is therefore:

\[
\begin{align*}
|\chi_T^{(0)}G_l,|\phi\rangle_G^{(0)} & \xrightarrow{Q_0} |\chi_T^{(-1)}G_l,|\phi\rangle_G^{(0)} \\
\mathcal{A} & \updownarrow \\
|\chi_T^{(0)}Q_l,|\phi\rangle_Q^{(0)} & \xrightarrow{G_0} |\chi_T^{(1)}Q_l,|\phi\rangle_Q^{(0)} \\
\end{align*}
\]

The untwisting of the uncharged singular vector $|\chi_T^{(0)}G_l,|\phi\rangle_G^{(0)}$; using $T^+ (2.3)$, produces two uncharged mirror-symmetric NS singular vectors located in mirror-symmetric Verma modules. Therefore, as shown in (3.1), the twisting of two uncharged mirror-symmetric NS singular vectors, using $T^+_W$ and $T^-_W$ respectively, produces the same uncharged topological singular vector of type $|\chi_T^{(0)}G_l,|\phi\rangle_G^{(0)}$. That is, one has the maps:

\[
|\chi_T^{(0)}G_l,|\phi\rangle_G^{(0)} = T^+_W |\chi_{NS}^{(0)}G_l,|\phi\rangle^{(0)} = T^-_W |\chi_{NS}^{(0)}G_l,|\phi\rangle^{(0)} \\
\]

where we have redefined $\Delta$ as the conformal weight of the NS primaries. The maps from the NS singular vectors to the remaining topological singular vectors in diagrams (3.2) and (3.3) can be derived now resulting in the following expressions:
|χT⟩_{l,|Δ+h/2,−h−c/3⟩Q} = ∂T^+_W |χNS⟩_{l,|Δ,−h⟩}
|χT⟩_{l,|Δ+h/2,2⟩C} = Q_0 T^+_W |χNS⟩_{l,|Δ,−h⟩}
|χT⟩_{l,|Δ+h/2,−h−c/3⟩Q} = ∂Q_0 T^+_W |χNS⟩_{l,|Δ,−h⟩}
|χT⟩_{l,|Δ−h⟩C} = ∂Q_0 T^+_W |χNS⟩_{l,|Δ,−h⟩}
|χT⟩_{l,|Δ−h⟩C} = ∂Q_0 T^+_W |χNS⟩_{l,|Δ,−h⟩}

(3.5)

and mirror-symmetric expressions using \( T^-_W |χNS⟩_{l,|Δ,−h⟩} \). Observe that the last two maps, to chiral singular vectors, are not invertible since the arrows \( Q_0, Q_0 \) in diagram (3.3) cannot be reversed.

One finds similar results associated to the charge \( q = 1 \) singular vector \(|χT⟩_{l,|Δ,−h⟩}⟩\) in the Verma module \( V(|Δ,−h⟩⟩\), as shown in diagrams (3.6) and (3.7). For \( Δ + l ≠ 0 \) the box diagram consists of singular vectors of types \(|χT⟩_{l,|Δ,−h⟩}⟩\) and \(|χT⟩_{l,|Δ,−h⟩}⟩\) at the same level \( l \) in the Verma module \( V(|Δ,−h⟩⟩\), and singular vectors of types \(|χT⟩_{l,|Δ,−h⟩}⟩\) and \(|χT⟩_{l,|Δ,−h⟩}⟩\) also at level \( l \) in the Verma module \( V(|Δ,−h−c/3⟩⟩\).

\[
|χT⟩_{l,|Δ,−h⟩C} \xrightarrow{Q_0} |χT⟩_{l,|Δ,−h⟩C}
\mathcal{A} \downarrow \hspace{1cm} \uparrow \mathcal{A}
|χT⟩_{l,|Δ,−h⟩C} \xrightarrow{Q_0} |χT⟩_{l,|Δ,−h⟩C}
\mathcal{A} \downarrow \hspace{1cm} \uparrow \mathcal{A}

(3.6)

