Les Houches Lectures on De Sitter Space*

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Abstract

These lectures present an elementary discussion of some background material relevant to the problem of de Sitter quantum gravity. The first two lectures discuss the classical geometry of de Sitter space and properties of quantum field theory on de Sitter space, especially the temperature and entropy of de Sitter space. The final lecture contains a pedagogical discussion of the appearance of the conformal group as an asymptotic symmetry group, which is central to the dS/CFT correspondence. A (previously lacking) derivation of asymptotically de Sitter boundary conditions is also given.

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1 Introduction

We begin these lectures with one of our favorite equations

\[ S = \frac{A}{4G}. \]  

(1)

This is the Bekenstein-Hawking area-entropy law, which says that the entropy \( S \) associated with an event horizon is its area \( A \) divided by \( 4G \), where \( G \) is Newton’s constant [1, 2]. This is a macroscopic formula. It should be viewed in the same light as the macroscopic thermodynamic formulae that were first studied in the 18th and 19th centuries. It describes how properties of event horizons in general relativity change as their parameters are varied. This behavior can be succinctly summarized by ascribing to them an entropy given by (1).

One of the surprising and impressive features of this formula is its universality. It applies to all kinds of black holes with all kinds of charges, shapes and rotation, as well as to black strings and to all of the strange new objects we’ve found in string theory. It also applies to cosmological horizons, like the event horizon in de Sitter space [3].

After Boltzmann’s work we tend to think of entropy in microscopic statistical terms as something which counts the number of microstates of a system. Such an interpretation for the entropy (1) was not given at the time that the law was discovered in the early 70s. A complete understanding of this law, and in particular of the statistical origin of this law, is undoubtedly one of the main keys to understanding what quantum gravity is and what the new notions are that replace space and time in quantum gravity.
There has been some definite but still limited progress in understanding the microscopic origin of (1) in very special cases of black holes which can be embedded into string theory [4]. That little piece of (1) that we have managed to understand has led to all kinds of interesting insights, ultimately culminating in the AdS/CFT correspondence [5]. Nevertheless the progress towards a complete understanding of (1) is still very limited, because we only understand special kinds of black holes—among which Schwarzschild black holes are not included—and we certainly don’t understand much about cosmological event horizons, such as the horizon in de Sitter space.

In some ways cosmological horizons are much more puzzling than black hole horizons because in the black hole case one may expect that the black hole is a localized object with some quantum microstates. Then if you could provide the correct description of that localized object, you would be able to count those microstates and compare your result to the Bekenstein-Hawking formula and see that they agree. In some stringy cases this agreement has been achieved. On the other hand in de Sitter space the event horizon is observer dependent, and it is difficult even to see where the quantum microstates that we would like to count are supposed to be.

Why has there been significant progress in understanding black hole entropy, but almost no progress in understanding the entropy of de Sitter space? One reason is that one of the principal tools we’ve used for understanding black hole entropy is supersymmetry. Black holes can be supersymmetric, and indeed the first black holes whose entropy was counted microscopically were supersymmetric. Since then we’ve managed to creep away from the supersymmetric limit a little bit, but not very far, and certainly we never managed to get all the way to Schwarzschild black holes. So supersymmetry is a crutch that we will need to throw away before we can do anything about de Sitter space. Indeed there is a very simple observation [6] that de Sitter space is inconsistent with supersymmetry in the sense that there is no supergroup that includes the isometries of de Sitter space and has unitary representations.\(^{1}\) A second, related, obstacle to progress in understanding de Sitter space is that so far we have not been able to embed it in a fully satisfactory manner into string theory.

