\[ \left( \frac{\lambda}{N} \right) \frac{N^d}{d} = \langle p \rangle N_d \]

The protocol \( \{ p \} \rangle N_d \) of finding a quantum random walk is to implement a large number of steps in the walk. However, if we consider for large dimensions it will not be possible to implement a large number of steps in the walk. Therefore, it is necessary to implement the quantum random walk on a single qubit. The quantum random walk on a single qubit can be implemented using a quantum random walk.

The idea of a computational device based on the quantum random walk is to implement a large number of steps in the walk. However, if we consider for large dimensions it will not be possible to implement a large number of steps in the walk. Therefore, it is necessary to implement the quantum random walk on a single qubit. The quantum random walk on a single qubit can be implemented using a quantum random walk.

The idea of a computational device based on the quantum random walk is to implement a large number of steps in the walk. However, if we consider for large dimensions it will not be possible to implement a large number of steps in the walk. Therefore, it is necessary to implement the quantum random walk on a single qubit. The quantum random walk on a single qubit can be implemented using a quantum random walk.

The idea of a computational device based on the quantum random walk is to implement a large number of steps in the walk. However, if we consider for large dimensions it will not be possible to implement a large number of steps in the walk. Therefore, it is necessary to implement the quantum random walk on a single qubit. The quantum random walk on a single qubit can be implemented using a quantum random walk.

The idea of a computational device based on the quantum random walk is to implement a large number of steps in the walk. However, if we consider for large dimensions it will not be possible to implement a large number of steps in the walk. Therefore, it is necessary to implement the quantum random walk on a single qubit. The quantum random walk on a single qubit can be implemented using a quantum random walk.
Table I contains the probabilities for the first few values of $N$. The non-zero elements of the distribution are simply terms from Pascal’s triangle, divided by the appropriate factor of two. There are two features of this random walk that we would like to compare to the quantum analogue. Firstly, the mean of the walk is zero. This is intuitively obvious, we are using a fair coin, so we are as likely to step left as we are to step right. The other property of the distribution that we are interested in is the standard deviation. It is not hard to calculate that the standard deviation of this distribution, $\sigma_c$, is given by

$$\sigma_c = \sqrt{N}. \tag{2}$$

B. Quantum walk on a line

Now let us consider a quantum version of the walk on a line. The first modification we can make is to replace the coin with a qubit. In this paper, we shall be representing the two levels of the qubit with the states $|\downarrow\rangle$ and $|\uparrow\rangle$ rather than $|0\rangle$ and $|1\rangle$. If we start with the qubit in the down state, and apply a Hadamard operation, we get an equal superposition of up and down,

$$H |\downarrow\rangle = \frac{1}{\sqrt{2}} |\downarrow\rangle + \frac{1}{\sqrt{2}} |\uparrow\rangle, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3}$$

If we were to measure the qubit, and step left or right depending upon the result, we would obtain exactly the classical walk described above. Now, rather than a person holding a coin, suppose we have a particle, whose motion is confined to one dimension. We can now treat the particle as a quantum system, and perform the quantum walk as follows. During each iteration, we apply the Hadamard operation, followed by the operation which steps right if the qubit is down, and steps left if the qubit is up. That is, we apply the operator,

$$U = e^{i \hat{\sigma}\hat{p}}, \quad \hat{H}, \tag{4}$$

where $\hat{p}$ is the momentum operator of the particle confined to one dimension, and $\hat{\sigma}_z$ is the pauli-$z$ operator acting on the qubit. Therefore, the state of the system after $N$ steps is

$$|\Psi_N\rangle = (e^{i \hat{\sigma}\hat{p}})^N |\Psi_0\rangle, \tag{5}$$

where $|\Psi_0\rangle$ is the initial state of the system. The mean of the distribution produced by this quantum random walk is not necessarily zero. It is dependent upon the initial state of the qubit. For example, choosing the initial state of the qubit to be down gives a non-zero mean after the second step. For the remainder of this paper, we shall only be considering the distribution created with the initial qubit state $\frac{1}{\sqrt{2}} |\downarrow\rangle + \frac{1}{\sqrt{2}} |\uparrow\rangle$ which has a mean of zero for all values of $N$.