For the case of zero conformal weight \( Δ + l = 0 \) the uncharged singular vectors on the right-hand side become secondary chiral singular vectors:

\[
|χT⟩_{l,|Δ,−h⟩C} \xrightarrow{Q_0} |χT⟩_{l,|Δ,−h⟩C}
\mathcal{A} \downarrow \hspace{1cm} \uparrow \mathcal{A}
|χT⟩_{l,|Δ,−h⟩C} \xrightarrow{Q_0} |χT⟩_{l,|Δ,−h⟩C}
\mathcal{A} \downarrow \hspace{1cm} \uparrow \mathcal{A}

(3.7)

The untwisting of the charged singular vector \(|χT⟩_{l,|Δ,−h⟩}⟩\), using \( T^+_W \) (2.3), produces two charged mirror-symmetric NS singular vectors located in mirror-symmetric Verma modules. Conversely, the twisting of two charged mirror-symmetric NS singular vectors, using \( T^+_W \) and \( T^-_W \) respectively, produces the same charged \(|q⟩ = 1 \) topological singular vector, as shown in (3.1). For charge \( q = 1 \) one finds the maps:

\[
|χT⟩_{l,|Δ+h/2,−h⟩C} = T^+_W |χNS⟩_{l,|Δ,−h⟩}⟩ = T^-_W |χNS⟩_{l,|Δ,−h⟩}⟩.

(3.8)
where we have redefined $\Delta$ again, for convenience. The maps from the NS singular vectors to the remaining topological singular vectors in diagrams (3.6) and (3.7) are given by:

\[
\begin{align*}
|\chi_T^{(1)Q}_{l,|\Delta+h/2, -h-\xi\rangle} &= A \ T_W^+ |\chi_{NS}^{(1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(2)Q}_{l,|\Delta+h/2, h\rangle} &= Q_0 \ T_W^+ |\chi_{NS}^{(1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(1)G}_L|\Delta+h/2, -h-\xi\rangle &= A \ Q_0 \ T_W^+ |\chi_{NS}^{(1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(2)G}_L|\Delta+h/2, h\rangle &= Q_0 \ T_W^+ |\chi_{NS}^{(1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(1)G}_L|\Delta+h/2, -h-\xi\rangle &= A \ Q_0 \ T_W^+ |\chi_{NS}^{(1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(2)G}_L|\Delta+h/2, h\rangle &= Q_0 \ T_W^+ |\chi_{NS}^{(1)}_{l+1/2,|\Delta, h}\rangle \\
\end{align*}
\]

and mirror-symmetric expressions using $T_W^- |\chi_{NS}^{(-1)}_{l+1/2,|\Delta, -h}\rangle$. As before, the last two maps, to chiral singular vectors, are not invertible since the arrows $Q_0, G_0$ in diagram (3.7) cannot be reversed.

Finally let us take a charge $q = -1$ singular vector $|\chi_T^{(-1)G}_{l,|\Delta, h\rangle}$ in the Verma module $V(|\Delta, h\rangle^G)$. For $\Delta + l \neq 0$ the box diagram (3.10) consists of singular vectors of the types $|\chi_T^{(-1)G}_{l,|\Delta, h\rangle}$ and $|\chi_T^{(-2)Q}_{l,|\Delta, h\rangle}$, at the same level $l$ in the Verma module $V(|\Delta, h\rangle^G)$, and singular vectors of the types $|\chi_T^{(1)Q}_{l,|\Delta, h\rangle}$ and $|\chi_T^{(2)G}_{l,|\Delta, h\rangle}$ also at level $l$ in the Verma module $V(|\Delta, h-c/3\rangle^G)$.

\[
\begin{align*}
|\chi_T^{(-1)G}_{l,|\Delta, h\rangle} &\xrightarrow{Q_0} |\chi_T^{(-2)Q}_{l,|\Delta, h\rangle} \\
&\downarrow A \\
|\chi_T^{(1)Q}_{l,|\Delta, h-\xi\rangle} &\xrightarrow{g_0} |\chi_T^{(2)G}_{l,|\Delta, h-\xi\rangle}
\end{align*}
\]

