Why do naked singularities form in gravitational collapse?

Pankaj S. Joshi*, Naresh Dadhich† and Roy Maartens‡

* Tata Institute for Fundamental Research, Mumbai, India
† Inter-University Centre for Astronomy and Astrophysics, Pune, India
‡ Relativity and Cosmology group, School of Computer Science and Mathematics, Portsmouth University, Portsmouth PO1 2EG, Britain

We investigate what are the key physical features that cause the development of a naked singularity, rather than a black hole, as the end-state of gravitational collapse. We show that inhomogeneity, and the associated shearing effects within the collapsing matter cloud, can play a crucial role in delaying formation of the apparent horizon in the vicinity of the singularity. This in turn can make the singularity visible to an external observer, in contrast to a black hole, which is hidden behind an event horizon of gravity.

In the past decade or so, several scenarios have been discovered where the gravitational collapse of a massive matter cloud results in the development of a naked singularity [1]. The final outcome of the gravitational collapse in general relativity is an issue of great importance and interest from the perspective of black hole physics as well as its astrophysical implications. When there is a continual collapse without any final equilibrium, either a black hole forms when the super-dense regions of matter are hidden from the outside observer within an event horizon of gravity, or a naked singularity results as the end product, depending on the nature of the initial data from which the collapse develops.

The theoretical and observational properties of a naked singularity would be quite different from those of a black hole (see [2] for further discussion of this). Thus it is of crucial importance to understand what are the key physical characteristics and dynamical features in collapse that give rise to a naked singularity, rather than a black hole. While many models of naked singularity formation within dynamically developing collapse scenarios have been found and analyzed [3], not much attention has been given to understanding this important aspect.

We begin here an investigation of this question and show that when there are no inhomogeneities present, the collapse necessarily ends in a black hole. This inhomogeneity and associated shear play a crucial role in determining the final fate of collapse. Shear effects can delay the formation of the apparent horizon so that the singularity becomes visible and communication from the very strong gravity regions to faraway observers becomes possible. This is in contrast to the case of a black hole, where the singularity is necessarily hidden behind the event horizon.

For spherical gravitational collapse of a massive matter cloud, the interior metric in comoving coordinates is

\[ ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\Omega^2. \tag{1} \]

The matter shear is

\[ \sigma_{ab} = e^{-\nu} \left( \frac{\dot{R}}{R} - \dot{\psi} \right) \left( \frac{1}{3} h_{ab} - n_a n_b \right), \tag{2} \]

where \( h_{ab} = g_{ab} + u_a u_b \) is the induced metric on 3-surfaces orthogonal to the fluid 4-velocity \( u^a \), and \( n^a \) is a unit radial vector.

The initial data for collapse are the values on \( t = t_i \) of the three metric functions, the density, the pressures, and the mass function that arises from integrating the Einstein equations (for details see e.g. [4]),

\[ F(t, r) = \int \rho(t, r) r^2 dr, \tag{3} \]

where \( 4\pi F(t, r_i) = M \), the total mass of the collapsing cloud, where \( r > r_i \) is a Schwarzschild spacetime. We use the rescaling freedom in \( r \) to set

\[ R(t_i, r) = r, \tag{4} \]

so that the physical area radius \( R \) increases monotonically in \( r \), and with \( R_i = 1 \) there are no shell-crossings on the initial surface. (We will be interested here only in the central shell-focussing singularity at \( R = 0, r = 0 \) which is a gravitationally strong singularity, as opposed to the shell-crossing ones which are weak, and through which the spacetime may sometimes be extended.) The evolution of the density and radial pressure are given by

\[ \rho = \frac{F'}{R^2 R'}, \quad p_r = \frac{F}{R^2 R} \tag{5} \]

The central singularity at \( r = 0 \), where density and curvature are infinite, is naked if there are outgoing non-spacelike geodesics which reach faraway observers in the future and terminate at the singularity in the past. Outgoing radial null geodesics of Eq. (1) are given by
Consider first the case of homogeneous collapse, $\rho = \rho(t)$. Writing $f = e^{-2\nu}R^2 - 1$, the Einstein equations give $f - e^{-2\nu}R^2 = -F/R$. Then Eq. (6) can be written as

$$\frac{dR}{du} = \left(1 - \sqrt{\frac{f + F/R}{1 + f}}\right)\frac{R'}{\alpha^{\alpha-1}} ,$$

where $u = r^\alpha (\alpha > 1)$. If there are outgoing radial null geodesics terminating in the past at the singularity with a definite tangent, then at the singularity we have $dR/du > 0$. For homogeneous collapse, the entire mass of the cloud collapses to the singularity simultaneously at the event $(t = t_s, r = 0)$, so that $F/R \to \infty$. By Eq. (7), $dR/du \to -\infty$, so that no radical null geodesics can emerge from the central singularity. It can be similarly shown that all the later epochs $t > t_s$ are similarly covered.