While the importance of understanding de Sitter quantum gravity has been evident for decades, it has recently been receiving more attention [8, 7, 23, 21, 22, 9–15, 17–20, 24–28, 16, 29–38]. One reason for this is the recent astronomical observations which indicate that the cosmological constant in our universe is positive [39–42]. A second reason is that recent progress in string theory and black holes provides new tools and suggests potentially fruitful new angles. So perhaps de Sitter quantum gravity is a nut ready to be cracked. These lectures are mostly an elementary discussion of the background material relevant to the problem of de Sitter quantum gravity. The classical geometry of de Sitter space is described in section 2. Scalar quantum field theory in a fixed de Sitter background is in section 3. Finally, in section 4 we turn to some recent work on de Sitter quantum gravity. A pedagogical derivation is given of the appearance

\(^{1}\)See, however [7].
of the two dimensional conformal group in three dimensional de Sitter space, which leads to the dS/CFT correspondence [23]. This section also contains a derivation, missing in previous treatments, of the asymptotically de Sitter boundary conditions on the metric. The appendix contains a calculation of the asymptotic form of the Brown-York stress tensor.

2 Classical Geometry of De Sitter Space

The \(d\)-dimensional de Sitter space \(dS_d\) may be realized as the hypersurface described by the equation

\[-X_0^2 + X_1^2 + \cdots + X_d^2 = \ell^2\]

in flat \(d+1\)-dimensional Minkowski space \(M^{d,1}\), where \(\ell\) is a parameter with units of length called the de Sitter radius. This hypersurface in flat Minkowski space is a hyperboloid, as shown in figure 1.

Figure 1: Hyperboloid illustrating de Sitter space. The dotted line represents an extremal volume \(S^{d-1}\).

The de Sitter metric is the induced metric from the standard flat metric on \(M^{d,1}\). The embedding (2) is a nice way of describing de Sitter space because the \(O(d,1)\) symmetry, which is the isometry group of \(dS_d\), is manifest. Furthermore one can show that \(dS_d\) is an Einstein manifold with positive scalar curvature, and the Einstein tensor satisfies

\[G_{ab} + \Lambda g_{ab} = 0,\]

where

\[\Lambda = \frac{(d - 2)(d - 1)}{2\ell^2}\]

is the cosmological constant. Henceforth we will set \(\ell = 1\).
2.1 Coordinate Systems and Penrose Diagram

We will now discuss a number of coordinate systems on $dS_d$ which give different insights into the structure of $dS_d$. We will frequently make use of coordinates on the sphere $S^{d-1}$, which is conveniently parametrized by setting

$$\omega_1 = \cos \theta_1,$$
$$\omega_2 = \sin \theta_1 \cos \theta_2,$$
$$\vdots$$
$$\omega_{d-1} = \sin \theta_1 \cdots \sin \theta_{d-2} \cos \theta_{d-1},$$
$$\omega_d = \sin \theta_1 \cdots \sin \theta_{d-2} \sin \theta_{d-1},$$

(5)

where $0 \leq \theta_i < \pi$ for $1 \leq i < d-1$, but $0 \leq \theta_{d-1} < 2\pi$. Then it is clear that $\sum_{i=1}^{d}(\omega^i)^2 = 1$, and the metric on $S^{d-1}$ is

$$d\Omega_{d-1}^2 = \sum_{i=1}^{d} (d\omega^i)^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + \sin^2 \theta_1 \cdots \sin^2 \theta_{d-2} d\theta_{d-1}^2.$$

(6)

a. Global coordinates $(\tau, \theta_i)$. This coordinate system is obtained by setting

$$X^0 = \sinh \tau,$$
$$X^i = \omega^i \cosh \tau, \quad i = 1, \ldots, d,$$

(7)

where $-\infty < \tau < \infty$ and the $\omega^i$ are as in (5). It is not hard to check that these satisfy (2) for any point $(\tau, \omega_i)$.

