We consider interaction of two lumps corresponding to 0-branes in noncommutative
gauge theory

Recent development of string theory has shown that noncommutative 
(NC) models can arise in certain limits of String Theory \(^1\).

Noncommutative field theory operates with operators defined on a Hilbert 
space rather than with ordinary functions. There is, however, a one-to-one 
correspondence between such operators and functions which is given by the 
Weyl map. Important solutions discovered recently are the so called non-
commutative solitons or lumps \(^2\). These configurations are represented by 
projector operators. Their Weyl symbols are localised functions.

In this short note we are going to consider the interaction of two such 
gauge field lumps. This is realised by the dynamics of the configuration which 
represents a superposition of two projectors to finite-dimensional subspaces 
of the Hilbert space which correspond to two copies of the oscillator vacuum 
state shifted along noncommutative plane. For more details the reader is 
referred to Ref. \(^3\).

\section{The Model.}

Consider the noncommutative gauge model described by the following action,

\[ S = \int dt \, tr \left( \frac{1}{2} \dot{X}^i \dot{X}^i + \frac{1}{4g^2} [X^i, X^j]^2 \right). \]  

(1)

Here fields \(X^i, i = 1, \ldots, D\) are time dependent Hermitian operators defined 
on some separable infinite-dimensional Hilbert space \(\mathcal{H}\).
In the two-dimensional Moyal–Weyl form some operators $f$ are represented by functions $f(x) = e^{i\theta x^\mu x_\mu}$ on $x^\mu \mu = 1, 2$, subject to the star-product,

$$f \ast g(x) = e^{i\theta x^\mu x_\mu} f(x) g(x') \bigg|_{x' = x},$$  

(2)

where $x^\mu$ satisfy, $[x^1, x^2] = i\theta$.

Equations of motion look as follows

$$\ddot{X}_i + \frac{1}{g^2} [X_i, [X_i, X_\lambda]] = 0.$$  

(3)

In the case of just two lumps one can take them to be:

$$X_1 = cV PV^{-1} \equiv c \langle -u/2 | -u/2 \rangle, \quad (4a)$$  

$$X_2 = cV^{-1} PV \equiv c |u/2 \rangle \langle u/2 | \quad (4b)$$  

$$X_i = \text{const}, \quad i = 3, \ldots, D, \quad (4c)$$

where we introduced the shorthand notations,

$$V = e^{(i/2)p_\mu w^\mu}, \quad (5)$$  

$$P = \langle 0 | 0 \rangle. \quad (6)$$

To find the time evolution of such a system one has to solve the equations of motion (3) with initial condition given by (4) and $X_i|_{t=0} = 0$.

Symmetries of the model and initial conditions allows one to reduce the system to a two-dimensional particle described by the equations,

$$\ddot{X} = -\frac{2}{g^2} Y^2 X, \quad (7a)$$

$$\ddot{Y} = -\frac{2}{g^2} X^2 Y, \quad (7b)$$

with the initial conditions,

$$X|_{t=0} = e^{-\frac{i}{2} |u|^2}, \quad Y|_{t=0} = -\sqrt{1 - e^{-|u|^2}}. \quad (7c)$$

This model has a number of interesting features. The motion resulting from above equations appears to be stochastic for all values of $u$ except $u = \sqrt{\theta} \ln 2$. When $u = \sqrt{\theta} \ln 2$ the motion is periodic but unstable, infinitesimal deviation from this position brings it back to stochasticity.

The solution to equations (3) is expressed in terms linear combination of some functions $\sigma_a(\tilde{z}, z)$ with coefficients depending on $X(t)$ and $Y(t),$

$$X_1(t, \tilde{z}, z) = \frac{1}{2} \sigma_0(\tilde{z}, z) + X(t)\sigma_3(\tilde{z}, z) + Y(t)\sigma_1(\tilde{z}, z), \quad (8)$$

$$X_2(t, \tilde{z}, z) = \frac{1}{2} \sigma_0(\tilde{z}, z) + X(t)\sigma_3(\tilde{z}, z) - Y(t)\sigma_1(\tilde{z}, z). \quad (9)$$
Functions $\sigma_{a}(\bar{z}, z)$ are the Weyl symbols of the two-dimensional Pauli matrices,

\begin{align}
\sigma_{1}(z, \bar{z}) &= \frac{2}{\sqrt{1 - e^{-|u|^2}}} \left( e^{-2|z|^{2} - \frac{4}{3}|\bar{z}|^{2}} - e^{-2|z|^{2} + \frac{4}{3}|\bar{z}|^{2}} \right), \\
\sigma_{2}(z, \bar{z}) &= \frac{2ie^{-2\bar{z}z}}{\sqrt{1 - e^{-|u|^2}}} (e^{\bar{z}u - \bar{u}z} - e^{-\bar{z}u + \bar{u}z}), \\
\sigma_{3}(z, \bar{z}) &= \frac{2e^{-2|z|^{2}}}{1 - e^{-|u|^2}} \left( e^{-2|z|^{2} - \frac{4}{3}|\bar{z}|^{2}} + e^{-2|z|^{2} + \frac{4}{3}|\bar{z}|^{2}} \right) + \frac{e^{-2\bar{z}z}}{1 - e^{-|u|^2}} (e^{\bar{z}u - \bar{u}z} + e^{-\bar{z}u + \bar{u}z}), \\
I(\bar{z}, z) &= \sigma_{0}(z, \bar{z}) = \frac{2}{1 - e^{-|u|^2}} \left( e^{-2|z|^{2} - \frac{4}{3}|\bar{z}|^{2}} + e^{-2|z|^{2} + \frac{4}{3}|\bar{z}|^{2}} \right) - \frac{2e^{-2|z|^{2} - \frac{4}{3}|\bar{z}|^{2}}}{1 - e^{-|u|^2}} (e^{\bar{z}u - \bar{u}z} + e^{-\bar{z}u + \bar{u}z}).
\end{align}

The solution describes lumps bouncing (stochastically) around “points” $z = 0, \pm u/2$ of the noncommutative plane. In the string theory language the heights of the lumps can be interpreted as transversal (to the noncommutative 2-brane) coordinates of two 0-branes.

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**References**