We have thus shown that for spherical gravitational collapse with homogeneous density (and arbitrary pressures), the final outcome is necessarily a black hole. We note that this conclusion does not require homogeneity of the pressures $p_r$ and $p_\perp$, and is independent of their behaviour. The result generalizes the well-known Oppenheimer-Snyder result for the special case of dust, where the homogeneous cloud collapses to form a black hole always.

An immediate consequence is that if the final outcome of spherical gravitational collapse is not a black hole, then the density must be inhomogeneous. In any physically realistic scenario, the density will be typically higher at the center, so that generically collapse is inhomogeneous.

Consider now a collapsing inhomogeneous dust cloud, with density higher at the center. The metric is Tolman-Bondi-Lemaître, given by Eq. (1) with $\nu = 0$ and $e^{2\psi} = R^2/(1 + f)$, and

$$R^2 = f(r) + \frac{F(r)}{R} .$$

These models are fully characterized by the initial data, specified on an initial surface $t = t_i$ from which the collapse develops, which consist of two free functions: the initial density $\rho_i(r) = \rho(t_i, r)$ (or equivalently, the mass function $F(r)$), and $f(r)$, which describes the initial velocities of collapsing matter shells. At the onset of collapse the spacetime is singularity-free, so that by Eq. (5),

$$F(r) = r^3\tilde{F}(r) , \ 0 < \tilde{F}(0) < \infty .$$

The initial density $\rho_i(r)$ must be at least $C^1$ for $r \geq 0$, so that

$$\rho_i(r) = \rho_c[1 - r\tilde{q}(r)] ,$$

where $\rho_c \equiv \rho_i(0)$ and $\tilde{q}(0)$ is finite.

The shell-focusing singularity appears along the curve $t = t_s(r)$ defined by

$$R(t_s(r), r) = 0 .$$

As the density grows without bound, trapped surfaces develop within the collapsing cloud. These can be traced explicitly via the outgoing null geodesics, and the equation of the apparent horizon, $t = t_{ah}(r)$, which marks the boundary of the trapped region, is given by

$$R(t_{ah}(r), r) = F(r) .$$

If the apparent horizon starts developing earlier than the epoch of singularity formation, then the event horizon can fully cover the strong gravity regions including the final singularity, which will thus be hidden within a black hole. On the other hand, if trapped surfaces form sufficiently later during the evolution of collapse, then it is possible for the singularity to communicate with faraway observers.

For the sake of clarity, we consider marginally bound collapse, $f = 0$, although the conclusions can be generalized to hold for the general case. Then Eq. (8) can be integrated to give

$$R^{3/2}(t, r) = r^{3/2} - \frac{3}{2}(t - t_i)F^{1/2}(r) ,$$

and Eqs. (11) and (12) lead to

$$t_s(r) = t_i + \frac{2}{3}\left[\frac{r^3}{F(r)}\right]^{1/2} ,$$

$$t_{ah}(r) = t_s(r) - \frac{2}{3}F(r) .$$

The central singularity at $r = 0$ appears at the time

$$t_0 = t_s(0) = t_i + \frac{2}{\sqrt{3}\rho_c} .$$

Clearly there is no simultaneous collapse now, and the singularity is described by a curve, the first point being $(t = t_0, r = 0)$.

For inhomogeneous dust, the shearing effects within the collapsing cloud are generically nonzero. By Eqs. (2) and (13),

$$\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} = \frac{r}{6R^4R^2F(3F - rF')^2} .$$

A generic (inhomogeneous) mass profile has the form

$$F(r) = F_0r^3 + F_1r^4 + F_2r^5 + \cdots ,$$

near $r = 0$, where $F_0 = 1/3$, $F_1 = -q(0)/4$, .... Homogeneous dust (Oppenheimer-Snyder) collapse has $F_n = 0$ for $n > 0$, and Eq. (17) implies $\sigma = 0$. The converse is also true in this case: if we impose vanishing shear $\sigma = 0$, we get $F_n = 0$. Whenever there is a negative density gradient, e.g., when there is higher density at the center, then $F_i \neq 0$ for some $n > 0$, and it follows from Eq. (17) that the shear is then necessarily nonzero.
The important question is: what is the effect of such a shear on the evolution and development of the trapped surfaces? In other words, we want to determine the behavior of the apparent horizon in the vicinity of the central singularity at \( R = 0, r = 0 \). To this end, let the first non-vanishing derivative of the density at \( r = 0 \) be the \( n \)-th one \((n > 0)\), i.e.,

