MASS MATRICES AND EIGENSTATES FOR SCALARS / PSEUDOSCALARS;  
INDIRECT CP VIOLATION, MASS HIERARCHIES  
AND SYMMETRY BREAKING

B. Machet 1 2

Laboratoire de Physique Théorique et Hautes Énergies 3
Universités Pierre et Marie Curie (Paris 6) et Denis Diderot (Paris 7)
Unité associée au CNRS UMR 7589

Abstract: I study indirect CP violation for neutral kaons, and extend it to large values of the CP-violating parameter (taken to be real). I show how and at which condition there can exist a continuous set of basis in which the kinetic and mass terms in the Lagrangian can be diagonalized simultaneously. An ambiguity results for the mass spectrum, which then depends on the basis. In particular, for fixed (positive) (mass)² of the CP eigenstates $K^0, K^0$, and for certain ranges of values of the CP-violating parameter, a negative (mass)² can arise in the CP-violating basis. Under certain conditions, even a small perturbation, by lifting the ambiguity, can strongly alter the pattern of masses.

These investigations extend in a natural way to indirect CP violation among a set of Higgs-like doublets.

The $C$-odd commutator $[K^0, \bar{K^0}]$, or its equivalent for Higgs multiplets, plays an important role. The condition for its vanishing and its consequences are among the main concerns of this work.

PACS: 11.30.Er 12.60.Fr 11.15.Ex

1Member of ‘Centre National de la Recherche Scientifique’
2E-mail: machet@lpthe.jussieu.fr
3LPTHE tour 16/1er étage, Université P. et M. Curie, BP 126, 4 place Jussieu, F-75252 Paris Cedex 05 (France)
1 Introduction

This study concerns indirect \( CP \) violation [1] for kaons [2] and Higgs-like doublets, when the \( CP \)-violating parameter is allowed to go beyond its small customary experimental values.

It starts with the simple example of (generalized) \( K_L(\theta) \) and \( K_S(\theta) \) mesons, in which \( \theta \) is a real parameter measuring indirect \( CP \) violation. The decays of kaon are not considered, such that their mass matrix can be taken to be hermitian. I show that there can exist a continuous set of basis, depending on \( \theta \), in which the kinetic and mass terms in the Lagrangian are both diagonal; if so, the mass splitting depends on the basis.

The origin of the ambiguity is traced down to the contribution of the commutator \([K^0, K^0]\) to the mass matrix, and to the basis in which the Lagrangian is written; in particular, the independence or not of the vectors of the basis is important.

I investigate the vanishing of this commutator, in which case the above-mentioned ambiguity in the mass spectrum arises. For a pair particle-antiparticle ((\( K^0, \bar{K}^0 \)) or alike) considered to be fundamental and not decaying, taking it as vanishing is legitimate; for composite particles, like in the quark model of mesons, it is in general untrue as soon as electroweak interactions are turned on.

I study the mass spectrum in the \( CP \)-violating basis as a function of the masses of the \( CP \) eigenstates and of the \( CP \)-violating parameter \( \theta \). I show that, even if, in the basis of \( C (CP) \) eigenstates \( K^0 \pm \bar{K}^0 \), all excitations are taken to be positive, the occurrence of one negative (mass)\(^2\) has nothing exceptional in the \( (CP) \) breaking) basis \((K_L, K_S)\).

Turning on electroweak interactions makes, in general, the commutator of composite mesons not vanish. If so, the hamiltonian can only be diagonalized in a \( CP \)-violating basis, and the masses of the \( CP \)-violating eigenstates \( \mu_L^2 \) and \( \mu_S^2 \) become “physical” observable quantities: the ambiguity in the mass spectrum disappears.

The investigation is straightforwardly extended to electroweak models with more than one Higgs doublet. First, it is shown that the peculiarities of the neutral kaon system can be extended to \( SU(2)_L \times U(1) \) Higgs-like doublets having opposite transformations by \( C \); then, on a more general ground, that the same phenomenon can take place among any pair of Higgs-like doublets with definite \( C \), whatever their \( C \) quantum numbers.

The role of the flavour singlet is emphasized. It can in particular occur that the Higgs mass, as it is usually defined, stays an undetermined quantity.

One is accustomed to considering both the mass difference between the \( K_L \) and \( K_S \) mesons and the \( CP \)-violating parameter \( \epsilon \) as small. When \( \epsilon \) increases unnoticed phenomena can occur. In particular, even a small perturbation can, under certain conditions, induce large effects on the mass spectrum of certain pairs of particles. While nature seems not to have pushed indirect \( CP \) violation to such extremes for pseudoscalar mesons, it cannot be excluded for other systems like, for example, Higgs multiplets; no information is indeed yet available concerning their properties by charge conjugation.

Unlike in other works dedicated to electroweak models with several Higgs-like doublets, no ad-hoc potential to trigger spontaneous \( CP \) breaking is introduced; this is in particular why such questions as “natural flavour conservation” [10], the presence or not of discrete symmetries [4] [10][11] [12] … will hardly be mentioned.

\(^1\)Since the pioneering work [3] [4], many investigations have been dedicated to \( CP \) violation in the framework of an extended scalar sector of the standard electroweak model (see for example [6][7][8][1] and references therein). It has been long recognized that, since there exists no bound on the number of Higgs doublets, it is likely to provide an unavoidable source of \( CP \) violation, in addition to the one which comes from a complex Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix for the quarks [9]. Much information, specially about models with two Higgs doublets, can be found in [5], and references therein.
2 How an ambiguity arises; fixing the masses of $CP$-violating states

2.1 A unitary change of basis ($T$ conserved, $CPT$ violated)

- We choose the phases for the neutral pseudoscalar kaon system such that
  \[ C(K^0) = \overline{K^0} \Rightarrow CP \left( K^0 \right) = -\overline{K^0}; \]  
  \[ \text{it corresponds to } \gamma = 0 \text{ in (27) below.} \]
  $K^0$ and $\overline{K^0}$ are independent fields.

- Let us consider the fields
  \[ \varphi_3 = \frac{K^0 + \overline{K^0}}{\sqrt{2}}, \varphi_4 = \frac{K^0 - \overline{K^0}}{\sqrt{2}} \]
  \[ \text{and the change of basis} \]
  \[ \begin{pmatrix} \varphi_L \\ \varphi_S \end{pmatrix} = V \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix}, \]
  \[ \text{where } V \text{ is a unitary matrix} \]
  \[ V = V^\dagger = V^{-1} = \begin{pmatrix} c \theta & s \theta \\ s \theta & -c \theta \end{pmatrix}, \]
  \[ \text{and } \theta \text{ a real parameter (angle).} \]

While $\varphi_3$ and $\varphi_4$ are $C$ (and $CP$) eigenstates with opposite $C$ quantum number
  \[ \varphi_3 \equiv C(\varphi_3) = \varphi_3, \varphi_4 \equiv C(\varphi_4) = -\varphi_4, \]
$\varphi_L(\theta)$ and $\varphi_S(\theta)$ are, in general, not $C$ ($CP$) eigenstates.

At the limit of small $\theta$ they become $^2$
  \[ \varphi_L \rightarrow \frac{K^0 + \overline{K^0}}{\sqrt{2}} + \theta \frac{K^0 - \overline{K^0}}{\sqrt{2}}, \varphi_S \rightarrow \theta \frac{K^0 + \overline{K^0}}{\sqrt{2}} - \frac{K^0 - \overline{K^0}}{\sqrt{2}}. \]

If $K^0$ and $\overline{K^0}$ are orthogonal and normalized to 1
  \[ < K^0 | K^0 > = 1 = < \overline{K^0} | \overline{K^0} >, \quad < K^0 | \overline{K^0} > = 0 = < \overline{K^0} | K^0 >, \]
one has the relations
  \[ < \varphi_3 | \varphi_3 > = 1 = < \varphi_4 | \varphi_4 >, \quad < \varphi_3 | \varphi_4 > = 0 = < \varphi_4 | \varphi_3 >, \]

$^2$This is to be compared with the usual expressions for the $K_L$ and $K_S$ mesons [13]

\[ K_L = \frac{1}{(1 + |\epsilon_1|^2)^{1/2}} \left( \frac{K^0 + \overline{K^0}}{\sqrt{2}} + \epsilon_1 \frac{K^0 - \overline{K^0}}{\sqrt{2}} \right), \]
\[ K_S = \frac{1}{(1 + |\epsilon_2|^2)^{1/2}} \left( \frac{K^0 - \overline{K^0}}{\sqrt{2}} + \epsilon_2 \frac{K^0 + \overline{K^0}}{\sqrt{2}} \right), \]
where $\epsilon_{1,2}$ are complex parameters. If $CPT$ is conserved, $CP$ and $T$ violated, $\epsilon_1 = \epsilon_2$; if $CPT$ is violated and $T$ is conserved, $\epsilon_1 = -\epsilon_2$.

The expression (7) for small $\theta$ matches this last case. In addition, $P$ is conserved; so $CP$ violation only occurs through $C$ violation.
\( < \varphi_S | \varphi_S > = 1 = < \varphi_L | \varphi_L >, \quad < \varphi_S | \varphi_L > = 0 = < \varphi_L | \varphi_S >. \)  

(9)

Because of the transformation properties (5) of \( \varphi_3 \) and \( \varphi_4 \) by \( C \), and from our choice of phase (see (27) below with \( \gamma = 0 \)), we have, for operators (see for example [14])

\[ K^0 = C(K^0)C^{-1} = K^0, \]

(10)

and for the fields in the Lagrangian

\[ K^0 = K^0, \]

(11)

such that (5) rewrites

\[ \varphi_3^* = \varphi_3, \quad \varphi_4^* = -\varphi_4; \]

(12)

\( \varphi_3 \) can be considered to be purely real, and \( \varphi_4 \) purely imaginary.

With the chosen conventions, there is equivalence between the two notations \( \overline{\psi} \) (charge conjugate) and \( \psi^* \) (complex conjugate); however the "*" notation is useful to keep trace of which fields are independent.

- The kinetic terms \( (\mathcal{L}_{\text{kin}}) \) are trivially diagonal in the three basis \( (K^0, K^0), (\varphi_3, \varphi_4) \) and \( (\varphi_S(\theta), \varphi_L(\theta)) \) since \(^3\)

\[ \varphi^*_L(\theta)\varphi_L(\theta) + \varphi^*_S(\theta)\varphi_S(\theta) = \varphi^*_3\varphi_3 + \varphi^*_4\varphi_4 = K^0*K^0 + K^0[K^0^]. \]

(13)

They can be written in the three equivalent forms \(^4\)

\[ \mathcal{L}_{\text{kin}} = \frac{1}{2} \left( \partial_\mu K^0* \partial^\mu K^0 + \partial_\mu K^0 \partial^\mu K^0^* \right) \]

\[ = \frac{1}{2} \left( \partial_\mu \varphi_3^* \partial^\mu \varphi_3 + \partial_\mu \varphi_4^* \partial^\mu \varphi_4 \right) \]

\[ = \frac{1}{2} \left( \partial_\mu \varphi_L^* \partial^\mu \varphi_L + \partial_\mu \varphi_S^* \partial^\mu \varphi_S \right). \]

(14)

- Like \( K^0 \) and \( K^0^\) \(, \varphi_3 \) and \( \varphi_4 \), \( \varphi_L \) and \( \varphi_S \), are considered as independent fields.

Accordingly, introduce, in the \( \varphi_L(\theta), \varphi_S(\theta) \) basis, the (hermitian) mass terms \(^5\)

\[ \mathcal{L}_m = -\frac{1}{2} \left( \mu^2_S \varphi^*_S(\theta)\varphi_S(\theta) + \mu^2_L \varphi^*_L(\theta)\varphi_L(\theta) \right). \]

(16)

\( \mu^2_L \) and \( \mu^2_S \) are considered as fixed.

\(^3\)In particular, no \( C \)-odd commutator occurs here as it will in the mass terms (see below and appendix A).

\(^4\)\( K^0 \) and \( K^0^\) being independent, their kinetic terms are distinct.

\(^5\)Consider a complex scalar field \( \phi = \phi_1 + i\phi_2 \), where \( \phi_1 \) and \( \phi_2 \) are real and independent. Writing \( \mathcal{L}_m = -m\phi^* \phi \) corresponds to a mass term for \( \phi \) alone, which re-expresses in terms of \( \phi_1 \) and \( \phi_2 \) as \( \mathcal{L}_m = -m(\phi_1^2 + \phi_2^2 + i[\phi_1, \phi_2]) \); a commutator arises; if instead it is written

\[ \mathcal{L}_m = -(m/2)(\phi^* \phi + \phi \phi^*), \]

(15)

which corresponds to a mass term for \( \phi \) and an identical one for \( \phi^* \), it rewrites \( \mathcal{L}_m = -(m/2)(\phi_1^2 + \phi_2^2) \) and no commutator arises. \([\phi_1, \phi_2] \) is not present when \( \phi \) and \( \phi^* \) are treated as two independent fields, like \( \phi_1 \) and \( \phi_2 \), \( \varphi_L, \varphi_S^* \) and \( \varphi_S \) are not independent. It is emphasized in (24) and (25), where charge conjugates can be replaced by complex conjugates. This is why the Lagrangian (16) should not be symmetrized like in (15) with respect to fields and their * conjugates, and why a commutator arises after using the \( C \) properties of \( \varphi_3 \) and \( \varphi_4 \) (this also traduces the non-independence of \( \varphi_3 \) and \( \varphi_3^*, \varphi_4 \) and \( \varphi_4^* \)).
\( \mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_m \), given by (14) (16), is a priori a suitable hermitian Lagrangian to describe one \( \varphi_L \) and one \( \varphi_S \) neutral meson, considered to be, like \( K^0 \) and \( \bar{K}^0 \), two independent fields, with \((\text{mass})^2 \mu_L^2 \) and \( \mu_S^2 \).