From the flat metric on $\mathcal{M}^{d,1}$

$$ds^2 = -dX_0^2 + dX_1^2 + \cdots + dX_d^2,$$

(8)

plugging in (2.1) we obtain the induced metric on $dS_d$,

$$ds^2 = -d\tau^2 + (\cosh^2 \tau) d\Omega_{d-1}^2.$$

(9)

In these coordinates $dS_d$ looks like a $d-1$-sphere which starts out infinitely large at $\tau = -\infty$, then shrinks to a minimal finite size at $\tau = 0$, then grows again to infinite size as $\tau \to +\infty$.

b. Conformal coordinates $(T, \theta_i)$. These coordinates are related to the global coordinates by

$$\cosh \tau = \frac{1}{\cos T},$$

(10)

so that we have $-\pi/2 < T < \pi/2$. The metric in these coordinates takes the form

$$ds^2 = \frac{1}{\cos^2 T} (-dT^2 + d\Omega_{d-1}^2).$$

(11)

This is a particularly useful coordinate system because it enables us to understand the causal structure of de Sitter space. If a geodesic is null with respect
to the metric (11), then it is also null with respect to the conformally related metric
\[ ds^2 = (\cos^2 T) ds^2 = -dT^2 + d\Omega^2_{d-1}. \] (12)
So from the point of view of analyzing what null geodesics do in dS, we are free to work with the metric (12), which looks simpler than (11).

\[ \text{Figure 2: Penrose diagram for } \text{dS}_d. \text{ The north and south poles are timelike lines, while every point in the interior represents an } S^{d-2}. \text{ A horizontal slice is an } S^{d-1}. \text{ The dashed lines are the past and future horizons of an observer at the south pole. The conformal time coordinate } T \text{ runs from } -\pi/2 \text{ at } \mathcal{I}^- \text{ to } +\pi/2 \text{ at } \mathcal{I}^+. \]

The Penrose diagram 2 contains all the information about the causal structure of dS although distances are highly distorted. In this diagram each point is actually an S^{d-2} except for points on the left or right sides, which lie on the north or south pole respectively. Light rays travel at 45° angles in this diagram, while timelike surfaces are more vertical than horizontal and spacelike surfaces are more horizontal than vertical.

The surfaces marked \( \mathcal{I}^-, \mathcal{I}^+ \) are called past and future null infinity. They are the surfaces where all null geodesics originate and terminate. Note that a light ray which starts at the north pole at \( \mathcal{I}^- \) will exactly reach the south pole by the time it reaches \( \mathcal{I}^+ \) infinitely far in the future.

One of the peculiar features of de Sitter space is that no single observer can access the entire spacetime. We often get into trouble in physics when we try to describe more than we are allowed to observe—position and momentum in quantum mechanics, for example. Therefore in attempting to develop de Sitter quantum gravity we should be aware of what can and cannot be observed. A classical observer sitting on the south pole will never be able to observe anything past the diagonal line stretching from the north pole at \( \mathcal{I}^- \) to the south pole at \( \mathcal{I}^+ \). This region is marked as \( \mathcal{O}^- \) in figure 3. This is qualitatively different from Minkowski space, for example, where a timelike observer will eventually have the entire history of the universe in his/her past light cone.

Also shown in figure 3 is the region \( \mathcal{O}^+ \), which is the only part of de Sitter space that an observer on the south pole will ever be able to send a message to. The intersection \( \mathcal{O}^+ \cap \mathcal{O}^- \) is called the (southern) causal diamond. It is this
Figure 3: These diagrams show the regions $\mathcal{O}^-$ and $\mathcal{O}^+$ corresponding respectively to the causal past and future of an observer at the south pole.

region that is fully accessible to the observer on the south pole. For example if she/he wishes to know the weather anywhere in the southern diamond, a query can be sent to the appropriately located weather station and the response received before $I^+$ is reached. This is not possible in the lower diamond of $\mathcal{O}^-$, to which a query can never be sent, or the upper diamond of $\mathcal{O}^+$, from which a response cannot be received. The northern diamond on the left of 3 is completely inaccessible to an observer on the south pole.

c. Planar coordinates $(t, x^i)$, $i = 1, \ldots, d - 1$. To define this coordinate system we take

\begin{align*}
X^0 &= \sinh t - \frac{1}{2} x_i x^i e^{-t}, \\
X^i &= x^i e^{-t}, \quad i = 1, \ldots, d - 1, \\
X^d &= \cosh t - \frac{1}{2} x_i x^i e^{-t}.
\end{align*}

(13)

The metric then takes the form

$$ds^2 = -dt^2 + e^{-2t} dx_i dx^i.$$  

(14)

These coordinates do not cover all of de Sitter space, but only the region $\mathcal{O}^-$ and are therefore appropriate for an observer on the south pole. The slices of constant $t$ are illustrated in figure 4.