In this case the twistings of the two charged mirror-symmetric NS singular vectors which produce the $q = -1$ charged topological singular vector read:

\[
|\chi_T^{(-1)G}_{l,|\Delta+h/2, h\rangle} = T_W^+ |\chi_{NS}^{(-1)}_{l+1/2,|\Delta, h}\rangle = T_W^- |\chi_{NS}^{(1)}_{l+1/2,|\Delta, -h}\rangle.
\]

The maps from the NS singular vectors to the remaining topological singular vectors in diagram (3.10) result as follows:

\[
\begin{align*}
|\chi_T^{(1)Q}_{l,|\Delta+h/2, -h-\xi\rangle} &= A \ T_W^+ |\chi_{NS}^{(-1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(-2)Q}_{l,|\Delta+h/2, h\rangle} &= Q_0 \ T_W^+ |\chi_{NS}^{(-1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(2)G}_{l,|\Delta+h/2, -h-\xi\rangle} &= A \ Q_0 \ T_W^+ |\chi_{NS}^{(-1)}_{l+1/2,|\Delta, h}\rangle \\
|\chi_T^{(2)G}_{l,|\Delta+h/2, h\rangle} &= Q_0 \ T_W^+ |\chi_{NS}^{(-1)}_{l+1/2,|\Delta, h}\rangle \\
\end{align*}
\]

and mirror-symmetric expressions using $T_W^- |\chi_{NS}^{(1)}_{l+1/2,|\Delta, -h}\rangle$.

Let us come back to diagram (3.10). For $\Delta + l = 0$ the ‘would be’ secondary chiral singular vectors with $|q| = 2$ simply do not exist, as follows from the results in tables (2.8)
and (2.9). As a consequence, the singular vectors of types $|\chi T^{(-1)G,Q}_{l,|l\rangle}^G\rangle$ and $|\chi T^{(1)G,Q}_{l,|l\rangle}^G\rangle$ ‘become’ actually chiral for zero conformal weight, i.e. of types $|\chi T^{(-1)G,Q}_{l,|l\rangle}^G\rangle$ and $|\chi T^{(1)G,Q}_{l,|l\rangle}^G\rangle$ instead, and the box diagram reduces to two chiral singular vectors, connected by $A$. Thus one has the maps:

$$
|\chi T^{(-1)G,Q}_{l,|l\rangle}^G\rangle = T^+_W |\chi NS_{l+1/2,|l-h/2,h\rangle}^{(-1)}\rangle = T^-_W |\chi NS_{l+1/2,|l-h/2,h\rangle}^{(1)}\rangle
$$

$$
|\chi T^{(1)G,Q}_{l,|l\rangle}^G\rangle = A T^+_W |\chi NS_{l+1/2,|l-h/2,h\rangle}^{(-1)}\rangle = A T^-_W |\chi NS_{l+1/2,|l-h/2,h\rangle}^{(1)}\rangle.
$$

(3.13)

An important observation now is the following. The fact that $|\chi T^{(-1)G,Q}_{l,|l\rangle}^G\rangle$ becomes chiral for $\Delta = -l$ implies necessarily that all the NS singular vectors of the type $|\chi NS_{l+1/2,|l-h/2,h\rangle}^{(-1)}\rangle$ are antichiral (annihilated by $G^-_{1/2}$), whereas all the NS singular vectors of the type $|\chi NS_{l+1/2,|l-h/2,h\rangle}^{(1)}\rangle$ are chiral (annihilated by $G^+_{1/2}$). The reason is that $Q_0 = T^+_W G^-_{1/2} = T^-_W G^+_{1/2}$, so that the condition of being annihilated by $Q_0$ is transformed into the conditions of being annihilated by $G^-_{1/2}$ and $G^+_{1/2}$, respectively, under the twists $T^+_W$ and $T^-_W$. Thus we have found that all the charged NS singular vectors $|\chi NS_{l,|l\rangle}^{(\pm 1)}\rangle$ with $\Delta' + l' = \pm \frac{h_0 \pm 1}{2}$ are chiral (upper signs) or antichiral (lower signs). Furthermore, the uncharged chiral singular vectors are equivalent to charged chiral singular vectors, as was pointed out in section 2. Namely

