II. BASIC FORMULAE

\[ \begin{align*}
(2) \quad & \frac{dx}{d\tau} = \frac{1}{x} \left( \frac{x}{\tau} + 1 \right) \frac{\dot{x}}{x} e^{-x} + \frac{1}{x} \frac{\dot{x}}{x} e^{-x} \\
& \quad \left[ \frac{1}{x} \left( \frac{x}{\tau} + 1 \right) + \frac{1}{x} \right] \frac{\dot{x}}{x} e^{-x} + \frac{1}{x} \frac{\dot{x}}{x} e^{-x} \\
& \quad \left( x \right) \cdot \frac{\dot{x}}{x} e^{-x}
\end{align*} \]

where

\[ \frac{1}{x} \left( \frac{x}{\tau} + 1 \right) \frac{\dot{x}}{x} e^{-x} + \frac{1}{x} \frac{\dot{x}}{x} e^{-x} = \frac{1}{\tau} \frac{\dot{x}}{x} e^{-x} \]

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\[ \frac{1}{x} \left( \frac{x}{\tau} + 1 \right) \frac{\dot{x}}{x} e^{-x} + \frac{1}{x} \frac{\dot{x}}{x} e^{-x} = \frac{1}{\tau} \frac{\dot{x}}{x} e^{-x} \]

The introduction potential recognition in the bimodal section.
brane tension. The mass scale $\mu$ is given by
\begin{equation}
\mu \equiv \sqrt{\frac{4\pi}{3}} \sqrt{\frac{1}{M_4}},
\end{equation}
prime indicates derivative with respect to the scalar field $\phi$, and dot a derivative with respect to time.

The Hubble parameter $H$ is related to the energy density $\rho$ by
\begin{equation}
H^2 = \frac{8\pi}{3M_4^2} \rho \left(1 + \frac{\rho}{2\lambda}\right) \simeq \frac{8\pi}{3M_4^2} V \left(1 + \frac{V}{2\lambda}\right),
\end{equation}
which reduces to the usual Friedmann equation for $\rho \ll \lambda$, the scalar field obeys the usual slow-roll equation
\begin{equation}
3H\dot{\phi} = -V',
\end{equation}
and the amount of expansion, in terms of $e$-foldings, is given by
\begin{equation}
\frac{dN}{d\phi} = \frac{H}{\dot{\phi}}.
\end{equation}
The expressions for the spectra are, as always, to be evaluated at Hubble radius crossing $k \equiv aH$, and the spectral indices of the scalars and tensors are defined as usual by
\begin{equation}
n - 1 \equiv \frac{d\ln A^2}{d\ln k} ; \quad n_T \equiv \frac{d\ln A^2_T}{d\ln k}.
\end{equation}
If one defines slow-roll parameters, generalizing the usual ones, by [7]
\begin{equation}
\epsilon_B \equiv \frac{M_4^2}{16\pi} \left(\frac{V'}{V}\right)^2 \frac{1 + V/\lambda}{(1 + V/2\lambda)^2},
\end{equation}
\begin{equation}
\eta_B \equiv \frac{M_4^2}{8\pi} \frac{V''}{V} \frac{1}{1 + V/2\lambda},
\end{equation}
then the scalar spectral index, in the slow-roll approximation, obeys the usual equation
\begin{equation}
n - 1 \simeq -6\epsilon_B + 2\eta_B.
\end{equation}
The tensor index obeys a more complicated equation, but remarkably it still obeys the usual consistency equation [9]
\begin{equation}
R \equiv \frac{A^2_T}{A^2} = -\frac{1}{2}n_T.
\end{equation}
This result is maintained because the normalization of the tensor mode function requires $F(x)$ to obey a particular differential equation. This tells us that the relation between scalar and tensor perturbations is unchanged in the braneworld scenario. In particular, this equation can be viewed as an expression giving the scalar spectrum corresponding to a given tensor spectrum, namely
\begin{equation}
A^2_S = -2\frac{A^2_T}{n_T} = -\frac{A^2_T}{dA_T/d\ln k}.
\end{equation}
Hence in the braneworld scenario, as in Einstein gravity, the scalars carry no additional information about the potential if the tensors are known. This invariance of the consistency relation under a change in gravitational physics is unexpected, since in most variations of standard inflation, e.g. warm inflation [10], this relation is substantially different.