The surfaces of constant $t$ are spatial sections of de Sitter space which are infinite volume $d-1$-planes with the flat metric. From the diagram it is clear that every surface of constant $t$ intersects $I^-$ at the north pole. It may seem puzzling—and is certainly one of the salient features of de Sitter space—that a spatial plane can make it to the infinite past. This happens because $I^-$ is very large, and you can get there along a spatial trajectory from anywhere in $\mathcal{O}^-$. In these coordinates the time $t$ is not a Killing vector, and the only manifest symmetries are translations and rotations of the $x^i$ coordinates.

d. Static coordinates $(t, r, \theta_a)$, $a = 1, \ldots, d - 2$. The $t$ in these coordinates
South Pole
North Pole
I
O
South Pole
I

Figure 4: The dashed lines are slices of constant $t$ in planar coordinates. Note that each slice is an infinite flat $d-1$-dimensional plane which extends all the way down to $I^-$. 

is not the same as the $t$ in planar coordinates, but we are running out of letters! Note also that for these coordinates and the following ones we will need a parametrization of $S^{d-2}$, not $S^{d-1}$. This coordinate system is constructed to have an explicit timelike Killing symmetry. If we write

$$
X^0 = \sqrt{1-r^2} \sinh t, \\
X^a = r \omega^a, \quad a = 1, \ldots, d-1, \\
X^d = \sqrt{1-r^2} \cosh t, 
$$

(15)

then the metric takes the form

$$
ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega_{d-2}^2.
$$

(16)

In this coordinate system $\partial/\partial t$ is a Killing vector and generates the symmetry $t \rightarrow t + \text{constant}$. The horizons are at $r^2 = 1$, and the southern causal diamond has $0 \leq r \leq 1$, with the south pole at $r = 0$.

One of the reasons to want a timelike Killing vector is so that we can use it to define time evolution, or in other words to define the Hamiltonian. But from (16) we see that at $r = 1$ the norm of $\partial/\partial t$ vanishes, so that it becomes null. In figure 5 we illustrate what the Killing vector field $\partial/\partial t$ is doing when extended to the various diamonds of the Penrose diagram. In the top and bottom diamonds, $\partial/\partial t$ is spacelike, while in the northern diamond the vector is pointing towards the past! Thus $\partial/\partial t$ in static coordinates can only be used to define a sensible time evolution in the southern diamond of de Sitter space. The absence of a globally timelike Killing vector in de Sitter space has important implications for the quantum theory.

e. Eddington-Finkelstein coordinates $(x^+, r, \theta^a)$. This coordinate system is the de Sitter analog of the (outgoing) Eddington-Finkelstein coordinates for
a Schwarzschild black hole. Starting from the static coordinates, we define $x^+$ by the equation

$$dt = dx^+ + \frac{dr}{1 - r^2},$$

which we can solve to obtain

$$x^+ = t + \frac{1}{2} \ln \frac{1 + r}{1 - r}. \quad (18)$$

In these coordinates the metric is

$$ds^2 = -(1 - r^2)(dx^+)^2 - 2dx^+ dr + r^2 d\Omega^2_{d-2}. \quad (19)$$

The same symmetries are manifest in this coordinate system as in the static coordinates since $\partial/\partial t$ at fixed $r$ is the same as $\partial/\partial x^+$ at fixed $r$. Lines of constant $x^+$ are the null lines connecting $I^-$ with the south pole depicted in figure 3. These coordinates cover the causal past $O^-$ of an observer at the south pole while still keeping the symmetry manifest.