$$
|\chi T^{(0)G,Q}_{l,|l\rangle}^G\rangle = |\chi T^{(-1)G,Q}_{l,|l\rangle}^G\rangle, \quad |\chi T^{(0)G,Q}_{l,|l\rangle}^G\rangle = |\chi T^{(1)G,Q}_{l,|l\rangle}^G\rangle
$$

by exchanging the primary states of the Verma module: $|-l, h\rangle^G = G_0 |-l, h - 1\rangle^Q$ and $|-l, -h - \frac{\Delta}{2}\rangle^Q = Q_0 |-l, -h - \frac{\Delta}{2} + 1\rangle^G$. As a consequence we have found invertible maps from the NS singular vectors to the chiral topological singular vectors (charged as well as uncharged). These are, in addition, simpler than the maps (3.5) and (3.9) deduced from diagrams (3.3) and (3.7).

### 3.2 Maps from NS to R singular vectors

The maps from the NS to the R singular vectors are derived using the spectral flows $U_{\pm 1/2}$ and $A_{\pm 1/2}$, as deduced from expressions (2.4) and (2.5). To be precise, the R singular vectors with the same helicities than the primaries on which they are built are directly connected to the NS singular vectors through the spectral flows, whereas the singular vectors with different helicities than their primaries are related to the NS singular vectors through the spectral flows plus one the modes $G^+_0$ or $G^-_0$. Similarly as for the topological singular vectors, in every case there are two mirror-symmetric NS singular vectors mapped to each R singular vector.
Let us start with the singular vectors of table (2.15) built on helicity (+) primaries \(|\Delta, h\rangle^+\). The helicity (+) R singular vectors \(|\chi_R\rangle_l^{(-1)+}, |\chi_R\rangle_l^{(0)+}\) and \(|\chi_R\rangle_l^{(1)+}\) are obtained from the NS singular vectors through the maps:

\[
|\chi_R\rangle_l^{(q)+} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(q)} |\Delta, h\rangle = \mathcal{A}_{-1/2} |\chi_{NS}\rangle_l^{(-q)\ast} |\Delta, -h\rangle, \tag{3.15}
\]

resulting in:

\[
|\chi_R\rangle_l^{(-1)+} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(-1)\ast} |\Delta, h\rangle
|\chi_R\rangle_l^{(0)+} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(0)\ast} |\Delta, h\rangle
|\chi_R\rangle_l^{(1)+} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(1)\ast} |\Delta, h\rangle, \tag{3.16}
\]

and mirror-symmetric expressions using \(\mathcal{A}_{-1/2} |\chi_{NS}\rangle_l^{(-q)\ast} |\Delta, -h\rangle\). The helicity (−) R singular vectors \(|\chi_R\rangle_l^{(-2)-}, |\chi_R\rangle_l^{(-1)-}\) and \(|\chi_R\rangle_l^{(0)-}\) are obtained from the previous by the action of \(G_0^-\) since

\[
|\chi_R\rangle_l^{(q-1)-} = G_0^- |\chi_R\rangle_l^{(q)+} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(-q)\ast} |\Delta, h\rangle
|\chi_R\rangle_l^{(-2)-} = G_0^- |\chi_R\rangle_l^{(-1)+} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(0)\ast} |\Delta, h\rangle
|\chi_R\rangle_l^{(0)-} = G_0^- |\chi_R\rangle_l^{(1)+} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(1)\ast} |\Delta, h\rangle, \tag{3.17}
\]

that is:

\[
|\chi_R\rangle_l^{(-2)-} = G_0^- \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(-1)\ast} |\Delta, h\rangle
|\chi_R\rangle_l^{(-1)-} = G_0^- \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(0)\ast} |\Delta, h\rangle
|\chi_R\rangle_l^{(0)-} = G_0^- \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(1)\ast} |\Delta, h\rangle, \tag{3.18}
\]

and mirror-symmetric expressions using \(G_0^- \mathcal{A}_{-1/2} |\chi_{NS}\rangle_l^{(-q)\ast} |\Delta, -h\rangle\). When the total conformal weight of the singular vectors of types \(|\chi_R\rangle_l^{(0)-}\) and \(|\chi_R\rangle_l^{(-1)+}\) is equal to \(\frac{c}{24}\) then they ‘become’ the chiral singular vectors \(|\chi_R\rangle_l^{(0)+,-}\) and \(|\chi_R\rangle_l^{(-1)+,-}\). As a result one gets the maps:

\[
|\chi_R\rangle_l^{(0)+,-} = G_0^- \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(1)\ast} |\Delta, -h\rangle
|\chi_R\rangle_l^{(-1)+,-} = \mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(-1)\ast} |\Delta, -h\rangle, \tag{3.19}
\]

and mirror-symmetric expressions using \(\mathcal{A}_{-1/2} |\chi_{NS}\rangle_l^{(-q)\ast} |\Delta, -h\rangle\).