In the \((\varphi_3, \varphi_4, \varphi_5)\) basis, \( \mathcal{L}_m \) rewrites

\[
\mathcal{L}_m = \frac{1}{2} \left( \varphi_3^* \varphi_3 \left( \mu_3^2 \sin^2 \theta + \mu_L^2 \cos^2 \theta \right) + \varphi_4^* \varphi_4 \left( \mu_L^2 \sin^2 \theta + \mu_S^2 \cos^2 \theta \right) + (\varphi_3^* \varphi_4 + \varphi_4^* \varphi_3) \left( -\mu_3^2 + \mu_L^2 \right) \sin \theta \cos \theta \right). \tag{17}
\]

The occurrence in (17) of the term proportional to \((\varphi_3^* \varphi_4 + \varphi_4^* \varphi_3)\), which in particular breaks (softly) the discrete symmetries \([4][10][11][12] \varphi_3 \rightarrow -\varphi_3 \) and \( \varphi_4 \rightarrow -\varphi_4 \), comes from the fact that only \( \varphi_S \) and \( \varphi_L \) are independent fields, while \( (\varphi_S^*, \varphi_S, \varphi_L^*, \varphi_L) \) are not (see footnote 5, (24), (25) and appendix A).

\((\varphi_3^* \varphi_4 + \varphi_4^* \varphi_3)\) rewrites as the commutator \([\varphi_3, \varphi_4] = \varphi_3 \varphi_4 - \varphi_4 \varphi_3\) which is presumably vanishing (see below and section 4); if so, in the \((\varphi_3, \varphi_4)\) basis, \( \mathcal{L}_m \) can be considered as diagonal, too. The masses of \( \varphi_3 \) and \( \varphi_4 \), which differ from \( \mu_S^2 \) and \( \mu_L^2 \), are

\[
\mu_3^2 = \mu_3^2 \sin^2 \theta + \mu_L^2 \cos^2 \theta, \quad \mu_4^2 = \mu_S^2 \cos^2 \theta + \mu_L^2 \sin^2 \theta. \tag{18}
\]

On Fig. 1 below are plotted \( \mu_3^2 \) (continuous line) and \( \mu_4^2 \) (dashed line) as functions of \( \theta \) for fixed values \( \mu_S^2 = 4 \) and \( \mu_L^2 = 2 \).

**Fig. 1:** \( \mu_3^2 \) and \( \mu_4^2 \) as functions of \( \theta \) for \( \mu_S^2 = 4 \) and \( \mu_L^2 = 2 \)

Shifting globally the pair of curves along the y axis is akin to changing \( \mu_3^2 + \mu_4^2 = \mu_3^2 + \mu_5^2 \); it is then easy to see that, even for, say, \( \mu_L^2 < 0 \), there can exist domains for which both \( \mu_3^2 \) and \( \mu_4^2 \) are positive (see also section 5.1).

(18) can be inverted into

\[
\mu_3^2 = \frac{\mu_3^2 \cos^2 \theta - \mu_4^2 \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta}, \quad \mu_4^2 = \frac{\mu_3^2 \cos^2 \theta - \mu_4^2 \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta}. \tag{19}
\]
One has

$$\mu_3^2 + \mu_4^2 = \mu_S^2 + \mu_L^2, \quad \frac{\mu_3^2 - \mu_4^2}{\mu_L^2 - \mu_S^2} = \cos 2\theta. \tag{20}$$

On Fig. 2 below is plotted \((\mu_4^2 - \mu_3^2)/(\mu_S^2 - \mu_L^2)\) as a function of \(\theta\).

\textit{Fig. 2:} \((\mu_4^2 - \mu_3^2)/(\mu_S^2 - \mu_L^2)\) as a function of \(\theta\)

Even for \(\mu_L^2 \neq \mu_S^2, \mu_3^2\) and \(\mu_4^2\) become degenerate for \(\cos 2\theta = 0\).

For \(\theta \to 0\),

$$\mu_4^2 \to \mu_S^2 - \theta^2(\mu_S^2 - \mu_L^2), \quad \mu_3^2 \to \mu_L^2 + \theta^2(\mu_S^2 - \mu_L^2). \tag{21}$$

- Phenomena appear more clearly if one writes the mass Lagrangian (16) in the basis of independent states \((K^0, \bar{K}^0)\), and also in the \((\varphi_3, \varphi_4)\) basis

\[
\mathcal{L}_m = -\frac{1}{2} \left( \begin{array}{cc}
\varphi_L^* & \varphi_S^*
\end{array} \right) \left( \begin{array}{cc}
\mu_L^2 & 0 \\
0 & \mu_S^2
\end{array} \right) \left( \begin{array}{c}
\varphi_L \\
\varphi_S
\end{array} \right)
\]

\[
= -\frac{1}{4} \left( \mu_L^2 + \mu_S^2 \right) (K^0 + \bar{K}^0 + \bar{K}^0 K^0 + \cos 2\theta (\mu_L^2 - \mu_S^2) \left( K^0 + \bar{K}^0 + \bar{K}^0 K^0 \right)
\]

\[
= -\frac{1}{4} \left( K^0 + \bar{K}^0 \right) \left[ \left( \begin{array}{cc}
\mu_L^2 + \mu_S^2 & (\mu_L^2 - \mu_S^2) \cos 2\theta \\
(\mu_L^2 - \mu_S^2) \cos 2\theta & \mu_L^2 + \mu_S^2
\end{array} \right)
\right]
\]

\[
= -\frac{1}{2} \left( \begin{array}{c}
\varphi_3^* \\
\varphi_4^*
\end{array} \right) \left( \begin{array}{cc}
\mu_L^2 \cos^2 \theta + \mu_S^2 \sin^2 \theta & 0 \\
0 & \mu_S^2 \cos^2 \theta + \mu_L^2 \sin^2 \theta
\end{array} \right)
\]
\[-\frac{1}{2}(\mu_L^2 - \mu_S^2) \sin 2\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] 
\begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix}. \hspace{1cm} (22)

The eigenvalues of the total mass matrix $M$ (i.e. the terms inside the brackets $|$ in the last two lines of (22)) are $\mu_S^2$ and $\mu_L^2$, as is conspicuous from its definition and the first line of (22); $M$ has been split in two parts $M = M_1 + (\mu_L^2 - \mu_S^2)M_2 \sin 2\theta$; the last one triggers indirect $C\,(CP)$ violation \(^6\).

As seen in (22), weak interactions are at the origin of two phenomena, through two types of terms, both proportional to $(\mu_L^2 - \mu_S^2)$:

- oscillations between the two states $K^0$ and $\bar{K}^0$, which have degenerate masses $(\mu_L^2 + \mu_S^2)/2$; they are induced by the $CP$ conserving $|\Delta S| = 2$ operator $K^0\bar{K}^0 + K^{0*}\bar{K}^0 = K_0^0 + \bar{K}_0^0 \equiv \varphi_3^2 + \varphi_4^2 \equiv \varphi_L^2(\theta) + \varphi_S^2(\theta)$, with a coefficient proportional to $\cos 2\theta$; they generate the mass splitting between the two $CP$ eigenstates $\varphi_3$ and $\varphi_4$ \(^7\); they share similarities with Majorana mass terms which are traditionally introduced for fermions;

- additional transitions between $\varphi_3$ and $\varphi_4$, giving rise to indirect $C\,(CP)$ violation; they are induced by the term proportional to $\varphi_3^2\varphi_4 + \varphi_4^2\varphi_3$, which also rewrites as the $CP$-odd term $K^0\bar{K}^0 - K^{0*}\bar{K}^0$, or as the commutator $[K_0, \bar{K}_0] \equiv [\varphi_4, \varphi_3] \equiv [\varphi_L(\theta), \varphi_S(\theta)]$; its coefficient is proportional to $\sin 2\theta$.

For small values of $\theta$, the second transitions correspond to a very small energy with respect to the $\phi_L - \phi_S$ mass difference. $\mu_L^2$ and $\mu_S^2$ are very close to $\mu_S^2$ and $\mu_L^2$. For large values of $\theta$, $\mu_3^2$ and $\mu_4^2$ can be very different from $\mu_L^2$ and $\mu_S^2$.

- In addition to $(\mu_L^2 + \mu_S^2)/2$ and $(\mu_L^2 - \mu_S^2) \cos 2\theta$, (22) exhibits a third energy scale

$$\kappa^2(\theta) = |(\mu_L^2 - \mu_S^2) \sin 2\theta|.$$ \hspace{1cm} (23)

- Whatever be $\theta$, the eigenvalues of $M_1$ are $\mu_3^2$ and $\mu_4^2$ given by (18) and its eigenvectors are $\varphi_3$ and $\varphi_4$ (identical, up to a phase, to their own antiparticles, like Majorana fermions); a variation of $\theta$ does not change the eigenvectors, which stay $CP$-eigenstates; $CP$ conservation is thus associated to a $U(1)$ symmetry with phase $\theta$.

Any choice for $\theta$ breaks the above $U(1)$ symmetry; when $\sin 2\theta \neq 0$, it also breaks indirect $C\,(CP)$ invariance, by adding a non-vanishing contribution to the mass matrix proportional to $M_2$; the eigenvalues of $M$ become $\mu_S^2$ and $\mu_L^2$, and its eigenvectors $\varphi_L(\theta)$ and $\varphi_S(\theta)$. Then, to continue the comparison with fermion masses, the mass eigenstates switch to states which are not their own antiparticles, like Dirac fermions.

When $\theta$ goes from 0 to $\pi/4$, indirect $C\,(CP)$ violation goes from 0 (in which case the mass eigenstates are $C\,(CP)$ eigenstates) to its maximal possibility, where $\varphi_S(\pi/4) = \bar{K}^0$, $\varphi_L(\pi/4) = K^0$.

The case $\theta = \pm \pi/4$ corresponds to $\varphi_S(\pm \pi/4) = \pm \varphi_L(\pm \pi/4)$. Then, $CPT$ requires the two particles to have the same mass. For instance, let us consider the case $\theta = +\pi/4$, for which $\varphi_L(+\pi/4) = K_0$, $\varphi_S(+\pi/4) = \bar{K}^0$. In the second line of (22), the coefficient $\cos 2\theta$ of the $|\Delta S| = 2$ terms vanishes. As far as the $CP$ violating term is concerned, if the commutator $[K_0, \bar{K}_0] \equiv [\varphi_3, \varphi_4]$ can be taken as identically vanishing, the condition $\mu_L^2 = \mu_S^2$ is not needed stricto sensu to get the identical masses required by $CPT$ for $K^0$ and $\bar{K}^0$; they are then equal to $(\mu_L^2 + \mu_S^2)/2$, in agreement with (18). If the commutator cannot be taken as vanishing, then the condition $\mu_L^2 = \mu_S^2$ is required, which also yields $\mu_3^2 = \mu_4^2$.

\(^6\)Notice that $M_2$ has always one negative $(mass)^2$ eigenvalue.

\(^7\)They are equivalent to the customary $K^0 - \bar{K}^0$ transitions triggered by weak interactions when no complex phase is present.
We shall come back further in the paper to the singular points, like the one considered above, which correspond to \( \tan^2 \theta = 1 \).

- Unlike \( K^0 \) and \( \bar{K}^0 \), \( \varphi_L \), \( \bar{\varphi}_L \), \( \varphi_S \) and \( \bar{\varphi}_S \) are not independent (see also Appendix A): they satisfy for example the relation

\[
(\cos \theta - \sin \theta)(\bar{\varphi}_S(\theta) + \varphi_L(\theta)) = (\cos \theta + \sin \theta)(\varphi_L(\theta) - \varphi_S(\theta)),
\]

and the general equations

\[
\begin{pmatrix}
\varphi_L \\
\bar{\varphi}_S
\end{pmatrix} = \begin{pmatrix}
\cos 2\theta & \sin 2\theta \\
\sin 2\theta & -\cos 2\theta
\end{pmatrix} \begin{pmatrix}
\varphi_L \\
\varphi_S
\end{pmatrix}.
\]

They form an over-complete basis; because of the equivalence between charge conjugation and complex conjugation, it is also the case, through (12), for \( \varphi_L, \bar{\varphi}_L, \varphi_S, \bar{\varphi}_S \).