We can also define

$$x^- = t - \frac{1}{2} \ln \frac{1 + r}{1 - r}, \quad (20)$$

so that the metric takes the form

$$ds^2 = -(1 - r^2(x^+, x^-))dx^+ dx^- + r^2 d\Omega^2_{d-2}, \quad (21)$$

where $r = \tanh \frac{x^+ - x^-}{2}$.

f. Kruskal coordinates $(U, V, \theta_a)$. Finally we take

$$x^- = \ln U, \quad x^+ = -\ln(-V), \quad (22)$$
in which case
\[ r = \frac{1 + UV}{1 - UV}. \] (23)

Then the metric takes the form
\[ ds^2 = \frac{1}{(1 - UV)^2}(-4dUdV + (1 + UV)^2d\Omega_{d-2}^2). \] (24)

These coordinates cover all of de Sitter space. The north and south poles correspond to \( UV = -1 \), the horizons correspond to \( UV = 0 \), and \( I^\pm \) correspond to \( UV = 1 \). The southern diamond is the region with \( U > 0 \) and \( V < 0 \).

**Exercise 1.** Find \( X^0, \ldots, X^d \) as functions of \( U, V \), and \( \theta \) for the Kruskal coordinates.

![Figure 6: The Kruskal coordinate system covers all of de Sitter space. In this Penrose diagram the coordinate axes \( U = 0 \) and \( V = 0 \) are the horizons, \( UV = -1 \) are the north and south poles, and \( UV = 1 \) are \( I^+ \) and \( I^- \). The arrows denote the directions of increasing \( U \) and \( V \).](image)

### 2.2 Schwarzschild-de Sitter

The simplest generalization of the de Sitter space solution is Schwarzschild-de Sitter, which we abbreviate as SdS. This solution represents a black hole in de Sitter space. In \( d \) dimensions in static coordinates the SdS_d metric takes the form
\[
ds^2 = -(1 - \frac{2m}{r^{d-3}} - r^2)\, dt^2 + \frac{1}{1 - \frac{2m}{r^{d-3}} - r^2}\, dr^2 + r^2 d\Omega_{d-2}^2,
\] (25)

where \( m \) is a parameter related to the black hole mass (up to some \( d \)-dependent normalization constant). In general there are two horizons (recall that these are places where the timelike Killing vector \( \partial/\partial t \) becomes null), one of which is the black hole horizon and the other of which is the de Sitter horizon. Note that the two horizons approach each other as \( m \) is increased, so that there is a maximum size black hole which can fit inside de Sitter space before the black hole horizon hits the de Sitter horizon.

One reason to introduce SdS is that it plays an important role in the work of Gibbons and Hawking [3] determining the entropy of pure de Sitter space,
which will be reviewed in subsection 3.3. For this purpose it will be convenient to focus on the three dimensional Schwarzschild-de Sitter solution [43]

\[
    ds^2 = -(1 - 8GE - r^2)dt^2 + \frac{dr^2}{1 - 8GE - r^2} + r^2d\phi^2,
\]

(26)

where we have normalized the energy \( E \) of the Schwarzschild black hole appropriately for three dimensions. In three dimensions there is only one horizon, at \( r_H = \sqrt{1 - 8GE} \), and as \( E \) goes to zero this reduces to the usual horizon in empty de Sitter space. The fact that there is only a de Sitter horizon and not a black hole horizon is not surprising in light of the fact that in three dimensional flat space there are no black holes.

We can learn a little more about the solution (26) by looking near \( r = 0 \), where \( ds^2 \) behaves like

\[
    ds^2 \sim -r_H^2 dt^2 + \frac{dr^2}{r_H^2} + r^2 d\phi^2.
\]

(27)

Now we can rescale the coordinates by defining

\[
    t' = r_H t, \quad r' = r/r_H, \quad \phi' = r_H \phi.
\]

(28)

In the rescaled coordinates the metric (27) is simply

\[
    ds^2 = -dt'^2 + dr'^2 + r'^2 d\phi'^2.
\]

(29)

This looks like flat space, but it is not quite flat space because while \( \phi \) was identified modulo \( 2\pi \), \( \phi' \) is identified modulo \( 2\pi r_H \). Therefore there is a conical singularity with a positive deficit angle at the origin.