Observe that the map to the charged chiral singular vector \(|\chi_R\rangle_l^{(-1)+,-}\) is invertible while the map to the uncharged \(|\chi_R\rangle_l^{(0)+,-}\) is not invertible as the action of \(G_0^-\) cannot be reversed in this case (since chiral singular vectors are annihilated by \(G_0^+\) as well as by \(G_0^-\)). Nevertheless, similarly as we showed in the topological case, the uncharged chiral singular vectors can be ‘converted’ into charged chiral singular vectors. We will come back to this issue a few paragraphs below.
Now let us move to the singular vectors of table (2.16) built on helicity \((-\Delta, h)^-\). The helicity \((+\Delta, h)^-\) R singular vectors \(|\chi_R|^{(-1)}_l\), \(|\chi_R|^{(0)}_l\) and \(|\chi_R|^{(1)}_l\) are obtained from the NS singular vectors through the maps:

\[
|\chi_R|^{(q-)}_{l,|\Delta-h|} = \mathcal{U}_{1/2} |\chi_{NS}|^{(q-)}_{l+\frac{\Delta}{2},|\Delta-h|},
\]

which give:

\[
|\chi_R|^{(-1)-}_{l,|\Delta-h|} = \mathcal{U}_{1/2} |\chi_{NS}|^{(-1)-}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

\[
|\chi_R|^{(0)-}_{l,|\Delta-h|} = \mathcal{U}_{1/2} |\chi_{NS}|^{(0)-}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

\[
|\chi_R|^{(1)-}_{l,|\Delta-h|} = \mathcal{U}_{1/2} |\chi_{NS}|^{(1)-}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

and mirror-symmetric expressions using \(\mathcal{A}_{1/2} |\chi_{NS}|^{(-q-)}_{l+\frac{\Delta}{2},|\Delta-h|}\). The helicity \((+\Delta, h)^+\) R singular vectors \(|\chi_R|^{(0)+}_l\), \(|\chi_R|^{(1)+}_l\) and \(|\chi_R|^{(2)+}_l\) are obtained now simply by acting with \(G^+_0\):

\[
|\chi_R|^{(q+1)+}_{l,|\Delta-h|} = G^+_0 |\chi_R|^{(q-)}_{l,|\Delta-h|} = G^+_0 \mathcal{U}_{1/2} |\chi_{NS}|^{(q-)}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

resulting in:

\[
|\chi_R|^{(1)+}_{l,|\Delta-h|} = G^+_0 \mathcal{U}_{1/2} |\chi_{NS}|^{(1)-}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

\[
|\chi_R|^{(0)+}_{l,|\Delta-h|} = G^+_0 \mathcal{U}_{1/2} |\chi_{NS}|^{(0)-}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

\[
|\chi_R|^{(2)+}_{l,|\Delta-h|} = G^+_0 \mathcal{U}_{1/2} |\chi_{NS}|^{(1)+}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

and mirror-symmetric expressions using \(G^+_0 \mathcal{A}_{1/2} |\chi_{NS}|^{(-q-)}_{l+\frac{\Delta}{2},|\Delta-h|}\). When the total conformal weight of the singular vectors of types \(|\chi_R|^{(0)+}_l\) and \(|\chi_R|^{(1)-}_l\) is equal to \(\frac{\Delta}{2}\) then they ‘become’ the chiral singular vectors \(|\chi_R|^{(0)+,-}_l\) and \(|\chi_R|^{(1)+,-}_l\). As a result one gets the maps:

\[
|\chi_R|^{(0)+,-}_{l,|\Delta-h|} = G^+_0 \mathcal{U}_{1/2} |\chi_{NS}|^{(1)-}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

\[
|\chi_R|^{(1)+,-}_{l,|\Delta-h|} = \mathcal{A}_{1/2} |\chi_{NS}|^{(1)-}_{l+\frac{\Delta}{2},|\Delta-h|}
\]

and mirror-symmetric expressions using \(\mathcal{A}_{1/2} |\chi_{NS}|^{(-q-)}_{l+\frac{\Delta}{2},|\Delta-h|}\).