\[
F(r) = F_0 r^{-3} + F_n r^{n+3} + \cdots, \quad F_n < 0, \quad \text{(19)}
\]

near the center. By Eqs. (17) and (15),

\[
\sigma^2(t, r) = \frac{n^2 F_0^2}{6 F_0} \left[ 1 - 3 F_0^{1/2} (t - t_i) + \frac{9}{4} F_0 (t - t_i)^2 \right] r^{2n} + O(r^{2n+1}), \quad \text{(20)}
\]

\[
t_{ah}(r) = t_0 - \frac{2}{3} F_0 r^3 - \frac{F_n}{3 F_0^{3/2}} r^n + O(r^{n+1}). \quad \text{(21)}
\]

The time-dependent factor in square brackets on the right of Eq. (20) decreases monotonically from 1 at \( t = t_i \) to 0 at \( t = t_0 \). Thus the qualitative role of the shear in singularity formation can be seen by looking at the initial shear. The initial shear \( \sigma_i = \sigma(t_i, r) \) on the surface \( t = t_i \) grows as \( r^n \), \( n \geq 1 \), near \( r = 0 \). A dimensionless and covariant measure of the shear is the relative shear, \( \sigma/\Theta \), where

\[
\Theta = 2 \frac{\dot{R}}{R} - \frac{\dot{R}'}{R'}, \quad \text{(22)}
\]

is the volume expansion. It follows that

\[
\left| \frac{\sigma}{\Theta} \right|_{t_i} = \frac{n F_n}{3 \sqrt{6 F_0}} r^n \left[ 1 + O(r) \right]. \quad \text{(23)}
\]

It is now possible to see how such an initial shear distribution determines the growth and evolution of the trapped surfaces, as prescribed by the apparent horizon curve \( t_{ah}(r) \). For a black hole, we require

\[
t_{ah}(r) \leq t_0 \text{ for } r > 0, \quad \text{near } r = 0. \quad \text{(24)}
\]

This condition is violated for \( n = 1, 2 \) by Eq. (21). The apparent horizon curve initiates at the singularity \( r = 0 \) at the epoch \( t_0 \) and increases with increasing \( r \), moving to the future, i.e. \( t_{ah} > t_0 \) for \( r > 0 \) near the center. The behavior of the outgoing families of null geodesics has been analyzed in detail in these cases, and it is known that the geodesics terminate at the singularity in the past \([4]\), which results in a naked singularity. In such cases the extreme strong gravity regions can communicate with faraway observers. For the case \( n = 3 \), Eq. (24) shows that we can have a black hole if \( F_3 \geq -2 F_0^{5/2} \), or a naked singularity, if \( F_3 < -2 F_0^{5/2} \). For \( n \geq 4 \), Eq. (24) is always satisfied, and a black hole forms.

When the dust is homogeneous, the apparent horizon starts developing earlier than the epoch of singularity formation, which is then fully hidden within a black hole. There is no density gradient, and no shear. On the other hand, if a density gradient is present at the center, then the trapped surface development is delayed via shear, and, depending on the “strength” of the density gradient/ shear at the center, this may expose the singularity. It is the rate of decrease of shear as we approach the center \( r = 0 \) on the initial surface \( t = t_i \), given by Eq. (23), that determines the end-state of collapse. When the shear falls rapidly to zero at the center, the result is necessarily a black hole; if shear falls more slowly, there is a naked singularity. It is thus seen that naked singularities are caused by the sufficiently strong shearing forces near the singularity, as generated by the inhomogeneities in density distribution of the collapsing configuration. When shear decays rapidly near the singularity, the situation is effectively like the shear-free (and homogeneous) case, with a black hole end-state.

The relation between density gradients and shear may be understood via the non-local (or free) gravitational field. Density gradients act as a source for the electric Weyl tensor [6]

\[
D^b E_{ab} = \frac{4}{3} D_a \rho, \quad \text{(25)}
\]

where \( D_a \) is the covariant spatial derivative. (The magnetic Weyl tensor vanishes for spherical symmetry.) In turn, the gravito-electric field is a source for shear (equivalently, the shear is a gravito-electric potential [6]):

\[
u^c \nabla_c \sigma_{ab} + \frac{2}{3} \Theta \sigma_{ab} + \sigma_{ac} \sigma^c_b - \frac{2}{3} \sigma^2 h_{ab} = -E_{ab}. \quad \text{(26)}
\]

Thus density gradients may be directly related to shear:

\[
D_a \rho = -4 \sigma D_a \sigma - 2 \Theta D^b \sigma_{ab} - 3 D^b (u^c \nabla_c \sigma_{ab}) - 3 \sigma^b D^c \sigma_{bc} - 3 D^b (\sigma_{ac} \sigma^c_b), \quad \text{(27)}
\]

where we have used the shear constraint \( D^b \sigma_{ab} = \frac{4}{3} D_a \Theta \).