You may be familiar with the fact that if you put a point-like mass in flat three dimensional Minkowski space you would also get a conical deficit angle at the location of the particle. Hence we recognize (26) as a point-like mass, rather than a black hole, at the south pole of dS\(_3\). If the solution is maximally extended one finds there is also point-like mass of the same size at the north pole [43].

Exercise 2. Show that SdS\(_3\) is a global identification of dS\(_3\).

2.3 Geodesics

Our last topic in the classical geometry of de Sitter space is geodesics. It is clear that if we take two points on the sphere \( S^n \) of radius \( R \), then there is only one independent SO\((n+1)\)-invariant quantity that we can associate to the two points. That is the geodesic distance \( D \), or equivalently the angle \( \theta \) between them, which are related by \( D = R\theta \). Let us think of the sphere as being embedded in flat Euclidean space, with the embedding equation \( \delta_{ij}X^iX^j = R^2 \), \( i, j = 1, \ldots, n+1 \). It is useful to define a quantity \( P \) by \( R^2 P(X, X') = \delta_{ij}X^iX'^j = R^2 \cos \theta \).
It is a little harder to visualize, but we can do something similar for dS. There we can define
\[ P(X, X') = \eta_{ij} X^i X'^j, \quad \eta_{ij} = \text{diag}(-1, 1, \ldots, 1) \] (30)
(recall that we have set the de Sitter radius \( \ell \) to one). For points in a common
causal diamond, this is related to the geodesic distance \( D(X, X') \) between \( X \) and \( X' \) by \( P = \cos D \). This quantity \( P \) will turn out to be a more convenient
invariant to associate to two points in de Sitter space. We can easily write explicit formulas for \( P(X, X') \) in the various coordinate systems discussed above. For example, in planar coordinates we have
\[ P(t, x^i; t', y^i) = \cosh(t - t') - \frac{1}{2} e^{-t-t'} \delta_{ij} (x^i - y^i)(x^j - y^j). \] (31)
The expression for \( P \) is simple in terms of the \( X \)'s but can get complicated when written in a particular coordinate system.

To conclude, we note a few important properties of \( P \) for later use. If \( P = 1 \),
then the geodesic distance is equal to zero, so the two points \( X \) and \( X' \) coincide
or are separated by a null geodesic. We can also consider taking antipodal points \( X' = -X \), in which case \( P = -1 \). In general \( P = -1 \) when the antipodal point
of \( X \) lies on the light cone of \( X' \). In general, the geodesic separating \( X \) and \( X' \)
is spacelike for \( P < 1 \) and timelike for \( P > 1 \), while for \( P < -1 \) the geodesic
between \( X \) and the antipodal point of \( X' \) is timelike.

3 Quantum Field Theory on De Sitter Space

Ultimately, a complete understanding of the entropy-area relation (1) in de
Sitter space will require an understanding of quantum gravity on de Sitter space.
In this section we will take a baby step in that direction by considering a single
free massive scalar field on a fixed background de Sitter spacetime. This turns
out to be a very rich subject which has been studied by many authors [44,3,45–
51,53,52,54].

3.1 Green Functions and Vacua

Let us consider a scalar field in dS with the action
\[ S = -\frac{1}{2} \int d^4x \sqrt{-g} \left[ (\nabla \phi)^2 + m^2 \phi^2 \right]. \] (32)
Since this is a free field theory, all information is encoded in the two-point
function of \( \phi \). We will study the Wightman function
\[ G(X, Y) = \langle 0 | \phi(X) \phi(Y) | 0 \rangle, \] (33)
which obeys the free field equation
\[ (\nabla^2 - m^2) G(X, Y) = 0, \] (34)
where $\nabla^2$ is the Laplacian on $dS_d$.