$$|\Psi_N\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle + |\uparrow\rangle). \tag{6}$$

Table II contains the probability distribution associated with the first few states $|\Psi_N\rangle$. The non-zero elements of the distribution are no longer simply terms from Pascal’s triangle which arose in the classical case. The deviations from the classical distribution are caused by quantum interference effects. Now it is no longer simple to calculate the standard deviation of the distribution. However, numerical simulations reveal that the standard deviation, $\sigma_q$, is almost independent of the initial state of the qubit, and is approximately linear in $N$,

$$\sigma_q \approx \frac{3}{5} N. \tag{7}$$

The standard deviation is plotted in Fig. 1 up to $N = 40$ for both the classical and quantum walk distributions.
Clearly, the standard deviation is significantly different for the quantum and classical random walks on a line. Now let’s consider the random walks which arise when periodic boundary conditions are applied to the random walks.

C. Classical walk on a circle

In the paper by Aharonov et al. [9], they consider random walks on the circle, where the step size is an irrational multiple of $\pi$. Here, we shall only be considering the simple distribution which arises when the step size is taken to be $\pi/2$. Let us assume that the particle is initially found, with probability one, at some point on a circle denoted by $\theta = 0$,

$$P_0(\theta = 0) = 1.$$  \hspace{1cm} (8)

After one step of the algorithm, the classical distribution is given by

$$P_1(\theta) = \begin{cases} 0 : & \theta = 0, \pi \\ \frac{1}{\pi} : & \theta = \pm \frac{\pi}{2} \end{cases},$$ \hspace{1cm} (9)

and after the second step,

$$P_2(\theta) = \begin{cases} \frac{1}{\pi} : & \theta = 0, \pi \\ 0 : & \theta = \pm \frac{\pi}{2} \end{cases}.$$ \hspace{1cm} (10)

It is not difficult to see that the probability distribution for all subsequent odd number of steps will be given by Eq. (9), and the distribution for all subsequent even number of steps will be given by Eq. (10).

D. Quantum walk on a circle

Let us consider the quantum random walk on a circle. Once again, we start with the particle at some point on a circle denoted by $\theta = 0$, thus the initial probability distribution is given by Eq. (8). The probability distributions after one and two steps are also given by Eqs. (9) and (10) respectively, however after the third step, interference effects results in the distribution

$$P_3(\theta = \frac{\pi}{2}) = 1.$$ \hspace{1cm} (11)

Calculation of the states after subsequent steps reveals that the quantum random walk around the circle, with a step size of $\pi/2$ is periodic with a period of eight. The eight probability distributions which are given in Table III.

III. IMPLEMENTING THE WALKS IN AN ION TRAP

The analysis thus far has assumed that all operations can be applied without error and the particle can exist in position eigenstates. Now we shall relax these assumptions, and describe how the algorithm can be implemented in an ion trap.

The ion trap provides a convenient setting for the quantum random walks we have described, as it contains the required discrete and continuous quantum variables. For the remainder of this paper we shall be discussing implementations based on a single $^{87}$Rb$^+$ ion, confined in a co-axial-resonator radio frequency (RF)-ion trap, as described in [13] and references therein.