Observe that, again, the map to the charged chiral singular vector \(|\chi_R|^{(1)+,-}_l\) is invertible while the map to the uncharged \(|\chi_R|^{(0)+,-}_l\) is not. Nevertheless the uncharged chiral singular vectors \(|\chi_R|^{(0)+,-}_l\), from tables (2.15) and (2.16), are equivalent to charged chiral singular vectors as there are always two h.w. vectors with opposite helicities in the Verma modules which contain chiral singular vectors. Thus one only has to express the uncharged singular vector as built on the h.w. vector with the opposite helicity than previously and it will pick up a nonzero \(q\). In this manner the singular vectors \(|\chi_R|^{(0)+,-}_l\) on helicity \((-\Delta, h)^-\) primaries are equivalent to singular vectors \(|\chi_R|^{(-1)+,-}_l\) on helicity \((+\Delta, h)^-\)
3.3 Charged NS singular vectors

and for the R singular vectors, in each case. – and we will summarize the maps that provide construction formulae for the topological construction formulae for the NS singular vectors – all the details are in refs. [8] and [9] just by expressing the singular vectors of the Neveu-Schwarz N=2 algebra in terms of 16 + 16 types of singular vectors of the Topological and of the Ramond N=2 algebras. The explicit expressions for the charged singular vectors at level

\[ |\chi_{NS}^{(0)}|_{\ell, l}^{(+,-)} = \frac{\omega_{\pm}^{(1)-}(-h_{\pm}^{(1)|\Delta, h} - h_{\pm}^{(1)|\Delta, h} + \frac{3}{2} l, h) = U_{1/2} |\chi_{NS}^{(1)}|_{\ell, l}^{(+,-)} = U_{1/2} |\chi_{NS}^{(1)}|_{\ell, l}^{(+,-)} \] (3.25)

and mirror-symmetric expressions using \( A_{1/2} \) and \( A_{-1/2} \), respectively.

3.3 Construction Formulae

The maps given in subsections 3.1 and 3.2 turn into construction formulae for the 16 + 16 types of singular vectors of the Topological and of the Ramond N=2 algebras just by expressing the singular vectors of the Neveu-Schwarz N=2 algebra in terms of their corresponding construction formulae. In what follows we will describe briefly the construction formulae for the NS singular vectors – all the details are in refs. [8] and [9] – and we will summarize the maps that provide construction formulae for the topological and for the R singular vectors, in each case.

3.3.1 Charged NS singular vectors \( |\chi_{NS}^{(\pm)}| \)

The explicit expressions for the charged singular vectors at level \( k \) read [8]

\[ |\chi_{NS}^{(\pm)}|_{k} = \mathcal{W}^{\pm} \mathcal{E}^{\pm}(k-1/2)\mathcal{T}^{\pm}(k-1)\mathcal{E}^{\pm}(k-3/2)\mathcal{T}^{\pm}(k-2)\ldots\mathcal{E}^{\pm}(1)\mathcal{T}^{\pm}(1/2)\Psi_{0}^{\pm}, \] (3.26)

where \( \mathcal{E}^{\pm}(k) \) and \( \mathcal{T}^{\pm}(k) \) are even and odd recursion step matrices, respectively, and \( \mathcal{W}^{\pm} \) and \( \Psi_{0}^{\pm} \) are vectors, the latter depending on the initial low level singular vectors. The spectrum of \( \Delta \) and \( h \) for which the NS Verma modules \( V_{NS}(\Delta, h) \) contain charged singular vectors \( |\chi_{NS}^{(\pm)}|_{k} \) is given (at least) by the zeroes of the NS determinant formula which are solutions to the vanishing planes \( g_{\pm k}(\Delta, h) = 0 \) [26][27].