### A. Low-energy limit

Provided $\rho \ll \lambda$, the equations all reduce to the usual Einstein gravity ones and the normal reconstruction equations apply. In particular, if one is able to observe the tensor spectrum, it gives a unique potential (under the single-field assumption), and if the scalars can additionally be measured they obey consistency relations. If only the scalars can be measured, a unique potential cannot be obtained and there is a one-parameter family of possible models. Measurement of the tensors at a single scale is sufficient to remove this degeneracy.

### B. High-energy limit

Before proceeding to the general case, it is instructive to analyze the high-energy limit. The key expressions are
\begin{equation}
A^2_S \simeq \frac{64\pi}{75M_4^2} \frac{V^6}{V^{12/3}},
\end{equation}
\begin{equation}
A^2_T \simeq \frac{8}{25M_4^2} \frac{V^3}{\lambda^2},
\end{equation}
\begin{equation}
n - 1 \simeq -\frac{M_4^2}{4\pi} \frac{\lambda}{V} \left[ \frac{6}{5} \left(\frac{V'}{V}\right)^2 - 2\frac{V''}{V} \right],
\end{equation}
\begin{equation}
n_T \simeq -\frac{3M_4^2}{4\pi} \frac{\lambda}{V} \left(\frac{V''}{V}\right)^2.
\end{equation}
Recalling that $n_T$ is redundant due to the consistency equation, we see that we now have only three observable quantities but four parameters to measure, namely $V$ and its first two derivatives and $\lambda$. It is clear therefore that a unique reconstruction is no longer possible.

This can be made explicit by redefining variables to absorb the degeneracy. Defining
\begin{equation}
\alpha \equiv V\lambda^{-2/3} ; \quad \beta \equiv V^{1/3} \lambda^{-1/3} ; \quad \gamma \equiv V^{1/3} \lambda^{-2/3},
\end{equation}
gives a closed system for these three variables in terms of the observables, which can be inverted to give
\begin{equation}
\alpha^3 \simeq \frac{25}{8} M_4^2 A^2_S,
\end{equation}
\begin{equation}
\beta^3 \simeq \frac{25\pi}{3} M_4^2 A_T^2 \frac{A^2_T}{A^2},
\end{equation}
\begin{equation}
\gamma \simeq \frac{25^2}{2} \frac{\pi}{3} M_4^2 A_T^{4/3} \frac{A^2}{A^2_T} \left[ \frac{4A_T^2}{A^2_T} + (n - 1) \right].
\end{equation}
These resemble the usual reconstruction equations (see e.g. Ref. [3]). However the dependence on $\lambda$ is not a linear one, so its presence alters the functional form of the reconstructed potential and is not simply a scaling.

Finally, the validity of the high-energy approximation requires $V \gg 2\lambda$, which equivalently can be written
\[
A_T^2 \gg \frac{64}{25} \frac{\lambda}{M_4^4}.
\]
\[\text{Eq. (22)}\]

That $\lambda$ cannot be determined in the high-energy limit is no surprise, because in that limit it appears only in the combination $M_4^2\lambda$ which determines the five-dimensional Planck mass. As observations determine only dimensionless quantities, the overall mass scale cannot be determined and so the degeneracy in $M_4$ must be exact. This is precisely the same reason that one does not expect to determine the Planck mass from the perturbations in the usual four-dimensional case.