The preparation involves laser-cooling the ion to the motional and electronic ground state, $|0\rangle\!\langle 0|$, as described in [14]. A sequence of four Raman beam pulses are then applied [13] to create the state $(|0\rangle\!\langle 1| + |1\rangle\!\langle 0|)/\sqrt{2}$, where |0\rangle denotes the coherent state of the trap oscillators,

$$|\alpha\rangle = \frac{e^{2\alpha} \pi^{-\frac{1}{4}}}{\sqrt{\pi}} \int dx e^{\pi \alpha x^2} \left[ e^{-\frac{1}{2}(x - \sqrt{\pi} \alpha)^2} \right] |x\rangle$$ \hspace{1cm} (12)

and $\alpha \equiv a_R + ia_I$. The first pulse is a $\pi/2$-pulse which creates an equal superposition of $|0\rangle\!\langle 1|$ and $|1\rangle\!\langle 0|$. A displacement beam is then applied which excites the motion correlated to the $|1\rangle\!\langle 1|$ internal state. The third pulse is a $\pi$-pulse which exchanges the internal states, and finally the displacement beam is applied again. The combined action of the four pulses is to effectively perform the operator $U$, defined in Eq. (4). The quantum random walk on the line is accomplished by repeating this sequence of pulses $N$ times. Fig. 2 contains the Wigner function obtained by tracing over the internal degree of freedom after five steps of the quantum random walk algorithm.

The quantum random walk on the circle can be implemented in an ion trap by 'walking' the particle around a circle in phase space, rather than a circle in real space. In order to accomplish this task, we need to generate an operator of the form

$$W = e^{i\pi \frac{a^2}{2} / \hat{a}^2} \hat{H}.$$ \hspace{1cm} (13)

where $a$ and $a^\dagger$ correspond to the annihilation and creation operators of the harmonic oscillator. This operator can be produced in an ion trap by applying far-detuned laser pulses to the ion [13], followed by a $\pi/2$-pulse.
IV. MEASURING THE WALKS

Using current ion trap technologies, wave packet dispersion is negligible [13], so the main source of decoherence is related to the internal levels of the ion. Decoherence of the electronic levels of the ion during the application of the algorithm has the effect of gradually transforming the quantum random walk to the classical random walk. Rather than considering this to be a negative effect, we can measure the degree to which the ion is acting as a quantum variable rather than a classical variable, and thereby effectively measure the level of decoherence in the ion trap.

The scheme that we envisage for measuring the random walk utilises similar operators to those employed in the application of the algorithm. After applying the random walk sequence for some number of steps, the internal state of the ion is decoupled from the motional state by an appropriate Raman pulse. An effective operator such as $\exp(i\sigma_y)$ is applied, before finally measuring the internal state of the ion. Thus we are using the internal state of the ion to supply as with information about the motional state.

In the case of the walk on the line, suppose we decouple the internal state from the motional state by measuring whether the ion is in the state $|\uparrow\rangle$ or $|\downarrow\rangle$. We then apply the operator

$$M^\pm = e^{\pm i\sigma_y}.$$  \hspace{1cm} (14)

The positive Hamiltonian is applied upon obtaining the results $|\uparrow\rangle$, whilst the negative Hamiltonian is applied otherwise. Finally, we again measure the internal state of the ion. If the quantum random walk has experienced no decoherence, then we measure $|\downarrow\rangle$ with the probabilities given by the solid line in Fig. 3, whereas if the ion suffers complete decoherence we would expect to measure $|\downarrow\rangle$ with probability of one half.

A similar scheme can be used to measure the level of decoherence in the quantum random walk on the circle. Fig. 4 again depicts the probability of measuring the ion in the ground state after decoupling the internal and motional states, however this time we then apply the operator

$$D = e^{i\omega t}.$$  \hspace{1cm} (15)

In this case, because we have total destructive interference of certain paths during the walk, the deviation of the quantum to classical walk is much larger at certain stages of the walk.
V. DISCUSSION

We have described ion trap implementation schemes for quantum random walks, both on the line and on the circle. We have also suggested a measurement process which allows the enhanced features of these walks to be experimentally observed.

At this point, it is unclear whether quantum random walks will have any useful algorithmic applications. However, we believe that they can provide a benchmarking protocol for ion trap quantum computers, and perhaps other implementation schemes which combine continuous and discrete quantum variables.

Acknowledgments

BCT acknowledges support from the University of Queensland Traveling Scholarship, and thanks T. Bracken, O. Biham and J. Kempe for useful discussions.