As follows from the results of subsections 3.1 and 3.2, the construction formulae (3.26) for charged NS singular vectors provide also construction formulae for the following topological singular vectors:

\[ |\chi_{T}^{(1,G)}|_{\ell, |\Delta+h/2, h|} = T_{W}^{+} |\chi_{NS}^{(1)}|_{\ell, 1/2, |\Delta, h|} = T_{W}^{+} |\chi_{NS}^{(1)}|_{\ell, 1/2, |\Delta, h|} = T_{W}^{+} |\chi_{NS}^{(1)}|_{\ell, 1/2, |\Delta, h|} \] (3.27)
\[ |X_{T_l}^{G}|_{\Delta + h/2, h}^{(-1)} G \]  
\[ |X_{T_l}^{Q}|_{\Delta + h/2, \frac{h}{2}}^{(-2)} Q \]  
\[ |X_{T_l}^{Q}|_{\Delta + h/2, \frac{h}{2}}^{(-1)} Q \]  
\[ |X_{T_l}^{Q}|_{\Delta + h/2, \frac{h}{2}}^{(-1)} Q \]  
\[ |X_{T_l}^{Q}|_{\Delta + h/2, \frac{h}{2}}^{(-1)} Q \]  
\[ |X_{T_l}^{Q}|_{\Delta + h/2, \frac{h}{2}}^{(-1)} Q \]  
\[ |X_{T_l}^{Q}|_{\Delta + h/2, \frac{h}{2}}^{(-1)} Q \]  
\[ |X_{T_l}^{Q}|_{\Delta + h/2, \frac{h}{2}}^{(-1)} Q \]  

and for the following R singular vectors:

\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  

\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  
\[ |X_{R_l}^{(1)+}|_{\Delta + h/2, h}^{(1)+} \]  

3.3.2 Uncharged NS singular vectors \(|\chi_{NS}^{(0)}\)\\n
The explicit expressions for the uncharged singular vectors at level \(l = \frac{r s}{2} \) read [9]

\[ |\chi_{NS}^{(0)}|_{r,s} = \epsilon_{r,s}^{+}(t, h) \Delta_{r,s}(1, 0) + \epsilon_{r,s}^{-}(t, h) \Delta_{r,s}(0, 1) , \]  

where \(\Delta_{r,s}(1, 0)\) and \(\Delta_{r,s}(0, 1)\) are two basis vectors taken from the analytically continued Verma module, \(t = \frac{3-s}{3}\) parametrizes the central charge, and

\[ \epsilon_{r,s}^{\pm}(t, h) = \prod_{m=1}^{r} \left( \pm \frac{s - rt}{2t} + \frac{h}{t} + \frac{1}{2} \pm m \right) \]  

\(r \in \mathbb{Z}^+, \ s \in 2\mathbb{Z}^+\).
The spectrum of $\Delta$ and $h$ for which the NS Verma modules $V_{NS}(\Delta, h)$ contain uncharged singular vectors $|\chi_{NS}(0)\rangle_{r,s}$ is given (at least) by the zeroes of the NS determinant formula which are solutions to the quadratic vanishing surface $f_{r,s}(\Delta, h) = 0$ [26][27]. Interestingly enough, the simultaneous vanishing of the two curves, $\epsilon_{r,s}(t, h) = 0$ and $\epsilon_{r,s}(t, h) = 0$, leads to the appearance of two linearly independent uncharged NS singular vectors at the same level, in the same Verma module [9]. The topological twists $T^\pm_{W}$ (2.3) let these conditions invariant, extending the existence of the two-dimensional space of singular vectors to the topological singular vectors of types $|\chi_T(0)^G\rangle_{(\phi)}$, $|\chi_T(-1)^G\rangle_{(\phi)}$ and $|\chi_T(0)^Q\rangle_{(\phi)}$, as the generic uncharged NS singular vectors are transformed necessarily into these four types of topological singular vectors via the mappings (3.4) and (3.5). Similarly, for the case of the R singular vectors the two-dimensional singular spaces exist for singular vectors of the types $|\chi_R(0)^+\rangle_{(\phi)}$, $|\chi_R(-1)^-\rangle_{(\phi)}$, $|\chi_R(1)^+\rangle_{(\phi)}$ and $|\chi_R(0)^-\rangle_{(\phi)}$.