C. The general case

In the general case, the perturbation spectra in the braneworld are given by
\[
A_S^2 \simeq \frac{32}{75 M_4^4} \frac{V}{c_B} \left( 1 + \frac{V}{2\lambda} \right) \left( 1 + \frac{V}{\lambda} \right), \quad \text{Eq. (23)}
\]
\[
A_T^2 \simeq \frac{32}{75 M_4^4} V \left( 1 + \frac{V}{2\lambda} \right) G^2(V/\lambda),
\]
\[\text{Eq. (24)}\]
\[
n - 1 \simeq -6 n_B + 2 n_M,
\]
\[\text{Eq. (25)}\]

where $G$ is a function obtained from Eqs. (2), (3) and (5) by $G^2(V/\lambda) \equiv F^2(H/\mu)$, and again the consistency equation renders an equation for $n_T$ unnecessary. In the high-energy limit $G^2(V/\lambda) \rightarrow 3V/2\lambda$.

The brane tension $\lambda$ can be eliminated from these equations by a set of redefinitions to dimensionless variables
\[
\tilde{V} \equiv \frac{V}{\lambda}; \quad \tilde{A}_S^2 \equiv A_S^2 \frac{M_4^4}{\lambda}; \quad \tilde{A}_T^2 \equiv A_T^2 \frac{M_4^4}{\lambda}.
\]
\[\text{Eq. (26)}\]

This is sufficient to demonstrate that even in the general case the degeneracy is exact; one is able to reconstruct a potential from a given set of observations for any value of $\lambda$. The relation between $\tilde{A}_T^2$ and $\tilde{V}$ is shown in Figure 1.

We now explicitly demonstrate the effects of the braneworld on the reconstruction of the inflaton potential. The aim in reconstruction is to take measurements of the various observables, corresponding to a particular wavenumber $k$, and use these to obtain the potential and its derivatives at the scalar field value $\phi$ (which without loss of generality can be taken to be zero) when that scale crossed the Hubble radius during inflation. As Eqs. (23)–(25) are not analytically invertible, we must proceed using numerical inversion.

In order to obtain $V$ from a measurement of the tensors, we invert Eq. (24) using a Newton–Raphson root-finding method. A different value of $V$ will be found for each choice of $\lambda$, and note from Figure 1 that the function is always uniquely invertible. In the limit $\lambda \rightarrow \infty$ we recover the results of standard inflation, while for small $\lambda$ the effects of the brane lead a given tensor amplitude to correspond to a lower potential magnitude, $V$.

To obtain the slope of the potential we invert the ratio of Eqs. (24) and (23) to give
\[
\frac{V''}{V} = \sqrt{\frac{16 \pi R}{M_4^4} \left[ 1 + \tilde{V}/2 \right] ^2},
\]
\[\text{Eq. (27)}\]

which depends on the observable $R$ and the degenerate combination $V/\lambda$. Figure 2 shows the general relation between $V''/V$ and $V/\lambda$. For $V \ll \lambda$, the term in the square
brackets tends to unity and we recover the standard inflationary result, while for $V \gg \lambda$ the relation approaches the high-energy limit $V' / V \approx \sqrt{8 \pi R V / 3 \lambda M_4^2}$, leading to a rapid increase in the obtained gradient of the inflation potential as $\lambda$ is reduced.

The second derivative of $V$ can be obtained from

$$
\frac{V''}{V} = \frac{4 \pi}{M_4^2} \left(1 + \frac{\bar{V}}{2}\right) \left[6 R \frac{1}{G^2(V)} + \frac{n}{2} \right],
$$

which is a function of the observables $R$ and $n - 1$, plus the degenerate combination $V / \lambda$. Figure 3 shows the recovered curvature of the potential as a function of $V / \lambda$ for a range of values of $R$ and $n - 1$. For $V \gg \lambda$ the magnitude of the curvature of the potential increases and asymptotes to Eq. (21).