- The property that \( \mathcal{L}_m \) is diagonal in the basis \((\varphi_3, \varphi_4)\) of \( C \) \((CP)\) eigenstates rests on the implicit assumption that the commutator \([K^0(x), \bar{K}^0(x)]\), for fields at the same space-time point, which occurs in \( M_2 \), gives a vanishing contribution to the action

\[
\int d^4x \left( \bar{K}^0(x)K^0(x) - K^0(x)\bar{K}^0(x) \right) = 0.
\]

This statement is confirmed by the standard expansion of \( K^0 \) and \( \bar{K}^0 \) considered as fundamental fields \((\gamma \) is an arbitrary phase that we have chosen equal to 0)

\[
K^0(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left( a(\vec{k})e^{-i\vec{k}\cdot r} + b(\vec{k})e^{i\vec{k}\cdot r} \right),
\]

\[
\bar{K}^0(x) = e^{-i\gamma} \int \frac{d^3l}{(2\pi)^3 2l_0} \left( b(\vec{l})e^{-i\vec{l}\cdot r} + a(\vec{l})e^{i\vec{l}\cdot r} \right),
\]

in terms of independent creation \((a, b)\) and annihilation \((a, b)\) operators satisfying the usual commutation relations

\[
[a(\vec{k}), a(\vec{l})^\dagger] = [b(\vec{k}), b(\vec{l})^\dagger] = (2\pi)^3 2k_0 \delta^3(\vec{k} - \vec{l}).
\]

We shall come again to this point in subsection 4.1, and show that, instead, their commutator is likely not to vanish when \( K^0 \) and \( \bar{K}^0 \) are taken, like in the quark model, as composite.

If so, \( \langle K^0 | H | K^0 \rangle \neq \langle \bar{K}^0 | H | K^0 \rangle \) \((H \) being the Hamiltonian), and \( CPT \) is expected to be broken. This is in agreement with our starting formulae for \( \varphi_L \) and \( \varphi_S \) (3) (see footnote 2). \( CP \) violation occurs here only through \( C \) violation, \( P \) and \( T \) being conserved.

### 2.2 A non-unitary transformation (\( CPT \) conserved, \( T \) violated)

One uses the same phase convention as in subsection 2.1, such that the notations \( \varphi^* \) and \( \bar{\varphi} \) are equivalent.

Subsection 2.1 dealt with a unitary transformation to go from \((\varphi_3, \varphi_4)\) to \((\varphi_S, \varphi_L)\). We have mentioned (see footnote 2) that this choice of \( \varphi_L \) and \( \varphi_S \) is akin to considering that \( CP \) is violated through a violation of \( CPT \), \( T \) being conserved (and \( P \) too).

This is not the only possibility, and the recent measurements of CPLEAR [15][16] showed that \( T \) is probably violated. Then, \( K_L \) and \( K_S \) write instead, for small \( \epsilon \) [13], that we shall suppose hereafter to be real

\[
K_L(\epsilon) = \frac{1}{(1 + |\epsilon|^2)^{1/2}} (\varphi_3 + \epsilon \varphi_4), K_S(\epsilon) = \frac{1}{(1 + |\epsilon|^2)^{1/2}} (\varphi_4 + \epsilon \varphi_3);
\]

\[
\]
it is the limit, for small $\epsilon$, of the non-unitary transformation

\[
\begin{pmatrix}
K_L(\epsilon) \\
K_S(\epsilon)
\end{pmatrix} = \frac{1}{\sqrt{\cosh^2 \epsilon + \sinh^2 \epsilon}} A \begin{pmatrix}
\varphi_3 \\
\varphi_4
\end{pmatrix},
\tag{30}
\]

with

\[
A = \begin{pmatrix}
\cosh \epsilon & \sinh \epsilon \\
\sinh \epsilon & \cosh \epsilon
\end{pmatrix}.
\tag{31}
\]

(30) inverts into

\[
\begin{pmatrix}
\varphi_3 \\
\varphi_4
\end{pmatrix} = \sqrt{\cosh^2 \epsilon + \sinh^2 \epsilon} \begin{pmatrix}
\cosh \epsilon & -\sinh \epsilon \\
-\sinh \epsilon & \cosh \epsilon
\end{pmatrix} \begin{pmatrix}
K_L(\epsilon) \\
K_S(\epsilon)
\end{pmatrix}. \tag{32}
\]

If $K^0$ and $\overline{K^0}$ are orthogonal and normalized to 1 like in (8), $K_L$ and $K_S$ are non-longer orthogonal and (9) is replaced by

\[
< K_S | K_S > = 1 = < K_L | K_L >, \quad \text{but} \quad < K_S | K_L > = \tanh(2\epsilon) = < K_L | K_S >. \tag{33}
\]

Like in the previous section, $K_L(\epsilon), \overline{K_L(\epsilon)}, K_S(\epsilon), \overline{K_S(\epsilon)}$ are not independent.

Performing a non-unitary transformation can have dramatic results on the kinetic terms; in most cases, if the latter are diagonal in a basis, there will not be after the transformation. Of course, the spectrum of the physical states is only readable from diagonal kinetic and mass terms.

Performing the same analysis as in section 2, we transform the massive hermitian Lagrangian

\[
\mathcal{L} = \frac{1}{2} (\partial_{\mu} K_S^*(\epsilon) \partial^\mu K_S(\epsilon) + \partial_{\mu} K_L^*(\epsilon) \partial^\mu K_L(\epsilon) - \mu_S^2 K_S^*(\epsilon) K_S(\epsilon) - \mu_L^2 K_L^*(\epsilon) K_L(\epsilon)) \tag{34}
\]

by the non-unitary transformation $A$. $\mathcal{L}$ becomes

\[
\mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \varphi_3^* \partial^\mu \varphi_3 + \partial_{\mu} \varphi_4^* \partial^\mu \varphi_4 + \frac{2 \sinh \epsilon \cosh \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon} (\partial_{\mu} \varphi_3^* \partial^\mu \varphi_4 + \partial_{\mu} \varphi_4^* \partial^\mu \varphi_3) - \frac{(\mu_L^2 \cosh^2 \epsilon + \mu_3^2 \sinh^2 \epsilon) \varphi_3^* \varphi_3}{\cosh^2 \epsilon + \sinh^2 \epsilon} - \frac{\mu_S^2 \sinh^2 \epsilon + \mu_3^2 \cosh^2 \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon} \varphi_4^* \varphi_4 - \left( \mu_L^2 + \mu_3^2 \right) \sinh \epsilon \cosh \epsilon \varphi_3^* \varphi_4 + \varphi_4^* \varphi_3 \right). \tag{35}
\]

Note the presence of a non-diagonal kinetic term in (35).

If one uses the $C$ transformation properties of $\varphi_3$ and $\varphi_4$ to transform into commutators $\partial_{\mu} \varphi_3^* \partial^\mu \varphi_4 + \partial_{\mu} \varphi_4^* \partial^\mu \varphi_3 = [\partial_{\mu} \varphi_3, \partial^\mu \varphi_4]$ and $\varphi_3^* \varphi_4 + \varphi_4^* \varphi_3 = [\varphi_3, \varphi_4]$, when $[\partial_{\mu} \varphi_3, \partial^\mu \varphi_4] = 0 = [\varphi_3, \varphi_4]$, the mass and kinetic terms can be both diagonal in the two “basis”, and one has the relations between the masses:

\[
\mu_3^2 = \frac{\mu_S^2 \sinh^2 \epsilon + \mu_L^2 \cosh^2 \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon},
\]

*In particular, when the decays of the kaons are included, their mass matrix becomes non-hermitian; the matrix which diagonalizes it in the usual way is non-unitary, which would spoil the diagonality of the kinetic terms. In this case, one has, instead, to use a bi-unitary transformation to perform the diagonalization [16]; then, like for fermions, the masses of the physical states are not the roots of the characteristic equation of their mass matrix.
\[ \mu_4^2 = \frac{\mu_3^2 \sinh^2 \epsilon + \mu_5^2 \cosh^2 \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon}. \] (36)

On Fig. 3 below are plotted \( \mu_3^2 \) and \( \mu_4^2 \) as functions of \( \epsilon \) for fixed \( \mu_5^2 = 4 \) and \( \mu_L^2 = 2 \).

**Fig. 3:** \( \mu_3^2 \) and \( \mu_4^2 \) as functions of \( \theta \) for \( \mu_5^2 = 4 \) and \( \mu_L^2 = 2 \)

Like in subsection 2.1, shifting the pair of curves along the \( y \) axis is akin to changing \( \mu_3^2 + \mu_4^2 = \mu_L^2 + \mu_S^2 \); it is then easy to see that, even for, say, \( \mu_L^2 < 0 \), there can exist domains for which both \( \mu_3^2 \) and \( \mu_4^2 \) are positive (see also section 5.2).

(36) inverts into
\[
\begin{align*}
\mu_L^2 &= \mu_3^2 \cosh^2 \epsilon - \mu_4^2 \sinh^2 \epsilon, \\
\mu_S^2 &= \mu_4^2 \cosh^2 \epsilon - \mu_3^2 \sinh^2 \epsilon.
\end{align*}
\] (37)

A similar ambiguity in the mass spectrum arises as in section 2.1.

One has
\[
\begin{align*}
\mu_L^2 + \mu_S^2 &= \mu_3^2 + \mu_4^2, \\
\frac{\mu_3^2 - \mu_4^2}{\mu_L^2 - \mu_S^2} &= \frac{1}{\cosh 2\epsilon}. \tag{38}
\end{align*}
\]

On Fig. 4 below is plotted \( (\mu_4^2 - \mu_3^2)/(\mu_5^2 - \mu_L^2) \) as a function of \( \epsilon \).
In particular, even when \( \mu^2_L \neq \mu^2_S, \mu^2_3 \) and \( \mu^2_4 \) become identical when \( \epsilon \to \infty \).

At this limit, \( K_L \to K^0, K_S \to \overline{K^0} \); whatever different are their masses (and they can be, as long as these two states are not exactly identical with the two conjugate neutral kaons), the \( CP \) eigenstates \( K^0_1 \) and \( K^0_2 \) become degenerate. So, a large \( CP \) violating parameter goes along with the degeneracy of \( CP \) eigenstates.

For \( \epsilon \to 0 \),

\[
\mu^2_4 \to \mu^2_S - \epsilon^2 (\mu^2_S - \mu^2_L), \quad \mu^2_3 \to \mu^2_L + \epsilon^2 (\mu^2_S - \mu^2_L),
\]

(39)

which is formally identical to (21).

In the basis \( (K^0, \overline{K^0}) \) and \( (\varphi_3, \varphi_4) \), the mass Lagrangian rewrites

\[
\mathcal{L}_m = -\frac{1}{4} \left( K^0* \overline{K^0}* \right) \begin{pmatrix}
\mu^2_L + \mu^2_S & (\mu^2_L - \mu^2_S)/\cosh 2\epsilon \\
(\mu^2_L - \mu^2_S)/\cosh 2\epsilon & \mu^2_L + \mu^2_S
\end{pmatrix}
\begin{pmatrix}
\mu^2_L + \mu^2_S \\
(\mu^2_L - \mu^2_S)/\cosh 2\epsilon
\end{pmatrix}
+ (\mu^2_L + \mu^2_S) \tanh 2\epsilon \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
K^0 \\
\overline{K^0}
\end{pmatrix},
\]

(40)

which is to be compared with (22).

In addition to the two energy scales \( (\mu^2_L + \mu^2_S) \) and \( (\mu^2_L - \mu^2_S)/\cosh 2\epsilon \), the third excitation \( \kappa^2(\epsilon) \) writes now:

\[
\kappa^2(\epsilon) = |(\mu^2_L + \mu^2_S) \tanh 2\epsilon|.
\]

(41)
2.3 Conclusion of this section

This section uncovered a connection between the mass spectrum and the charge conjugation properties of scalar mesons, and exhibited the following peculiarities – that we shall show to be common to more general systems –.

A $CP$-odd, potentially vanishing, commutator $\mathcal{C} \equiv i[K_1^0, K_2^0] = i[K_0^0, K_0^0]$ can alter the spectrum of the theory and trigger indirect $CP$ violation; $\mathcal{C} = i[K_L, K_S]$ when $K_L$ and $K_S$ are deduced from $K_1^0$ and $K_2^0$ by a unitary transformation – $CP$ broken, $T$ conserved –, and $\mathcal{C} = i(\sinh^2 \epsilon + \cosh^2 \epsilon)[K_L, K_S]$ when $K_L$ and $K_S$ are deduced from $K_1^0$ and $K_2^0$ by a non-unitary transformation – $CP$ broken, $T$ broken, $CPT$ conserved –.

It can be looked at from two different points of view.

At the level of operators, it induces transitions between the two (independent) types of particles on which it operates.

At the level of the fields in the Lagrangian, the question of its vanishing arises, and, linked to it, the existence of an ambiguity in the mass spectrum.

- If it vanishes, which seems a legitimate assumption when $\varphi_3$ and $\varphi_4$ describe fundamental non-decaying particles – see section 4 –, an angle $\theta$, characterizing indirect $CP$-violation, defines a continuous set of basis in which the mass terms and the kinetic terms in the Lagrangian can be diagonalized; the mass splittings depend on the basis and their ratio (20) depend only on $\theta$; $\mu_L^2$ and $\mu_S^2$ (or $\mu_3^2$ and $\mu_4^2$) are then ambiguous quantities. Whatever be $\theta$, $CP$, which cannot be explicitly broken by the term in the Lagrangian proportional to this commutator (since it vanishes), can only be spontaneously broken$^9$;

- if it does not vanish, which in particular occurs, as we shall see in section 4, when the fields are considered as composite and as soon as electroweak interactions are turned on, $CP$ invariance is explicitly broken in the Lagrangian and the $CP$-violating basis is the only diagonal one. The masses of the $CP$-violating eigenstates are then no longer ambiguous and become true observable.