Equation (27) makes explicit the link between the behavior of density derivatives and shear near the center, which was discussed above. The free gravitational field, which mediates this link, can also provide a covariant characterization of singularity formation. By Eqs. (23) and (26), the relative gravito-electric field \( E/\Theta^2 \) is given near \( r = 0 \) at \( t = t_i \) by

\[
\left( \frac{E}{\Theta^2} \right) = \frac{-7 n F_n}{18 \sqrt{6 F_0}} r^n \left[ 1 + O(r) \right]. \quad \text{(28)}
\]

Thus naked singularities in spherical dust collapse are signalled by a less rapid fall-off of the relative gravito-electric field as we approach the singularity. Equations (23) and (28) provide two equivalent ways of expressing the result.

For the case of dust collapse, the role of shear in deciding the end-state of collapse is fairly transparent. To understand how shear affects the formation of the apparent horizon for general matter fields with pressures included is much more complicated, in particular since
then Eq. (5) shows that $\rho$ is singularity free cosmological models \[7\]. Depending on crucial role in avoidance of the big-bang singularity in dispersive. (It is this feature which also plays the role of delaying formation of the apparent horizon, without directly hampering the process of collapse. The dispersive effect of shear always tends to delay formation of the apparent horizon, but is only able to expose the singularity when the shear is strong enough near the singularity.

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In summary, we have shown how shearing effects in spherical collapsing matter can delay formation of the apparent horizon, thus exposing the strong gravity regions to the outside world. There is then a causal connection from the singularity to faraway observers, i.e. a naked singularity is the end product of gravitational collapse. An important point that emerges is that a naked singularity develops in quite a natural manner, very much within the standard framework of general relativity, as caused by the shearing effects present in the collapsing matter cloud.

In the case of spherical dust collapse, shear and inhomogeneity are equivalent, i.e., the one implies the other. Although shear contributes positively to the focusing effect via the Raychaudhuri equation,

$$\dot{\Theta} + \frac{1}{3} \Theta^2 = -\frac{1}{2} \rho - 2\sigma^2,$$

its dynamical action can make the collapse incoherent and dispersive. (It is this feature which also plays the crucial role in avoidance of the big-bang singularity in singularity free cosmological models \[7\].) Depending on the rate of fall-off of shear near the singularity, its dispersive effect can play the critical role of delaying formation of the apparent horizon, without directly hampering the process of collapse. The dispersive effect of shear always tends to delay formation of the apparent horizon, but is only able to expose the singularity when the shear is strong enough near the singularity.

$$F = F(t, r),$$ whereas $\dot{F} = 0$ for dust. In some special non-dust cases, it is possible to characterize collapse co-variantly. Above we showed that homogeneous density implies a black hole end-state. The next step could be to consider models for which the initial density is homogeneous. For example, if we choose the mass function as

$$F(t, r) = f(r) - R^3(t, r), \quad f(r) = 2r^3,$$ (29)

then Eq. (5) shows that $\rho_0$ and $(\rho_0)$ are constants. The density and pressure may however develop inhomogeneities as the collapse proceeds, depending on the choice of the remaining functions, including in particular the initial velocities of the collapsing shells, and the collapse may then end up in either a black hole or a naked singularity, depending on that (for a discussion on this for the case of dust collapse, we refer to \[5\]). In fact, we can show that zero shear implies a black hole for these models. By Eqs. (2), (5) and (29), the shear-free condition is

$$\frac{\dot{F}'}{F'} = \frac{2F' \dot{R}}{f'R} - \frac{F'R'}{f'R} + \frac{3\dot{R}}{R} = 0.$$

This leads to $R'/R = 1/r$, and Eq. (5) then shows that $\rho = \rho(t)$, i.e. the density evolution is necessarily homogeneous. As shown above, the collapse necessarily ends in a black hole and no naked singularity will form. For this class of models, whenever the collapse ends in a naked singularity, the shear of the collapsing cloud must necessarily be non-vanishing. Shear is thus a necessity for the creation of a naked singularity, due to its role in delaying the formation of trapped surfaces. In the absence of shear, the collapse must necessarily end in a black hole.