Finally we demonstrate these degeneracies for an example set of observables. We choose our observables to be $A_s^2 = 4 \times 10^{-12}$, $R = 0.01$, and $n - 1 = -0.05$; these numbers correspond roughly to the predictions from a quartic potential and are consistent with current observations [11] (the ratio of contributions to the large-angle microwave anisotropies is about $4 \pi R$ in our conventions). We reconstruct potentials for choices of $\lambda$ evenly spaced logarithmically in the range from $10^{-15} M_4^2$ to $10^{-9} M_4^2$. In each case, we plot only the portion of the potential accessible to observations; the relation between $\Delta \phi$ and the range of scales probed by observations depends on $\lambda$ and is computed via Eq. (7) as

$$
\frac{\Delta \phi / M_4}{\Delta \ln k} = \sqrt{\frac{R}{4 \pi}} \frac{1}{G(V)}.
$$

We take the Planck satellite as our guideline, which will have $\Delta \ln k \simeq 3.5$ on either side of the central point.

Figure 4 shows a set of reconstructed model potentials for the different assumed values of $\lambda$, each of which reproduces our model observations. The ratio $V / \lambda$ obtained ranges from 0.01 to 22. For $V \ll \lambda$ the reconstructed potential is nearly independent of $\lambda$, closely approximating the Einstein gravity result. As $\lambda$ is decreased, the magnitude of the potential begins to decrease while its gradient steepens; at the same time the amount of potential constrained shrinks as the extra friction leads to slower rolling of the field.

To end, we mention that as well as reproducing the correct perturbations, a viable potential must be able to support enough subsequent inflation to stretch those perturbations to the observable scales. If the recovered potential develops a minimum or goes negative within the constrained range, our approach will have broken down, and refinement becomes necessary (going beyond the quadratic potential approximation and/or slow-roll) to test whether there is still a viable potential. However the approximate condition for the reconstruction to break down, $\Delta \phi \gtrsim |V''/V|^2$, does not change significantly as $\lambda$ is decreased, because in the high-energy limit $\Delta \phi$ reduces at the same rate as $V''/V$ (for fixed values of the observables). Hence if a viable potential exists in the $\lambda \to \infty$ limit, it is unlikely that the problem will become ill-defined for low values of $\lambda$.

### III. Conclusions

One of the most anticipated results of forthcoming high-accuracy CMB experiments is the probing of the physics of inflation, and in particular empirically recon-
structing the form of the inflaton potential. However, it is important to be aware of the possible degeneracies that may arise. To date, attention has been focused on degeneracies between initial perturbation parameters and cosmological parameters such as reionization, suggesting that combinations of observations (for example CMB polarization as well as temperature, or completely different types of observation) are required to lift these degeneracies.

For early Universe cosmologists, more worrying are degeneracies that arise in predictions for the initial perturbations, which represent a fundamental limitation to the constraints we can extract and which are not broken by polarization. In this paper we have described how such a degeneracy arises in a braneworld scenario based on the Randall–Sundrum type-II model. We have shown that the unique reconstruction of the potential from scalar and tensor perturbation spectra in this scenario is no longer possible, with a different possible potential arising for each choice of brane tension. Accordingly, observations of the perturbation spectra cannot distinguish between the braneworld and standard inflation. It would be interesting to know if this is unique to the simplest braneworld scenario, or if it remains true in other versions. It would also be interesting to know if this result persists at higher order in the slow-roll expansion for the perturbations.

We end by stressing that our results refer to the initial perturbation spectra. Whether or not there might be significant braneworld effects on the subsequent evolution of the perturbations is presently unknown and is likely to be model dependent; for example in general the short-scale perturbation behaviour on the brane can be influenced by bulk perturbations which cannot be predicted on the brane (see Ref. [12] for an overview). It may well be that the braneworld might manifest itself through such effects. If, however, the perturbation evolution turns out to be unaffected (for example if inflation is successful in diluting the effect of bulk perturbations), then finding observable traces of the braneworld in the low-energy universe may not be easy.

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