For real kaons, $\epsilon$ (or $\theta$) is small, of order $10^{-3}$. The “masses” of the pairs $(K_1^0, K_2^0)$ and $(K_L, K_S)$ (if they can be defined), only differ by $\theta^2$ or $\epsilon^2$ (see (21) and (39)). Furthermore, the individual masses of $K_1^0$ and $K_2^0$ (or of $K_L$ and $K_S$) are extremely close. This makes the two sets experimentally indistinguishable, and the discussion concerning which set of masses are really measured purely academic. There may exist systems, however, for which the question could be relevant.

3 Extension to Higgs-like doublets

The same ambiguity as the one studied in section 2 occurs for more general systems, in particular $SU(2)_L \times U(1)$ Higgs-like multiplets.

We shall study below the case of a unitary transformation, keeping in mind that a similar discussion can be made for a non-unitary transformation like the one performed in subsection 2.2.

The phase convention for charge conjugation is the same as before, such that there is equivalence between $\bar{\varphi}$ (charge conjugation) and $\varphi^*$ (complex conjugation).

$^9$If the vanishing of $[K_0^0, K_0^0]$ is true at the quantum level, the two basis $(K_L, K_S)$ and $(K_1^0, K_2^0)$ are equivalent, which means that the neutral kaons can be truly measured in two different states, with two different mass spectra, a $CP$ conserving state and a $CP$ violating state. This is not a contradictory statement in the absence of electroweak interactions since the states can only be identified through their decays.
3.1 Scalar multiplets

We deal with $SU(2)_L \times U(1)$ multiplets isomorphic to the Higgs multiplets of the Standard Model [17].

If one considers quadruplets [18]

$$\phi = (\phi^0, \phi^3, \phi^+, \phi^-)$$

with $\phi^+ = \frac{\phi^3 + i\phi^0}{\sqrt{2}}, \phi^- = \frac{\phi^3 - i\phi^0}{\sqrt{2}}$, transforming by $SU(2)_L$ with generators $T^3, T^+ \equiv T^1 + iT^2, T^- \equiv T^1 - iT^2$ according to

$$T^i_L \cdot \phi^j = -\frac{i}{2} \left( \epsilon_{ijk} \phi^k + \delta_{ij} \phi^0 \right),$$

$$T^i_L \cdot \phi^0 = \frac{i}{2} \phi^i,$$

then the two complex doublets

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^3 - i\phi^0 \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} \phi^3 + i\phi^0 \\ -\phi^- \end{pmatrix}$$

(44)

are isomorphic to the standard Higgs doublets $\Phi$ and $\tilde{\Phi} = -i(\Phi^T \tau_2)$ of the Glashow-Salam-Weinberg model (the $\tau$'s are the Pauli matrices and the superscript “$T$” means “transposed”).

Those quadruplets have been explicitly constructed in [18] as quark-antiquark composite fields. If $N/2$ is the number of generations of quarks, there exist $N^2/2$ such multiplets. They can always be arranged in such a way that the parity of $\phi^0$ is the opposite of the parity of $\tilde{\phi}$, and there are consequently two types of such multiplets, $(S^0, \tilde{P})$ and $(P^0, \tilde{S})$, “$S$” and “$P$” meaning respectively “scalar” and “pseudoscalar” 10.

As soon as there are more than one generation of fermions, they can also be classified according to their transformation by charge conjugation $C$ [19]; in particular, for two generations, which we shall deal with in this work, among the total number of eight quadruplets, there are six with $C = +1$ and two with $C = -1$; that those numbers are all even corresponds to the classification according to parity mentioned above.

The law of transformation (43) entails that for any two quadruplets $\phi$ and $\phi'$, the quadratic expression

$$\phi \phi' = \phi^0 \phi'^0 + \tilde{\phi} \tilde{\phi}' = \Phi^T \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \Phi$$

(45)

is invariant by $SU(2)_L$.

3.2 A system of two quadruplets with opposite $C$ quantum numbers

For the reader to easily make an link with the simple example of section 2, we start by investigating the case of two $SU(2)_L \times U(1)$ quadruplets which have opposite transformations by $C$. They are generalizations, within an $SU(2)_L \times U(1)$ group structure, of the states $K^0 \pm \bar{K}^0$ used there; the Cabibbo mixing angle [9] plays now an important role [18][19].

For the sake of simplicity, we also postpone the general demonstration to the next subsection.

10The notation “$P$” is also used in this work for the parity operator, but confusion should not arise.
We consider the quadratic Lagrangian for two quadruplets \( \phi_3 \) and \( \phi_4 \) of the same type with \( C = +1 \) and \( C = -1 \) respectively
\[
\bar{\phi}_3 = \phi_3, \quad \bar{\phi}_4 = - \phi_4.
\] (46)

We suppose for example that they are both of the \((S^0, \bar{P})\) type.

In the \( \phi \) basis, let the quadratic Lagrangian be
\[
L = \frac{1}{2} \left( \partial \mu \left( \phi_3^* \phi_4^* \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \partial^\mu \left( \phi_3 \phi_4 \right) - \left( \phi_3^* \phi_4^* \right) M \left( \phi_3 \phi_4 \right) \right).
\] (47)

The choice of the kinetic term is guided by the fact that, for composite quadruplets of definite \( C \),
\[
\sum_{\text{all quadruplets}} \bar{\phi} \phi = \sum_{\text{all quadruplets}} \phi^* \phi
\] (48)
is diagonal in both basis of flavour and electroweak eigenstates \([18][19]\) \footnote{13}. Let \( M \) be a real symmetric mass matrix
\[
M = \left( \begin{array}{cc} h_3 & h \\ h & h_4 \end{array} \right)
\] (49)
in which all \( h \)'s have dimension \([\text{mass}]^2\).

Its non-diagonal elements contribute to the Lagrangian by
\[
-(1/2)h(\phi_3^* \phi_4 + \phi_4^* \phi_3) = -(1/2)h(\bar{\phi}_3 \phi_4 + \phi_4 \bar{\phi}_3),
\]
which is likely to vanish. When it does, \( \phi_3 \) and \( \phi_4 \) are mass eigenstates with masses \( h_3 \) and \( h_4 \).

We perform a change of basis
\[
\left( \begin{array}{c} \phi_3 \\ \phi_4 \end{array} \right) = V \left( \begin{array}{c} \phi_L \\ \phi_S \end{array} \right),
\] (50)
with \( V \) a unitary transformation (\( c_\theta \) and \( s_\theta \) stand respectively for \( \cos \theta \) and \( \sin \theta \))
\[
V = V^\dagger = V^{-1} = \left( \begin{array}{cc} c_\theta & s_\theta \\ s_\theta & - c_\theta \end{array} \right)
\] (51)
which keeps the kinetic terms diagonal.

In the \( \xi \) basis, the mass matrix
\[
V^\dagger M V = \left( \begin{array}{cc} c_\theta^2 h_3 + s_\theta^2 h_4 + 2 s_\theta c_\theta h & (s_\theta^2 - c_\theta^2) h + s_\theta c_\theta (h_3 - h_4) \\ (s_\theta^2 - c_\theta^2) h + s_\theta c_\theta (h_3 - h_4) & s_\theta^2 h_3 + c_\theta^2 h_4 - 2 s_\theta c_\theta h \end{array} \right)
\] (52)

\footnote{We shall not consider the more general case of three or more quadruplets.}\footnote{The subscripts 3 and 4 are used for compatibility with the notations used in previous works. Accordingly, \( \phi_3 \) is linked with the matrix \( D_3 \) and \( \phi_4 \) with the matrix \( D_4 \) of \([18]\); that they are respectively symmetric and antisymmetric in flavour space is at the origin of the transformation of the quadruplets by \( C \). The states chosen in the simple example also reflect this fact.}\footnote{According to (45), it is \( \phi \phi = \phi^0 \phi^0 + \phi^3 \phi^3 + \phi^+ \phi^- + \phi^- \phi^+ \) which is invariant by the gauge group. For quadruplets of given \( C \), \( \bar{\phi} \phi = \phi^* \phi = \pm \phi \phi \); accordingly, the sum (48) includes alternate signs when expressed in terms of the \( \phi \)'s alone.}
is diagonal for
\[ \tan 2\theta = \frac{2h}{h_3 - h_4}, \]  
(53)
and the masses of \( \phi_L \) and \( \phi_S \) are, then
\[ \mu_L^2 = c_\theta^2 h_3 + s_\theta^2 h_4 + 2 s_\theta c_\theta h \frac{c_\theta^2 h_3 - s_\theta^2 h_4}{c_\theta^2 - s_\theta^2}, \]
\[ \mu_S^2 = c_\theta^2 h_4 + s_\theta^2 h_3 - 2 s_\theta c_\theta h \frac{c_\theta^2 h_4 - s_\theta^2 h_3}{c_\theta^2 - s_\theta^2}, \]
(54)
which can be inverted into (see also (18))
\[ h_3 = c_\theta^2 \mu_L^2 + s_\theta^2 \mu_S^2, \]
\[ h_4 = s_\theta^2 \mu_L^2 + c_\theta^2 \mu_S^2. \]
(55)
One deduces from (54)
\[ \mu_L^2 + \mu_S^2 = h_3 + h_4, \]
(56)
and, like in (20)
\[ \frac{\mu_L^2 - \mu_S^2}{h_3 - h_4} = \frac{1}{\cos 2\theta}. \]
(57)
One has the relation:
\[ \left( \frac{\phi_L}{\phi_S} \right) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \left( \begin{array}{c} \phi_L \\ \phi_S \end{array} \right), \]
(58)
such that the mass splitting in the \( \xi \) basis, \( \mu_L^2 - \mu_S^2 \) cannot diverge.

The divergence in (57) could only happen for \( \cos 2\theta = 0 \). Then, \( s_\theta^2 = c_\theta^2 \), and \( V^\dagger MV \) in (52) can only be diagonal for \( h_3 = h_4 \); its eigenvalues are then \( h_3 \pm h = h_4 \pm h \), corresponding to \( |\mu_L^2 - \mu_S^2| = 2|h| \); a vanishing \( \cos 2\theta \) also corresponds, by (58), to \( \phi_L = \pm \phi_S \), and CPT requires then that the two states have the same mass. As was already mentioned in section 2, this constrains \( h \) to be vanishing only if the commutator \( [\phi_3, \phi_4] \) is not; if it does vanish, then \( h \) can be non-vanishing. This last remark can easily be checked directly by noticing that the combination \( h(\phi_L(\pm \pi/4)\phi_L(\pm \pi/4) - \phi_S(\pm \pi/4)\phi_S(\pm \pi/4)) \) which occurs, factorised by \( 2s_\theta c_\theta \), in the diagonal terms of the mass matrix (52) is identical to \( h[\phi_3, \phi_4] \). So, whether the commutator is vanishing or not, the bound \( |\mu_L^2 - \mu_S^2| \leq |2h| \) for \( \cos 2\theta = 0 \) always exists, which shows the absence of divergence for \( |\mu_L^2 - \mu_S^2| \).

(57) shows that the mass splitting in the basis of states which are not \( C \) eigenstates is always larger than the one in the basis of \( C \) eigenstates.

If \( h_3 = h_4 \), (54) entails \( \mu_L^2 = \mu_S^2 = h_3 = h_4 \), except (see Fig. 1, with the replacement \( \mu_3^2 \rightarrow h_3 \) and \( \mu_4^2 \rightarrow h_4 \)) for \( \sin^2 \theta = \cos^2 \theta \); in this last case, \( \mu_L^2 \neq \mu_S^2 \) becomes compatible with \( h_3 = h_4 \). Reciprocally, (57) shows that \( \mu_L^2 = \mu_S^2 \) only when \( h_3 = h_4 \); two non-degenerate \( \phi_3 \) and \( \phi_4 \) states (supposing \( [\phi_3, \phi_4] = 0 \)) cannot be rotated into degenerate ones.

For \( \theta \) small, \( \sin \theta \approx \theta \) is the \( \epsilon \)-like parameter describing indirect \( C \) (\( CP \)) violation in the neutral \( \phi \) system; from (53) (57), one gets
\[ \theta \approx \sin \theta = \frac{2h}{\mu_L^2 - \mu_S^2}. \]
(59)
Thus, a knowledge of the \( C \) (\( CP \)) violating parameter \( \theta \) and of \( \mu_L^2 - \mu_S^2 \) determines the non-diagonal entry \( h \) of the mass matrix; \( h_3 - h_4 \) can then obtained from (53).

The two basis \( (\phi_3, \phi_3, \phi_4, \phi_4) \) and \( (\phi_L, \phi_L, \phi_S, \phi_S) \) are over-complete; only \( (\phi_3, \phi_4) \) and \( (\phi_L, \phi_S) \) are not.
3.3 A system of two quadruplets with identical $C$ quantum numbers

We consider now the case of two quadruplets (of the same type, $(S^0, \vec{P})$ or $(P^0, \vec{S})$) with the same $C$, and show that one reaches the same conclusions. We make here the general demonstration, which can also be used in the previous section.