As follows from the results of subsections 3.1 and 3.2, the construction formulae (3.30) for uncharged NS singular vectors provide also construction formulae for the following topological singular vectors:

$$
\begin{align*}
|\chi_T(0)^G\rangle_{(L|\Delta+h/2, h)^G} &= T^+_W|\chi_{NS}(0)\rangle_{L|\Delta, h} &= T^-_W|\chi_{NS}(0)\rangle_{L|\Delta-h}, \\
|\chi_T(0)^Q\rangle_{(L|\Delta+h/2, -h-h^\pm)^Q} &= AT^+_W|\chi_{NS}(0)\rangle_{L|\Delta, h} &= AT^-_W|\chi_{NS}(0)\rangle_{L|\Delta-h}, \\
|\chi_T(-1)^Q\rangle_{(L|\Delta-h/2, h)^Q} &= \mathcal{Q}_0T^+_W|\chi_{NS}(0)\rangle_{L|\Delta, h} &= \mathcal{Q}_0T^-_W|\chi_{NS}(0)\rangle_{L|\Delta-h}, \\
|\chi_T(1)^G\rangle_{(L|\Delta-h/2, -h-h^\pm)^Q} &= \mathcal{A}\mathcal{Q}_0T^+_W|\chi_{NS}(0)\rangle_{L|\Delta, h} &= \mathcal{A}\mathcal{Q}_0T^-_W|\chi_{NS}(0)\rangle_{L|\Delta-h},
\end{align*}
$$

(3.32)

and for the following R singular vectors:

$$
\begin{align*}
|\chi_R(0)^+\rangle_{(L|\Delta+h/2, h)^+} &= \mathcal{U}_{1/2}|\chi_{NS}(0)\rangle_{L|\Delta, h} &= \mathcal{A}_{-1/2}|\chi_{NS}(0)\rangle_{L|\Delta-h}, \\
|\chi_R(-1)^-\rangle_{(L|\Delta+h/2, -h-h^\pm)^+} &= \mathcal{G}_0\mathcal{U}_{-1/2}|\chi_{NS}(0)\rangle_{L|\Delta, h} &= \mathcal{G}_0\mathcal{A}_{-1/2}|\chi_{NS}(0)\rangle_{L|\Delta-h}, \\
|\chi_R(0)^-\rangle_{(L|\Delta-h/2, h)^-} &= \mathcal{U}_{1/2}|\chi_{NS}(0)\rangle_{L|\Delta, h} &= \mathcal{A}_{1/2}|\chi_{NS}(0)\rangle_{L|\Delta-h}, \\
|\chi_R(1)^+\rangle_{(L|\Delta-h/2, -h-h^\pm)^-} &= \mathcal{G}_0\mathcal{U}_{1/2}|\chi_{NS}(0)\rangle_{L|\Delta, h} &= \mathcal{G}_0\mathcal{A}_{1/2}|\chi_{NS}(0)\rangle_{L|\Delta-h}.
\end{align*}
$$

(3.33)

4 Final Remarks

We have written down one-to-one invertible maps between the singular vectors of the Neveu-Schwarz N=2 algebra and $16+16$ types of singular vectors of the Topological and of the Ramond N=2 algebras. To be precise, we have derived two mirror-symmetric maps from the NS singular vectors to each of the $16+16$ topological and R singular vectors. These maps provide construction formulae for the latter once the NS singular vectors
are expressed in terms of their construction formulae themselves. The indecomposable no-label and no-helicity singular vectors can only be mapped to NS subsingular vectors [32], for which no construction formulae exist.

One remarkable finding resulting from our analysis is that the charged NS singular vectors $|\chi_{NS}\rangle_{l,|\Delta, h\rangle}^{(\pm 1)}$ with $\Delta + l = \pm \frac{(h\pm 1)}{2}$ are necessarily chiral (upper signs) or antichiral (lower signs).

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References