There are other two point functions that one can discuss: retarded, advanced, Feynman, Hadamard and so on, but these can all be obtained from the Wightman function (33), for example by taking the real or imaginary part, and/or by multiplying by a step function in time.

Let us assume that the state $|0\rangle$ in (33) is invariant under the $SO(d, 1)$ de Sitter group. Then $G(X,Y)$ will be de Sitter invariant, and so at generic points can only depend on the de Sitter invariant length $P(X,Y)$ between $X$ and $Y$. Writing $G(X,Y) = G(P(X,Y))$, (34) reduces to a differential equation in one variable $P$

$$ (1 - P^2) \partial^2_P G - dP \partial_P G - m^2 G = 0. \quad (35) $$

With the change of variable $z = \frac{1+P}{2}$ this becomes a hypergeometric equation

$$ z(1-z)G'' + \left( \frac{d}{2} - dz \right) G' - m^2 G = 0, \quad (36) $$

whose solution is

$$ G = c_{m,d} F(h_+, h_-, d^2, z), \quad (37) $$

where $c_{m,d}$ is a normalization constant to be fixed shortly, and

$$ h_{\pm} = \frac{1}{2} \left[ (d-1) \pm \sqrt{(d-1)^2 - 4m^2} \right]. \quad (38) $$

The hypergeometric function (37) has a singularity at $z = 1$, or $P = 1$, and a branch cut for $1 < P < \infty$. The singularity occurs when the points $X$ and $Y$ are separated by a null geodesic. At short distances the scalar field is insensitive to the fact that it is in de Sitter space and the form of the singularity is precisely the same as that of the propagator in flat $d$-dimensional Minkowski space. We can use this fact to fix the normalization constant $c_{m,d}$. Near $z = 1$ the hypergeometric function behaves as

$$ F(h_+, h_-, d^2, \frac{1+P}{2}) \sim \left( \frac{D^2}{4} \right)^{1-d/2} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} - 1)}{\Gamma(h_+)\Gamma(h_-)}, \quad (39) $$

where $D = \cos^{-1} P$ is the geodesic separation between the two points. Comparing with the usual short-distance singularity $\frac{\Gamma(\frac{d}{2})}{2(d-2)\pi^{d/2}}(D^2)^{1-d/2}$ fixes the coefficient to be

$$ c_{m,d} = 4^{1-d/2} \frac{\Gamma(h_+)\Gamma(h_-)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} - 1)} \times \frac{\Gamma(\frac{d}{2})}{2(d-2)\pi^{d/2}} = \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}. \quad (40) $$

The prescription for going around the singularity in the complex plane is also the same as in Minkowski space, namely replacing $X^0 - Y^0$ with $X^0 - Y^0 - i\epsilon$.\footnote{$P(X,Y) = P(Y,X)$ is insensitive to the time ordering between points, which is $SO(d, 1)$ (but not $O(d, 1)$) invariant. Because of this the $i\epsilon$ prescription for $G$, as discussed below, cannot be written as a function of $P$ alone.}
The equation (35) clearly has a $P \rightarrow -P$ symmetry, so if $G(P)$ is a solution then $G(-P)$ is also a solution. The second linearly independent solution to (35) is therefore

$$F(h_+, h_-, \frac{d}{2}, \frac{1 - P}{2}). \tag{41}$$

The singularity is now at $P = -1$, which corresponds to $X$ being null separated from the antipodal point to $Y$. This singularity sounds rather unphysical at first, but we should recall that antipodal points in de Sitter space are always separated by a horizon. The Green function (41) can be thought of as arising from an image source behind the horizon, and (41) is nonsingular everywhere within an observer's horizon. Hence the “unphysical” singularity can not be detected by any experiment.