Let for example $\phi_2$ and $\phi_3$ be two quadruplets with $C = +1$

$$\bar{\phi}_2 = \phi_2, \quad \bar{\phi}_3 = \phi_3,$$  \hspace{1cm} (60)

transforming by $SU(2)_L$ according to (43).

The quadratic Lagrangian is chosen to be

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \left( \begin{array}{c} \phi_2^* \\ \phi_3^* \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \partial^\mu \left( \begin{array}{c} \phi_2 \\ \phi_3 \end{array} \right) - \left( \begin{array}{c} \phi_2^* \\ \phi_3^* \end{array} \right) M \left( \begin{array}{c} \phi_2 \\ \phi_3 \end{array} \right) \right),$$  \hspace{1cm} (61)

with

$$M = \left( \begin{array}{cc} \lambda_2 & \rho \\ \sigma & \lambda_3 \end{array} \right).$$  \hspace{1cm} (62)

$\lambda_2, \lambda_3, \sigma, \rho$ all have dimension $[\text{mass}]^2$.

$\mathcal{L}$ is hermitian for $\lambda_2$ and $\lambda_3$ real, and for $\sigma = \bar{\rho}$. The non-diagonal mass term $\rho(\phi_2^* \phi_3) + \sigma(\phi_3^* \phi_2)$ presumably vanishes, owing to (60), for $\sigma = -\rho$, and we thus choose

$$\rho = -\sigma = i\nu, \quad \lambda_2 \text{ and } \lambda_3 \text{ real},$$  \hspace{1cm} (63)

where $\nu$ has the dimension $[\text{mass}]^2$. When the commutator $[\phi_2, \phi_3]$ vanishes, $\phi_2$ and $\phi_3$ are mass eigenstates with masses $\lambda_2$ and $\lambda_3$.

One goes from the $\phi$ basis to the $\xi$ basis according to

$$\left( \begin{array}{c} \phi_2 \\ \phi_3 \end{array} \right) = V \left( \begin{array}{c} \phi_L \\ \phi_S \end{array} \right),$$  \hspace{1cm} (64)

where $V$ is a general unitary matrix

$$V = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).$$  \hspace{1cm} (65)

That

$$V^\dagger V = \left( \begin{array}{cc} |a|^2 + |c|^2 & \bar{a}b + \bar{c}d \\ \bar{a}b + \bar{c}d & |b|^2 + |d|^2 \end{array} \right) = 1$$  \hspace{1cm} (66)

requires

$$\bar{a}b + \bar{c}d = 0 = a\bar{b} + c\bar{d}, \quad |a|^2 + |c|^2 = 1 = |b|^2 + |d|^2.$$  \hspace{1cm} (67)
The mass matrix in the $\xi$ basis

$$V^\dagger MV = \begin{pmatrix} |a|^2\lambda_2 + |c|^2\lambda_3 + a\bar{c}\sigma + \bar{a}c\rho & \bar{a}b\lambda_2 + \bar{a}d\rho + b\bar{c}\sigma + \bar{c}d\lambda_3 \\ ab\lambda_2 + ad\sigma + bc\rho + cd\lambda_3 & |b|^2\lambda_2 + |d|^2\lambda_3 + bd\sigma + \bar{b}d\rho \end{pmatrix}$$

(68)

is diagonal for

$$\bar{a}b\lambda_2 + \bar{a}d\rho + b\bar{c}\sigma + \bar{c}d\lambda_3 = 0, \quad ab\lambda_2 + ad\sigma + bc\rho + cd\lambda_3 = 0,$$

which are two self-conjugate equations when (63) is satisfied. (67), (69) and (63) combine into

$$ac(\lambda_2 - \lambda_3) + i\nu(a^2 + c^2) = 0.$$  

(70)

The masses of $\phi_L$ and $\phi_S$ are

$$\mu^2_L = \frac{|a|^2\lambda_2 + |c|^2\lambda_3 + a\bar{c}\rho + \bar{a}c\sigma}{|a|^2 + |c|^2} = |a|^2\lambda_2 + |c|^2\lambda_3 + i\nu(\bar{a}c - a\bar{c}),$$

$$\mu^2_S = \frac{|b|^2\lambda_2 + |d|^2\lambda_3 + bd\sigma + \bar{b}d\rho}{|b|^2 + |d|^2} = |c|^2\lambda_2 + |a|^2\lambda_3 - i\nu(\bar{a}c - a\bar{c}),$$

(71)

where we have again made use of (67) to write the r.h.s. of (71).

- The first trivial solution of (70)

$$\lambda_2 = \lambda_3, \quad c^2 + a^2 = 0$$  

(72)

yields also through (67) $b^2 + d^2 = 0$. Choosing, in order that $\Delta \equiv \det V = ad - bc$ be non-vanishing,

$$c = +ia, \quad d = -ib,$$

(73)

the masses of $\phi_L$ and $\phi_S$ become

$$\mu^2_L = \lambda_2 - \nu, \quad \mu^2_S = \lambda_2 + \nu.$$  

(74)

However, (73) entails

$$\phi_L = \frac{\phi_2 - i\phi_3}{2a}, \quad \phi_S = \frac{\phi_2 + i\phi_3}{2b},$$

(75)

giving

$$a^*\phi_L^* = b\phi_S \quad \text{or} \quad a\phi_L = b\phi_S.$$  

(76)

$a\phi_L$ and $b\phi_S$, being charge conjugate, must have the same mass, which is also the mass of $\phi_L$ and $\phi_S$; indeed, one checks explicitly that the contribution of $\rho$ in the quadratic Lagrangian for $\phi_L$ and $\phi_S$ identically vanishes when one makes use of (76), leaving a single state ($\phi_L$ or $\phi_S$) with a mass equal to $\lambda_2 \equiv \lambda_3$. This solution we consequently discard.

- We consider a more general solution to (70). Without loss of generality, we take $a$ to be real. Writing

$$\frac{c}{a} = r = r_1 + ir_2,$$

(77)

(70) yields the two equations

$$r_1(\lambda_2 - \lambda_3) - 2\nu r_1 r_2 = 0,$$
\[ r_2(\lambda_2 - \lambda_3) + \nu(1 + r_2^2 - r_2^2) = 0. \] (78)

The first equation entails that, either \( r_1 = 0 \) or \( r_2 = (\lambda_2 - \lambda_3)/2\nu \). The second option is easily discarded since, plugged into the second equation of (78) it yields \( \lambda_2 = \lambda_3, \nu = 0 \), which is a trivial uninteresting solution.

So the solution of (78) is

\[ r_1 = 0, \quad r_2 = \frac{\lambda_2 - \lambda_3 \pm \sqrt{(\lambda_2 - \lambda_3)^2 + 4\nu^2}}{2\nu}. \] (79)

As, from (77), \( c = iar_2 \) with a real, (67) yields \( b = idr_2, a^2(1 + r_2^2) = 1 = |d|^2(1 + r_2^2) \). Parameterizing \( r_2 = \tan \beta \), one gets \( a^2 = |d|^2 = c_\beta^2 \). \(^{14}\)

The masses become

\begin{align*}
\mu_L^2 &= \lambda_2 c_\beta^2 + \lambda_3 s_\beta^2 - 2\nu s_\beta c_\beta = \frac{\lambda_2 c_\beta^2 - \lambda_3 s_\beta^2}{c_\beta^2 - s_\beta^2}, \\
\mu_S^2 &= \lambda_2 s_\beta^2 + \lambda_3 c_\beta^2 + 2\nu s_\beta c_\beta = \frac{\lambda_3 c_\beta^2 - \lambda_2 s_\beta^2}{c_\beta^2 - s_\beta^2},
\end{align*}

(80)

where we have used the second equation of (78) with \( r_1 = 0 \) to write the last members of the r.h.s. (80) can be inverted for \( \lambda_2 \) and \( \lambda_3 \) exactly like (54) has been inverted into (55) for \( h_3 \) and \( h_4 \).

One has the relations, analogous to (56) and (57):

\[ \frac{\mu_L^2 + \mu_S^2}{\lambda_2 + \lambda_3}, \quad \frac{\mu_L^2 - \mu_S^2}{\lambda_2 - \lambda_3} = \frac{1}{\cos 2\beta}. \] (81)

The eigenstates \( \phi_L \) and \( \phi_S \) write

\[ \begin{align*}
\phi_L &= \frac{1}{\Delta}(d\phi_2 - b\phi_3) = -\frac{1}{\Delta} \frac{b}{c}(\bar{a}\phi_2 + \bar{c}\phi_3) = c_\beta \phi_2 - is_\beta \phi_3, \\
\phi_S &= \frac{1}{\Delta}(-c\phi_2 + a\phi_3) = \frac{c_\beta}{d}(-is_\beta \phi_2 + c_\beta \phi_3).
\end{align*} \] (82)

The equivalent of (58) is

\[ \begin{pmatrix} \bar{\phi}_L \\ \bar{\phi}_S \end{pmatrix} = \begin{pmatrix} \cos 2\beta & i(d/c_\beta) \sin 2\beta \\ i(1 + c_\beta/d) \sin 2\beta & (d/c_\beta) \cos 2\beta \end{pmatrix} \begin{pmatrix} \phi_L \\ \phi_S \end{pmatrix}, \] (83)

which simplifies, for \( d = c_\beta \) into

\[ \begin{pmatrix} \bar{\phi}_L \\ \bar{\phi}_S \end{pmatrix} = \begin{pmatrix} \cos 2\beta & i \sin 2\beta \\ i \sin 2\beta & \cos 2\beta \end{pmatrix} \begin{pmatrix} \phi_L \\ \phi_S \end{pmatrix}, \] (84)

showing again that the ratio of mass splittings (80) cannot diverge, since, for \( \cos 2\beta = 0 \) the states \( \phi_L \) and \( \phi_S \), connected by charge conjugation, must have the same mass.

(82) shows that \( \phi_L \) and \( \phi_S \) are not \( C \) eigenstates; for \( \beta \neq \pm \pi/4 \) they are not related to each other by charge conjugation and their masses may thus be different.

Like in the previous section, two non-degenerate \( \phi_2, \phi_3 \) (supposing \( [\phi_2, \phi_3] = 0 \)) cannot be rotated into degenerate ones. Reciprocally, \( \lambda_2 = \lambda_3 \) entails \( \mu_L^2 = \mu_S^2 \), except when \( \epsilon \to \infty \), where \( \lambda_2 = \lambda_3 \) is compatible with \( \mu_L^2 \neq \mu_S^2 \).

The two basis \( (\bar{\phi}_2, \phi_2, \phi_3, \phi_3) \) and \( (\bar{\phi}_L, \phi_L, \bar{\phi}_S, \phi_S) \) are over-complete; only \( (\phi_2, \phi_3) \) and \( (\phi_L, \phi_S) \) are not.

\(^{14}\) \( c_\beta \) and \( s_\beta \) stand respectively for \( \cos \beta \) and \( \sin \beta \).
4 Commutators; fundamental versus composite scalars

In all examples given, an ambiguity arose from introducing, in the Lagrangian, a term proportional to a $C$-odd commutator of two scalar fields.

We shall investigate here the cases when such a commutator vanishes or not.

4.1 The case of $K^0$ and $\bar{K}^0$

In the simple example of section 2, we assumed that (26) was true, and checked the legitimacy of this statement when $K^0$ and $\bar{K}^0$ are fundamental fields which can be expanded according to (27).

Suppose now that one considers them as composite fields of the type $\bar{q}_i \gamma_5 q_j$, where the $q$’s are fundamental fermions (quarks). Up to a normalization constant, it is natural to take, in agreement with PCAC

$$K^0 = \frac{i}{\rho^2} \bar{d} \gamma_5 s, \quad \bar{K}^0 = \frac{i}{\rho^2} \bar{s} \gamma_5 d,$$

where $\rho$ is a mass scale introduced to restore the correct dimension. By using the standard anticommutation relations of the quark fields \{\(q_i(\vec{x}, t), q_j^\dagger(\vec{x}', t)\)\} = $\delta^3(\vec{x} - \vec{x}')\delta_{ij}$, it is then straightforward to calculate the commutator 15

$$[K^0, \bar{K}^0](x) \equiv [K_L, K_S](x) = \frac{1}{\rho^4}(d^\dagger d - s^\dagger s)(x)\delta^3(\vec{0})$$

and its contribution to the action ($\nu^2 \equiv \sin 2\theta(\mu^2_L - \mu^2_S)$ in (22))

$$\nu^2 \int d^4x [K^0, \bar{K}^0](x) = \frac{\nu^2}{\rho^4}\delta^3(\vec{0}) \int dt \left(N_d(t) - N_s(t)\right)$$

where we have defined the “charge” $N_d(t)$ as

$$N_d(t) = \int d^3x \, d^\dagger(x) d(x) = \int d^3x \, J^0_d(x) \text{ with } J^\mu_d(x) = \bar{d}(x) \gamma^\mu d(x),$$

and a similar expression for $N_s(t)$.

$N_d(t)$ and $N_s(t)$ are not conserved as soon as electroweak interactions are turned on, since they do not conserve the number of $d$ quarks nor the one of $s$ quarks; for example, the so called “penguin” diagrams induce $d \leftrightarrow s$ transitions 16.

So, for composite neutral kaons, the $[K^0, \bar{K}^0]$ commutator is expected not to vanish and to explicitly break $C$ invariance in the Lagrangian (17).

