Cluster Percolation and Explicit Symmetry Breaking in Spin Models

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Many features of spin models can be interpreted in geometrical terms by means of the properties of well defined clusters of spins. In case of spontaneous symmetry breaking, the phase transition of models like the q-state Potts model, $O(n)$, etc., can be equivalently described as a percolation transition of clusters. We study here the behaviour of such clusters when the presence of an external field $H$ breaks explicitly the global symmetry of the Hamiltonian of the theory. We find that these clusters have still some interesting relationships with thermal features of the model.

1. INTRODUCTION

Since the early 40’s [1] the interest of several physicists studying critical phenomena has focused on the investigation of geometrical structures of the system whose interplay could represent the mechanism for the occurrence of a phase transition.

A breakthrough in this course of studies was the discovery of Fortuin and Kasteleyn (FK) [2] that the $q$-state Potts model with vanishing external field can be reformulated as a geometrical model, with the spin configurations turning into clusters configurations by connecting nearest-neighbouring spins of the same orientation with a bond probability $p_B = 1 - \exp(-J/kT)$ ($J$ is the Potts coupling, $T$ the temperature). A remarkable consequence of this mapping is that the FK clusters percolate at the critical point of the thermal transition and, in case of second order phase transitions, the percolation exponents coincide with the thermal exponents. This result has been since then extended to a wide variety of models, including continuous spin models like $O(n)$ [3].

The absence of an external field endows the Hamiltonian of all these models with a global symmetry, which is spontaneously broken by the state of the system at low temperatures. The spontaneous breaking of such global symmetry is, ultimately, the reason of the phase transition. If we switch on an external field $H$, the Hamiltonian of the system breaks explicitly the global symmetry one has for $H = 0$, and this has deep consequences on the possibility for the model to show critical behaviour. One has, in general, two possible cases, according to the order of the phase transition when there is no external field:

- The model undergoes a second order phase transition for $H = 0$.
- The model undergoes a first order phase transition for $H = 0$.

Here we want to see whether the clusters that describe the critical behaviour at $H = 0$ have any relationship with thermal properties of the model also in presence of an external field. For this purpose we will analyse separately the two cases listed above.

2. SECOND ORDER PHASE TRANSITION FOR $H=0$

If the system undergoes a second order phase transition for $H = 0$, for any $H \neq 0$ the partition function of the system is not singular and one has at most a rapid crossover from the ordered to the disordered phase.

From the renormalization group ansatz of the magnetization as a function of $h = H/kT$ and $t = (T - T_c)/T_c$ one derives simple scaling laws for $t, h \ll 1$. Particularly interesting is the so called

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pseudocritical line, which is given by the temperature \( t_\chi \) at which the susceptibility \( \chi \) peaks for a given \( h \). The equation of the pseudocritical line is given by

\[
t_\chi \propto h^{1/\beta \delta},
\]

where \( \beta \) and \( \delta \) are critical exponents.

Figure 1. Comparison of the Kertész and the pseudocritical line for the 2D Ising model.

On the other hand, the FK clusters\(^2\) percolate for any \( H \neq 0 \): the corresponding geometrical variables show divergences as in the case \( H = 0 \) and percolation remains a genuine critical phenomenon. The line of percolation thresholds is called Kertész line [4].

We want to compare the Kertész and the pseudocritical line. Fig. 1 shows the two lines for the 2D Ising model. In the double logarithmic scale of the plot, they look parallel to each other. A fit of the four points of the percolation line leads indeed to the same scaling law as Eq. 1; the relative exponent is \( \kappa = 0.534(3) \), in excellent agreement with \( 1/\beta \delta = 8/15 = 0.533 \). We have also found that the same result holds for the 3D \( O(2) \) model [5], so that it is likely to be general.

3. FIRST ORDER PHASE TRANSITION FOR \( H=0 \)

In the case of a first order transition for \( H = 0 \), the model shows a discontinuous phase change also in the presence of an external field, as long as \( H \) is smaller than some critical \( H_c \). One has then a line of first order phase transition thresholds, from \( H = 0, T = T_c(0) \) to \( H = H_c \) and \( T = T_c(H_c) \) (endpoint), where the transition becomes continuous and the exponents are conjectured to be the Ising exponents.

We have examined the case of the 3D 3-state Potts model [6], which is subject of intense investigations because its phase transition is closely related to the deconfinement transition of finite temperature QCD [7]. The critical value of the field \( H_c \) was recently determined with great precision [8].

In order to see how the FK clusters behave for \( H \neq 0 \) we have analyzed the time history of the percolation order parameter, the percolation strength \( P \), which is the probability that a randomly chosen site of the lattice belongs to a percolating cluster. Fig. 2 shows the time history of \( P \) compared to the one of the magnetization \( M \) at a point of the first order phase transitions line, quite close to the endpoint. From the figure we can see that \( P \) makes a jump from zero to a non-zero value: there is then a discontinuous geometrical transition between a percolation and a non-percolation phase. We repeated the procedure for several points of the first order transitions line, obtaining any time the same result.

\(^2\)The result is valid for any model which can be mapped onto a cluster model, e.g. \( O(n) \).
For \( H = H_c \) we expect that the percolation transition of the FK clusters becomes continuous like the thermal one. It is then interesting to check whether the two thresholds coincide and, in this case, whether the percolation exponents belong to the 3D Ising universality class as the thermal exponents. Fig. 3 shows the percolation cumulant, that is the fraction of configurations with a percolating cluster, as a function of \( \beta = J/kT \) for different lattice sizes. The crossing point of the curves is the critical point of the geometrical transition and it is in good agreement with the thermal threshold determined in [8] (vertical dashed lines in the figure).

Figure 3. Percolation cumulant at \( H = H_c \) for FK clusters as a function of \( \beta \) for four lattice sizes.

The height of the crossing point is a universal number: the figure shows that it considerably differs from the values corresponding to the 3D Ising and random percolation (RP) universality classes (horizontal lines in the figure). A finite size scaling analysis of the cluster variables at the critical point leads to the following values for the percolation exponents: \( \beta/\nu = 0.32(3), \gamma/\nu = 2.32(2), \nu = 0.45(3) \). As expected, these exponents belong neither to the 3D Ising nor to the RP universality class.

4. CONCLUSIONS

We have seen that the clusters which describe the critical behaviour of thermal models in absence of an external field \( H \) maintain a close relationship with thermal features of the models also when \( H \neq 0 \).

If the spontaneous symmetry breaking transition is second order, in the limit of small fields the Kertész line is described by the same function as the pseudocritical line of the model, a power law with exponent \( 1/\beta \delta \).

If the spontaneous symmetry breaking transition is first order, the line of thermal first order phase transitions is also a line of first order percolation transitions for the clusters. In contrast to the magnetization, the percolation strength \( P \) is a good order parameter for the geometrical transition all along the line. The endpoint is a percolation point, but the corresponding critical exponents are neither in the Ising nor in the random percolation universality class. That suggests that one might need to change the cluster definition, eventually introducing an explicit field dependence in the bond probability \( p_{B,s} \), to obtain the correct critical exponents of the thermal transition at the endpoint. Such definition could turn out to be fruitful also in the crossover region.

REFERENCES