It would instead vanish for a neutral particle and its antiparticle if they are identical, like the neutral pion.

The $\delta^3(\vec{0})$ in (86) (87) originate from anticommuting fermions at the same point in space, and the question of its regularization arises 17. This is however outside the limits of this study since taking the kaons as composite transforms their Lagrangian into a set of (non-renormalizable) 4-fermions operators, which can only get an eventual meaning by introducing an ultraviolet cut-off. What we

---

15 One gets of course the standard result of Current Algebra for the commutator of two charges.
16 This non-conservation also occurs in the decays of the kaons, but then, as already mentioned, their mass matrix is no longer hermitian.
17 Since $\langle 0 | d^\dagger d | 0 \rangle = 0 = \langle 0 | s^\dagger s | 0 \rangle$, the commutator cannot be regularized by just subtracting its vacuum expectation value.
can nevertheless say is that the singularities present in a commutator should be less severe than the ones occurring in other 4-fermions operators. So, if the kinetic terms can be given a signification, *a fortiori* the commutator also can. It then occurs that the “strength” of the perturbation induced by the commutator is controlled by $\int dt \left( N_d(t) - N_s(t) \right)$, which can be in principle very small. We shall come back in section 7 to the fact that a small perturbation can induce large modifications of the mass spectrum.

### 4.2 General case; role of the scalar flavour singlet

Though the case may have looked academic, we saw that the commutator of mesons considered to be fundamental and independent fields vanishes. This is also the case in the standard electroweak model when several fundamental Higgs-like doublets are included. But, as soon as they as considered as composite, the commutator gets non-vanishing contributions from electroweak interactions.

We investigate along this line Higgs-like doublets built as composite quark-antiquark fields like in subsection 3.1 [18][19]. In this case, it turns out that there is one among the $N^2/2$ quadruplets which, as can be easily verified, commutes with all $N^2/4$ other multiplets of the same type and, thus, plays a special role: it is the one called $\Phi_1 = (S^0_1, \vec{P}_1)$ in [18][19], constructed from the unit matrix in flavour space; it includes the scalar flavour singlet $S^0_1 \propto (\bar{u}u + \bar{c}c + \cdots + \bar{d}d + \bar{s}s + \cdots)$ and its three pseudoscalar partners\(^\text{18}\). It satisfies the relations\(^\text{19}\)

\[
\text{For all } k \neq 1, \ [S^0_1, S^0_k] = 0; [P^3_1, P^3_k] = 0; [P^+_1, P^-_k] + [P^-_1, P^+_k] = 0. \tag{89}
\]

The spectrum of the theory can then be ambiguous; in this precise case, it is related to the freedom to add to the electroweak Lagrangian, like in subsection 2.1, an arbitrary mass term proportional to the commutator of $\Phi_1$ with another composite Higgs-like (complex) doublet $\Phi_k, k \neq 1$.

### 5 Fixing the masses of $CP$ eigenstates

As shown below, even when the $(mass)^2$ of the $CP$ eigenstates are fixed to be positive, a negative $(mass)^2$ can arise in a $C (CP)$ violating basis. It is thus natural to investigate this phenomenon in relation with the Higgs mechanism for the breaking of a continuous (gauge) symmetry.

In all this section, the $CP$-odd commutator $C$ defined in subsection 2.3, or its equivalent for Higgs multiplets, is supposed to vanish, such that there is a continuous set of basis, labeled by the value the $CP$-violating parameter, in which the mass matrix can be diagonal.

#### 5.1 The case $T$ conserved, $CPT$ violated

In this subsection, the analysis is performed for a unitary change of basis, like in subsection 2.1 and section 3. The case of a non-unitary transformation like the one studied in subsection 2.2 will be examined in subsection 5.2.

Since (54) and (80) are formally identical, the discussion that we make below for two quadruplets $\phi_3$ and $\phi_4$ with opposite $C$’s also applies to quadruplets with the same $C$ quantum number.

\(^{18}\)It was often chosen in other works of the author as “the” Higgs quadruplet (complex doublet).

\(^{19}\)In the same way, one checks that the $(P^0, \vec{S})$ quadruplet associated with the unit matrix commutes with all $N^2/4$ quadruplets of the $9P^0, \vec{S}$ type.
5.1.1 The spectrum in the basis of $CP$-violating eigenstates

Consider two sets of $C$-eigenstates $J = 0$ fields $\phi_3$ and $\phi_4$, each of them being stable by a continuous (gauge) symmetry group $G$, and such that the quadratic forms $\phi_3^2, \phi_4^2, \phi_3 \phi_4$ and $\phi_4 \phi_3$ (the last two being identical by commutativity) are $G$-invariant.

Suppose for example that their charge conjugates $\overline{\phi}_3$ and $\overline{\phi}_4$ satisfy $\overline{\phi}_3 = \phi_3$ and $\overline{\phi}_4 = -\phi_4$ (an analogous demonstration can be made if $\phi_4$ is $C$-even too).

For each set, a $G$-invariant hermitian mass term can be written, corresponding respectively to the positive (mass)$^2 \mu_3^2$ and $\mu_4^2$, and the Lagrangian $L$ is:

$$ L = \frac{1}{2} \left( \partial_\mu \phi_3^* \partial^\mu \phi_3 + \partial_\mu \phi_4^* \partial^\mu \phi_4 - \mu_3^2 \phi_3^* \phi_3 - \mu_4^2 \phi_4^* \phi_4 \right). $$

In the basis $L$ rewrites

$$ L = L_{kin} - \frac{1}{2} \left( \begin{array}{cc} \phi_L^* & \phi_S^* \end{array} \right) \mathbb{M} \left( \begin{array}{c} \phi_L \\ \phi_S \end{array} \right), $$

with

$$ \mathbb{M} = \left( \begin{array}{cc} c_\theta^2 \mu_3^2 + s_\theta^2 \mu_4^2 & \frac{1}{2} (\mu_3^2 - \mu_4^2) \sin 2\theta \\ \frac{1}{2} (\mu_3^2 - \mu_4^2) \sin 2\theta & c_\theta^2 \mu_4^2 + s_\theta^2 \mu_3^2 \end{array} \right). $$

The multiplets $\phi_L(\theta)$ and $\phi_S(\theta)$ are also stable by $G$ but violate $C$ indirectly.

Consider now $L = L + l_m$ with $l_m$ hermitian and $C$-odd given by (it is the “commutator” term)

$$ l_m = \frac{1}{4} (\mu_3^2 - \mu_4^2) \tan 2\theta (\phi_3^* \phi_4 + \phi_4^* \phi_3); $$

$l_m$ has both effects of canceling the $\phi_L - \phi_S$ transitions of (93) and to shift the diagonal mass terms for $\phi_L$ and $\phi_S$, as is seen when expressing it in the ($\phi_L, \phi_S$) basis

$$ l_m = \frac{1}{4} (\mu_3^2 - \mu_4^2) \tan 2\theta \left( 2s_\theta c_\theta (\phi_L^* \phi_L - \phi_S^* \phi_S) - (c_\theta^2 - s_\theta^2) (\phi_L^* \phi_S + \phi_S^* \phi_L) \right). $$

If one writes the total mass Lagrangian

$$ \mathcal{L}_m = -\frac{1}{2} \left( \phi_3^* \phi_4^* \right) \mathcal{M} \left( \begin{array}{c} \phi_3 \\ \phi_4 \end{array} \right), $$

$l_m$ transforms in particular the total mass matrix from $M$ to $\mathcal{M}$ with

$$ M = \left( \begin{array}{cc} \mu_3^2 & 0 \\ 0 & \mu_4^2 \end{array} \right), \quad \mathcal{M} = M - \frac{1}{2} (\mu_3^2 - \mu_4^2) \tan 2\theta \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right). $$

The eigenvalues of $\mathcal{M}$ are $\mu_L^2$ and $\mu_S^2$ given by (19); they correspond to the eigenvectors (91).
\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi^*_L(\theta) \partial^\mu \phi_L(\theta) + \partial_\mu \phi^*_S(\theta) \partial^\mu \phi_S(\theta) - \mu^2_L \phi^*_L(\theta) \phi_L(\theta) - \mu^2_S \phi^*_S(\theta) \phi_S(\theta) \right). \] (98)

In all cases below, one among \((\mu^2_L, \mu^2_S)\) becomes negative

\[ \text{For } 1 < \tan^2 \theta < \mu^2_3/\mu^2_4, \quad \mu^2_S < 0 \text{ and } \mu^2_L > 0, \]
\[ \text{For } 1 < \tan^2 \theta < \mu^2_3/\mu^2_1, \quad \mu^2_S > 0 \text{ and } \mu^2_L < 0, \]
\[ \text{For } \mu^2_3/\mu^2_1 < \tan^2 \theta < 1, \quad \mu^2_S > 0 \text{ and } \mu^2_L < 0, \]
\[ \text{For } \mu^2_3/\mu^2_1 < \tan^2 \theta < 1, \quad \mu^2_S < 0 \text{ and } \mu^2_L > 0; \] (99)

this is summarized on Fig. 5 (the dashed curve corresponds to \(\mu^2_S\) and the continuous one to \(\mu^2_L\)), and in the formula

\[ \mu^2_L \leq 0 \text{ or } \mu^2_S \leq 0 \iff \inf(\mu^2_3/\mu^2_4, \mu^2_3/\mu^2_1) \leq \tan^2 \theta \leq \sup(\mu^2_3/\mu^2_4, \mu^2_3/\mu^2_1). \] (100)

**Fig. 5: An example: \(\mu^2_L\) and \(\mu^2_S\) as functions of \(\theta\) for \(\mu^2_3 = 2\) and \(\mu^2_4 = 4\)**

![Graph showing \(\mu^2_L\) and \(\mu^2_S\) as functions of \(\theta\).]

It is straightforward to show from (19) that the modulus of the negative \((\text{mass})^2\) is always smaller than the one of the positive \((\text{mass})^2\)

\[ |\mu^2_S| < \mu^2_L. \] (101)

Notice from (99) that a very small value for \(\theta\) \((\tan^2 \theta \ll 1)\) in the phase where \(\mu^2_L < 0\) requires the existence of a very abrupt hierarchy \(\mu^2_3/\mu^2_4 \ll 1\) or \(\mu^2_3/\mu^2_1 \ll 1\). The other extreme case is when the two states are nearly degenerate; then, the domain for which one of the \((\text{mass})^2\) becomes negative concentrates on a very small interval \(\Delta \theta \approx 2\Delta \mu^2/\mu^2\) centered at the singular point(s) \(\theta = \pi/4 + n\pi/2\). For exactly degenerate states, this domain shrinks to 0 and the \(U(1)\) symmetry with angle \(\theta\) evoked in subsection 2.1 stays unbroken too.

In all cases, the critical values \(\theta = \pm \pi/4 + n\pi/2\), equivalent to \(\sin^2 \theta = \cos^2 \theta\) yield singularities which must be studied separately; they correspond to states \(\phi_L(\theta), \phi_S(\theta)\) which transform into each
other by charge conjugation and which, accordingly, must have the same mass by CPT (see section 2 and subsection 3.2). In these cases, we have seen that, indeed, \( \phi_L(\pm \pi/4) \) and \( \phi_S(\pm \pi/4) \) can be made to have identical masses \( \mu^2 = (\mu^2_L(\pm \pi/4) + \mu^2_S(\pm \pi/4))/2 \), but that \( \mu^2_L(\pm \pi/4) - \mu^2_S(\pm \pi/4) \), though its modulus stays bounded, can be arbitrary.

Now, if one uses the identity between charge conjugation and complex conjugation to set \( \phi_3^* = \phi_4 = \phi \) and \( \phi^*_4 = -\phi \), \( l_m \) can also be written (see appendix A.1)

\[
l_m = \frac{1}{4}(\mu^2_3 - \mu^2_4)[\phi_3, \phi_4] \tan 2\theta = \frac{1}{4}(\mu^2_3 - \mu^2_4)[\phi_S, \phi_L] \tan 2\theta = \frac{1}{4}(\mu^2_3 - \mu^2_4)[\phi, \phi] \tan 2\theta,
\]

where we have introduced the independent (sets of) charge conjugate fields \( \phi \) and \( \overline{\phi} \).

The last form of \( l_m \) in (102) involves the equivalent of the \([K^0, K^0]\) commutator of section 2. If this commutator \([\phi_3, \phi_4] = [\phi_L, \phi_S] = 0 \) vanishes, \( L \equiv L - l_m \) is diagonal is the two basis \((\phi_3, \phi_4)\) and \((\phi_L, \phi_S)\). In the \((\phi_L, \phi_S)\) basis, the masses are \((\mu^2_L, \mu^2_S)\).

### 5.1.2 The Higgs mechanism

Let us consider the first line of (98) when \( \mu^2_L < 0 \) from a conservative viewpoint; we forget in particular about the \([\phi_3, \phi_4]\) commutator such that \( \phi_L^* \phi_L = c_3^2 \phi_3^2 + s_3^2 \phi_4^2 \), where \( \phi_4 = i \phi_1 \) is real. To stabilize the theory, let us introduce an additional term \( L_{4L} \) to the Lagrangian \( L \)

\[
L_{4L} = -\frac{\lambda_L}{4}(\phi^*_L(\theta)\phi_L(\theta))^2.
\]

In the “broken” phase,

\[
< \phi_L^*(\theta)\phi_L(\theta) > = \frac{|\mu_L^2|}{\lambda_L} \neq 0, \quad < \phi_S^*(\theta)\phi_S(\theta) > = 0,
\]

such that, writing \( \phi_3 \equiv (S_3^0, \tilde{P_3}) \) and \( \phi_4 \equiv (S_4^0, \tilde{P_4}) \) and imposing \( < \tilde{P_3} > = 0 = < \tilde{P_4} > \), one has \( < S_3^0 > \neq 0 \) and/or \( < S_4^0 > \neq 0 \). The gauge symmetry is spontaneously broken.

The Higgs mass squared is, like in the standard model, twice the modulus of the negative mass squared in the symmetry breaking potential; hence, from (101) one gets for it an absolute upper bound which is twice the mass squared of the heaviest \( J = 0 \) (scalar or pseudoscalar) composite mesons which make up the other Higgs-like multiplets:

\[
M^2_H(\theta) = \sup(M^2_{S,P}).
\]

\( M^2_H \) is given by

\[
M^2_H(\theta) = 2|\mu^2_L(\theta)| = 2|\mu^2_4 - \mu^2_3| \left| \frac{\tan^2 \theta - \tan^2 \theta_c}{(1 - \tan^2 \theta)(1 - \tan^2 \theta_c)} \right| = 2(\mu^2_3 + \mu^2_4) \frac{\tan^2 \theta - \tan^2 \theta_c}{(1 - \tan^2 \theta)(1 + \tan^2 \theta_c)}
\]

where

\[
\tan^2 \theta_c = \frac{\mu^2_3}{\mu^2_4}.
\]
We have made $\mu_3^2 + \mu_4^2$ appear in (107) because it is invariant by the change of basis (see (20)).

For a simple potential like above, the condition $\langle \phi_S(\theta) \rangle = 0$ entails $\langle \phi_L(\theta) \rangle = \langle \phi_3 \rangle / \cos \theta$.

$\phi_L(\theta)$ and $\phi_S(\theta)$ correspond to the so-called Georgi’s basis [20]. The Higgs boson is the neutral component of the set $\phi_L(\theta)$ which gets a non-vanishing vacuum expectation value; it can be a $P$ eigenstate but it is never a $C \ (CP)$ eigenstate, nor are the three Goldstones which are eaten by the gauge bosons to become massive $^{20}$.

In the case when $\mu_3^2 < 0$, $M_H^2$ is given by (107) after changing $\tan^2 \theta$ into $1 / \tan^2 \theta$.

### 5.2 The case $T$ violated, $CPT$ conserved

We perform the same study as in section 5.1 in the case of the non-unitary transformation of subsection 2.2.

#### 5.2.1 The spectrum of states

We consider the same Lagrangian $L$ as in (90).

In the complex basis $(\phi_L, \phi_S)$

$$
\begin{align*}
\phi_L(\epsilon) &= \frac{1}{\sqrt{\cosh^2 \epsilon + \sinh^2 \epsilon}} (\phi_3 \cosh \epsilon + \phi_4 \sinh \epsilon), \\
\phi_S(\epsilon) &= \frac{1}{\sqrt{\cosh^2 \epsilon + \sinh^2 \epsilon}} (\phi_3 \sinh \epsilon + \phi_4 \cosh \epsilon).
\end{align*}
$$

(109)

it writes

$$
L = \frac{1}{2} \cosh^2 2\epsilon \left( \begin{array}{cc} 1 & - \tanh 2\epsilon \\ - \tanh 2\epsilon & 1 \end{array} \right) \left( \begin{array}{c} \phi_L^* \\
\phi_S^* \end{array} \right) \left( \begin{array}{c} \phi_L \\
\phi_S \end{array} \right)
$$

(110)

with

$$
K = p^2 \left( \begin{array}{cc} 1 & - \tanh 2\epsilon \\ - \tanh 2\epsilon & 1 \end{array} \right)
$$

(111)

and

$$
M = \left( \begin{array}{cc}
\frac{\mu_3^2 \cosh^2 \epsilon + \mu_4^2 \sinh^2 \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon} & - \frac{\mu_4^2 + \mu_3^2}{2 \tanh 2\epsilon} \\
\frac{\mu_3^2 \cosh^2 \epsilon + \mu_4^2 \sinh^2 \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon} & \frac{\mu_4^2 \cosh^2 \epsilon + \mu_3^2 \sinh^2 \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon}
\end{array} \right)
$$

(112)

If we add to it $l_m$ and $l_\chi$ given by

$$
\begin{align*}
l_m &= - \frac{1}{4} (\mu_3^2 + \mu_4^2) \tanh 2\epsilon (\phi_3^* \phi_4 + \phi_4^* \phi_3), \\
l_\chi &= \frac{1}{2} \tanh 2\epsilon (\partial_\mu \phi_3^* \partial^\mu \phi_4 + \partial_\mu \phi_4^* \partial^\mu \phi_3),
\end{align*}
$$

(113)

to reconstruct $\mathcal{L} = L + l_m + l_\chi$ which is the analog of (35), $\mathcal{L}$ can be diagonalized in the complex basis $(\phi_L, \phi_S)$, and rewrites

$$
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi_L^* (\epsilon) \partial^\mu \phi_L (\epsilon) + \partial_\mu \phi_S^* (\epsilon) \partial^\mu \phi_S (\epsilon) - \mu_3^2 \phi_L^* (\epsilon) \phi_L (\epsilon) - \mu_4^2 \phi_S^* (\epsilon) \phi_S (\epsilon) \right).
$$

(114)

$^{20}$Consequently, the $C$ properties of the massive gauge bosons could be suspected in this framework to be more subtle than usually considered (see also [21]).
\( \mu_L^2 \) and \( \mu_S^2 \) are given by (37).

The effects of \( l_m \) and \( l_\chi \) are conspicuous when they are rewritten in the \((\phi_L, \phi_S)\) basis (see appendix A.2).

\[
\begin{align*}
l_m &= -\frac{1}{4}(\mu_3^2 + \mu_4^2) \sinh 2\epsilon \\
&\quad \times \left( -2 \sinh \epsilon \cosh \epsilon (\phi_L^* \phi_L + \phi_S^* \phi_S) + (\cosh^2 \epsilon + \sinh^2 \epsilon) (\phi_L^* \phi_S + \phi_S^* \phi_L) \right), \\
l_\chi &= \frac{1}{2} \sinh 2\epsilon \\
&\quad \times \left( -2 \sinh \epsilon \cosh \epsilon (\partial_\mu \phi_L^* \partial^\mu \phi_L + \partial_\mu \phi_S^* \partial^\mu \phi_S) + (\cosh^2 \epsilon + \sinh^2 \epsilon) (\partial_\mu \phi_L^* \partial^\mu \phi_S + \partial_\mu \phi_S^* \partial^\mu \phi_L) \right).
\end{align*}
\] (115)

Again, if one uses the identity between complex conjugation and charge conjugation, and the \( C \) properties of \( \phi_3 \) and \( \phi_4 \), \( l_m \) and \( l_\chi \) become proportional to commutators \( [\phi, \overline{\phi}] \) and \( [\partial_\mu \phi, \partial^\mu \overline{\phi}] \), where the independent charge conjugate fields \( \phi \) and \( \overline{\phi} \) are the equivalent of (103) which are likely to vanish.

If this is so, \( L \equiv L - l_m - l_\chi \) can be diagonalized in the two basis \((\phi_3, \phi_4)\) and \((\phi_L, \phi_S)\). In the \((\phi_L, \phi_S)\) basis, the masses are \( \mu_L^2 \) and \( \mu_S^2 \).

On Fig. 6 are displayed \( \mu_L^2 \) and \( \mu_S^2 \) as functions of \( \epsilon \) for \( \mu_3^2 = 2 \) and \( \mu_4^2 = 4 \).

**Fig. 6:** \( \mu_L^2 \) and \( \mu_S^2 \) as functions of \( \epsilon \) for \( \mu_3^2 = 2 \) and \( \mu_4^2 = 4 \)

\( \mu_L^2 \) becomes negative for \( \tanh^2 \epsilon > \tanh^2 \epsilon_c \) with

\[
\tanh^2 \epsilon_c = \frac{\mu_3^2}{\mu_4^2}.
\] (116)

### 5.2.2 The Higgs mechanism

For \( \mu_L^2 < 0 \), we are led to introduce, like previously, a “stabilizing” potential given by (104). One still use the notation \( \phi_4 = i\phi_4 \).
The Higgs mass is then $M_H^2 = 2|\mu_L^2|$ and writes

$$M_H^2(\epsilon) = 2(\mu_3^2 + \mu_4^2) \cosh^2 \epsilon \left| \frac{\tanh^2 \epsilon - \tanh^2 \epsilon_c}{1 + \tanh^2 \epsilon_c} \right|. \quad (117)$$

$\mu_3^2 + \mu_4^2$ is again invariant by the change of basis (see (38)).

6 Mass hierarchies

We have paid attention to the ratios between mass splittings in the different basis which can diagonalize the Lagrangian when the commutator $\mathcal{C}$ vanishes.

Also of interest are the hierarchies of masses $\mu_4^2 / \mu_3^2$ for fixed $\mu_L^2$ and $\mu_S^2$ (Figs. 7,8), and, inversely, the ratio $\mu_S^2 / \mu_L^2$ for fixed $\mu_3^2$ and $\mu_4^2$ (Figs. 9,10). In each of the two sets of figures, the first corresponds to a unitary change of basis and the second to a non-unitary transformation.

The hierarchy of masses is highly dependent on the basis, or more precisely of the parameters $\theta$ or $\epsilon$ measuring indirect CP violation. Huge mass hierarchies (between states which are not CP eigenstates) can be brought back to small ones (by going to another basis of CP eigenstates), or vice-versa.

Fig. 7: unitary change of basis; $\mu_4^2 / \mu_3^2$ as a function of $\theta$ for $\mu_S^2 = 4$ and $\mu_L^2 = 2$
Fig. 8: non-unitary transformation; $\mu_4^2/\mu_3^2$ as a function of $\epsilon$ for $\mu_S^2 = 4$ and $\mu_L^2 = 2$.

Fig. 9: unitary change of basis; $\mu_3^2/\mu_4^2$ as a function of $\theta$ for $\mu_S^2 = 2$ and $\mu_L^2 = 4$. 
7 Ambiguous or observable masses

The ambiguity that eventually arises in the mass spectrum of commuting fields has been connected to the existence of a symmetry; for example, in subsection 2.1, we mentioned the role of the $U(1)$ symmetry with angle $\theta$. In general, no observable can ever be associated with an unbroken symmetry; in the case at hand, due to the freedom to enlarge their mass matrix, the masses of commuting states do not correspond to observable quantities. A perturbation that breaks the symmetry in question is needed to lift this ambiguity.

Suppose indeed that one turns on electroweak interactions. The commutator $[\phi, \bar{\phi}]$ is then likely not to vanish; the Lagrangian $\mathcal{L} = L + l_m$ (98) can no longer be diagonalized in a continuous set of basis, but only in the $CP$-violating basis $(\phi_L, \phi_S)$; $\mu^2_L$, $\mu^2_S$ and $\theta$ become observable. The perturbation $l_m$ (94) changes the mass spectrum from $(\mu^2_3, \mu^2_4)$ to $(\mu^2_L, \mu^2_S)$, and, in particular, alters the hierarchy pattern.

A noticeable point is that the spectrum is independent of the precise value of the commutator. So, by tuning it, one can consider the possibility that the perturbation $l_m$ (94) can be made very small. Since it is, as we stressed in section 4, sensitive to electroweak interactions, this can be achieved, presumably, by setting their coupling constant to small enough values. If, in this process, the $CP$ violating parameter keeps to high enough values, a small perturbation is likely to induce large modifications in the mass spectrum.

The question is thus whether and how the $CP$-violating parameter(s) depends on the strength of electroweak interactions. When computed in the Standard Model (see for example [2], p. 104 and 108)

---

21 A typical example is colour.
7.1.1 Unitary change of basis

It shows that splitting the states by the term of the CKM matrix. While the Cabibbo angle is indeed expected to go to zero when the electroweak parameter(s) are expected to differ from the customary small values observed in systems like neutral $l_v$-violating parameter is large enough such that, at the same time, $l_m$ stays a “very small” perturbation, and its effects on the mass spectrum are large, possibly even inducing “spontaneous symmetry breaking”. If large hierarchies are observed in a “physical” basis of $CP$-violating states, we suggest that they can be “slightly” perturbed $CP$ eigenstates for which, in the absence of perturbation, the hierarchies are much smaller or ever equal to 1. Accordingly, for such particles, the $CP$-violating parameter(s) are expected to differ from the customary small values observed in systems like neutral kaons.

7.1 CPT constraint

There is a case where the spectrum is constrained (by CPT) to be unambiguous: the one corresponding to a pair of neutral charge-conjugate commuting states. This corresponds to maximal indirect $CP$-violation ($\theta = \pi / 4 + n \pi$ or $\epsilon = \pm \infty$). I show below how this constraint is recovered.

7.1.1 Unitary change of basis

Suppose that, owing to the independence and the charge conjugation properties of $\phi_3$ and $\phi_4$, $\phi_3^* \phi_4 + \phi_4^* \phi_3 = [\phi_3, \phi_4] = 0$; in the ($\phi_L, \phi_S$) basis this rewrites (see (124), (125))

$$\sin 2 \theta (\phi_L^* \phi_L - \phi_S^* \phi_S) = \cos 2 \theta (\phi_S^* \phi_L + \phi_L^* \phi_S) = |\phi_S, \phi_L| = 0.$$

This last identity can also be checked with the help of (25).

It shows that splitting the states by the term ($\phi_L^* \phi_L - \phi_S^* \phi_S$) is equivalent to making them oscillate by ($\phi_L^* \phi_L + \phi_S^* \phi_L$), with a proportionality factor $1 / \tan 2 \theta$:

$$\phi_L^* \phi_L - \phi_S^* \phi_S = \frac{1}{\tan 2 \theta} (\phi_L^* \phi_S + \phi_S^* \phi_L).$$

The r.h.s. of (118) goes to 0 when $\theta \to \pm \pi / 4$. So, at this limit, which is the one where $\phi_L = \pm \phi$ and $\phi_S = \pm \phi$ are conjugate – see (103) –, one can write, fixing $\mu_L^2$ and $\mu_S^2$ to finite values (eventually negative)

$$\mu_L^2 \phi_L^* \phi_L + \mu_S^2 \phi_S^* \phi_S = \mu_L^2 \phi_L^* \phi_L + \mu_S^2 \phi_S^* \phi_S - \mu_L^2 (\phi_L^* \phi_L - \phi_S^* \phi_S) + \mu_L^2 (\phi_L^* \phi_L - \phi_S^* \phi_S) = (\mu_L^2 + \mu_S^2) \phi_L^* \phi_S + \frac{\mu_L^2}{\tan 2 \theta} (\phi_L^* \phi_S + \phi_S^* \phi_L) \rightarrow (\mu_L^2 + \mu_S^2) \phi_L^* \phi_S$$

or

$$\mu_L^2 \phi_L^* \phi_L + \mu_S^2 \phi_S^* \phi_S = \mu_L^2 \phi_L^* \phi_L + \mu_S^2 \phi_S^* \phi_S + \mu_S^2 (\phi_L^* \phi_L - \phi_S^* \phi_S) - \mu_S^2 (\phi_L^* \phi_L - \phi_S^* \phi_S) = (\mu_L^2 + \mu_S^2) \phi_L^* \phi_L - \frac{\mu_S^2}{\tan 2 \theta} (\phi_L^* \phi_S + \phi_S^* \phi_L) \rightarrow (\mu_L^2 + \mu_S^2) \phi_L^* \phi_L.$$

\[\text{(119)}\]
For independent, commuting $\phi$ and $\overline{\phi}$ such that $\overline{\phi} = \phi^*$, (119) is nothing more than the trivial identity

$$\mu_L^2 \overline{\phi} \phi + \mu_S^2 \phi^* \phi = (\mu_L^2 + \mu_S^2) \phi^* \phi = (\mu_L^2 + \mu_S^2) \phi \phi^* = \frac{\mu_L^2 + \mu_S^2}{2} (\phi^* \phi + \phi \phi^*).$$

(120)

On the r.h.s. of (119) only one among the two fields $\phi_L$, $\phi_S$ remains, and it has a mass $(\mu_S^2 + \mu_L^2)/2$ (which can be positive even if $\mu_L^2$ was negative); the same occurs for the kinetic terms, since $\phi^*_L \phi_L + \phi^*_S \phi_S \rightarrow 2 \phi^*_S \phi_S$ or $\phi^*_L \phi_L + \phi^*_S \phi_S \rightarrow 2 \phi^*_L \phi_L$. So, at this limit, as expected, $\phi$ and $\overline{\phi}$ are degenerate and their mass is the one written above.

### 7.1.2 Non-unitary transformation

When $[\phi_3, \phi_4] = 0$ and since $\cosh 2\epsilon$ never vanishes, (129) entails

$$\phi^*_L \phi_L + \phi^*_S \phi_S = \frac{1}{\tanh 2\epsilon}(\phi^*_L \phi_S + \phi^*_S \phi_L).$$

(121)

Unlike in the case of a unitary change of basis, the r.h.s. of (121) never vanishes, such that, now, shifting $\mu_L^2$ and $\mu_S^2$ always goes with a “rotation” of the states.

The rest of the discussion, including now the non-diagonal kinetic term, follows the same lines as above.

### 7.2 The special case of the flavour singlet

The case of the flavour singlet stays a special one since it always commutes with other $J = 0$ mesons: either as a fundamental field, which commutes with other independent similar fields, or, as shown in section 4, as a composite quark-antiquark field.

So, for a mixture of two multiplets of the same type with definite $C$, the first including the scalar flavour singlet, turning on electroweak interactions is likely not to remove an ambiguity in the mass spectrum, such that the equivalent of $\mu_L^2$ and $\mu_S^2$ keep undetermined; this can be in particular the case of the Higgs boson mass, $M_H^2 = 2|\mu_{\phi_0}|^2$, which stays, in this framework, an unobservable quantity.

Are there ways to lift this ambiguity? A possibility is that the states carry quantum numbers other than electroweak, associated with interactions which are mis-aligned with the former (for example flavour-diagonal “strong” interactions); if a detector, being mostly sensitive to these interactions, signs the corresponding eigenstates, the process of measure can determine which linear combinations of electroweak $CP$ eigenstates are detected: this eventually fixes an orientation in the space that they span, can determine the $CP$-violating parameters, and select, among all possibilities, a precise mass pattern. The latter can, accordingly, depend on the quantum numbers that are detected.

### 7.3 Another possible attitude

One should not put arbitrarily aside the other reasonable attitude which simply refutes the existence of an ambiguity. Then, that the mass spectrum of a commuting pair be uniquely defined constrains the

---

24 This can however be considered unrealistic as soon as the Higgs boson is expected to decay.

25 Here, the process of detection is considered to be similar to the production mechanism. For example, charged kaons, which are considered to be flavour eigenstates, are commonly produced by strong interactions, which are flavour-diagonal.

26 It is not at all guaranteed that the sole freedom in the mass matrix studied in this work is enough to switch from the basis of electroweak $CP$ eigenstates to the basis of outgoing states that are detected through the other type of interactions.
A weaker constraint comes from only considering negative \((\text{mass})^2\) as nonphysical; this forbids certain ranges of values for the \(CP\)-violating parameter. For example, in the case when \(CP\) is violated while \(CPT\) is preserved, this yields an upper bound for \(|\epsilon|\).

8 Conclusion. Perspective

We have studied \(CP\) violation for the neutral kaon system and for electroweak Higgs-like doublets, emphasizing their analogy, and extended it to all possible values of the \(CP\) violating parameter.

Our attention was drawn to ambiguities that arise in the spectrum of states when the \(CP\)-odd commutator \([K^0, \bar{K}^0]\) vanishes. The mass spectrum turns out to heavily depend on the basis and on the \(CP\)-violating pattern attached to it.

This can look an academic problem since, in the real world and, in particular, when electroweak interactions are turned on, such a commutator is not expected to vanish. However, adding to a Lagrangian, which is diagonal in a basis of \(CP\) eigenstates, a term \(\kappa^2\mathcal{C}\), where \(\kappa^2\) is a function of the masses and of the \(CP\)-violating parameter, alters the starting hierarchy in a way that does not depend on the precise value of \(\mathcal{C}\). In this framework, for small enough \(\mathcal{C}\), a small perturbation is not excluded to trigger large hierarchies.

In particular, for certain ranges of values of the \(CP\)-violating parameter, a negative \((\text{mass})^2\) can occur in the basis of \(CP\) violating states, and the theory becomes unstable.

We saw that there exist cases when the commutator always vanishes, which maintains an ambiguity. The Higgs boson might fall into this framework. This needs a special investigation.

This work also suggests that discrete symmetries have to be handled with care when reducing the number of degrees of freedom or constraining the couplings in the Lagrangian. Some effects can be overlooked which play, in particular, a role in determining, even at the classical level, the vacuum structure of the theory.

Our point of view has been different from other studies in that we did not investigate the origin of indirect \(CP\)-violation. In particular, we paid no special attention to the potential that is introduced in the Lagrangian.

This simple study concerns the case where only two particles or multiplets are “rotated”. Since one is free to perform a change of basis for any pair among the set of Higgs-like doublets, various hierarchies can be expected among these multiplets (see also section 6). This is left for a subsequent work [23].
A Dependent versus independent states

All relations written below for the kaon fields $\phi$ of section 2 are also true for the Higgs-like multiplets $\phi$ of the next sections.

All relations written with charge conjugate fields are also valid with their complex conjugates.

A.1 The case of a unitary change of basis

We come back to subsection 2.1 and examine some combinations of fields relevant for writing the kinetic terms. From the definition (3), it is trivial to calculate,

$$\varphi^2_3 - \varphi^2_4 = (c^2_\theta - s^2_\theta)(\varphi^2_L - \varphi^2_S) + 2s_\theta c_\theta (\varphi_L \varphi_S + \varphi_S \varphi_L),$$

(122)

and

$$\overline{\varphi^3_3 \varphi_3} + \overline{\varphi^4_4 \varphi_4} = \overline{\varphi^3_3 \varphi_3} + \overline{\varphi^4_4 \varphi_4}. \quad (123)$$

The l.h.s’s of (122) and (123) are identical if one uses the $C$ conjugation properties $\overline{\varphi_3} = \varphi_3$ and $\overline{\varphi_4} = -\varphi_4$; the two r.h.s. are also identical if one uses the relations (25) between $(\varphi_S, \overline{\varphi_S}, \varphi_L, \overline{\varphi_L})$.

The only difference is that (122) is written with independent fields, while (123) is written with the two over-complete sets $(\varphi_3, \varphi_3, \varphi_4, \varphi_4)$ and $(\varphi_S, \overline{\varphi_S}, \varphi_L, \overline{\varphi_L})$.

The form (123), in which hermiticity is manifest, is the one that we used, in particular, to write the kinetic terms.

One has also

$$\overline{\varphi^3_3 \varphi_4} + \overline{\varphi^4_4 \varphi_4} = 2s_\theta c_\theta (\overline{\varphi^3_3 \varphi_3} - \overline{\varphi^4_4 \varphi_4}) - (c^2_\theta - s^2_\theta)(\overline{\varphi^3_3 \varphi_3} + \overline{\varphi^4_4 \varphi_4}),$$

(124)

and

$$[\varphi_3, \varphi_4] = [\varphi_S, \varphi_L]. \quad (125)$$

A.2 The case of a non-unitary transformation

In the same way, for subsection 2.2, one has the relations

$$\varphi^2_3 - \varphi^2_4 = (\cosh^2 \epsilon + \sinh^2 \epsilon)(K^2_L - K^2_S),$$

(126)

and

$$\overline{\varphi^3_3 \varphi_3} + \overline{\varphi^4_4 \varphi_4} = (\cosh^2 \epsilon + \sinh^2 \epsilon) \left( (\cosh^2 \epsilon + \sinh^2 \epsilon)(K^2_L K^2_S + K^2_S K^2_L) - 2 \sinh \epsilon \cosh \epsilon (K^2_L K^2_S + K^2_S K^2_L) \right),$$

(127)

and

$$K^2_L K^2_S + K^2_S K^2_L = (\overline{\varphi^3_3 \varphi_3} + \overline{\varphi^4_4 \varphi_4}) + \frac{2 \sinh \epsilon \cosh \epsilon}{\cosh^2 \epsilon + \sinh^2 \epsilon} (\overline{\varphi^3_3 \varphi_4} + \overline{\varphi^4_4 \varphi_3}),$$

(128)

One has also
\[
\varphi_3 \varphi_4 + \varphi_4 \varphi_3 = (\cosh^2 \epsilon + \sinh^2 \epsilon) \left( -2 \sinh \epsilon \cosh \epsilon (K_L K_L + K_S K_S) \right) \\
+ (\cosh^2 \epsilon + \sinh^2 \epsilon) (K_L K_S + K_S K_L),
\]

(129)

and

\[
[\varphi_3, \varphi_4] = (\sinh^2 \epsilon + \cosh^2 \epsilon) [K_L(\epsilon), K_S(\epsilon)].
\]

(130)

This shows again that the choice of the basis is important and that working with non-independent states is ambiguous.
References


A. SALAM: “Weak and electromagnetic interactions”, in “Elementary Particle Theory: Relativistic Groups and Analyticity” (Nobel symposium No 8), edited by N. Svartholm (Almquist and Wiksell, Stockholm 1968);


\( \mu_L^2 = 2, \mu_S^2 = 4 \)
$\mu_3^2 = 2, \mu_4^2 = 4$
\[ \mu L^2 = 2, \mu S^2 = 4 \]
\( \mu L^2 = 2, \mu S^2 = 4 \)
\( \mu L^2 = 2, \mu S^2 = 4 \)
$\mu_1^{**2} = 2, \mu_2^{**2} = 4$
\[ \mu_3^{**2} = 2, \mu_4^{**2} = 4 \]
\[ \text{muL}^2 = 2, \text{muS}^2 = 4 \]
\( \mu L^2 = 2, \mu S^2 = 4 \)
$\mu_1^{**2} = 2, \mu_2^{**2} = 4$