De Sitter space therefore has a one parameter family of de Sitter invariant Green functions $G_\alpha$ corresponding to a linear combination of the solutions (37) and (41). Corresponding to this one-parameter family of Green functions is a one-parameter family of de Sitter invariant vacuum states $|\alpha\rangle$ such that $G_\alpha(X,Y) = \langle \alpha | \phi(X) \phi(Y) | \alpha \rangle$. These vacua are discussed in detail in [53, 52], but are usually discarded as somehow “unphysical”. However, as we try to understand the quantum theory of de Sitter space these funny extra vacua will surely turn out to have some purpose in life.

De Sitter Green functions are often discussed in the context of analytic continuation to the Euclidean sphere. If we work in static coordinates and take $t \rightarrow i\tau$, the dS metric becomes the metric on the sphere $S^d$. On the sphere there is a unique Green function, which when analytically continued back to de Sitter space yields (37).

Let us say a few more words about the vacuum states. A vacuum state $|0\rangle$ is defined as usual by saying that it is annihilated by all annihilation operators

$$a_\alpha |0\rangle = 0. \tag{42}$$

That is, we write an expansion for the scalar field in terms of creation and annihilation operators of the form

$$\phi(X) = \sum_k \left[ a_k u_k(X) + a_k^\dagger u_k^*(X) \right], \tag{43}$$

where $a_k$ and $a_k^\dagger$ satisfy

$$[a_k, a_l^\dagger] = \delta_{kl}. \tag{44}$$

The modes $u_k(X)$ satisfy the wave equation

$$(\nabla^2 - m^2)u_k = 0, \tag{45}$$

and are normalized with respect to the invariant Klein-Gordon inner product

$$(u_k, u_l) = -i \int d\Sigma^\mu \left( u_k \overrightarrow{\partial_\mu} u_l^* \right) = \delta_{kl}. \tag{46}$$

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where the integral is taken over a complete spherical spacelike slice in \( \text{dS}_d \) and the result is independent of the choice of this slice.

The question is, which modes do we associate with creation operators in (43) and which do we associate with annihilation operators? In Minkowski space we take positive and negative frequency modes,

\[
u \sim e^{-iEt} f(x), \quad u^* \sim e^{iEt} f^*(x),
\]

respectively to multiply the annihilation and creation operators. But in a general curved spacetime there is no canonical choice of a time variable with respect to which one can classify modes as being positive or negative frequency. If we make a choice of time coordinate, we can get a vacuum state \(|0\rangle\) and then the state \((a^\dagger)^n|0\rangle \equiv |n\rangle\) is said to have \(n\) particles in it. But if we had made some other choice of time coordinate then we would have a different vacuum \(|0'\rangle\), which we could express as a linear combination of the \(|n\rangle\)'s. Hence the question “How many particles are present?” is not well-defined independently of a choice of coordinates. This is an important and general feature of quantum field theory in curved spacetime.

In order to preserve classical symmetries of \( \text{dS}_d \) in the quantum theory, we would like to find a way to divide the modes into \(u\) and \(u^*\) that is invariant under \(\text{SO}(d,1)\). The resulting vacuum will then be de Sitter invariant. It turns out \([45, 53, 52]\) that there is a family of such divisions, and a corresponding family of Green functions such as \(G_\alpha\).

### 3.2 Temperature

In this section we will show that an observer moving along a timelike geodesic observes a thermal bath of particles when the scalar field \(\phi\) is in the vacuum state \(|0\rangle\). Thus we will conclude that de Sitter space is naturally associated with a temperature \([47]\), which we will calculate.

Since the notion of a particle is observer-dependent in a curved spacetime, we must be careful to give a coordinate invariant characterization of the temperature. A good way to achieve this is to consider an observer equipped with a detector. The detector will have some internal energy states and can interact with the scalar field by exchanging energy, i.e. by emitting or absorbing scalar particles. The detector could for example be constructed so that it emits a ‘bing’ whenever its internal energy state changes. All observers will agree on whether or not the detector has binged, although they may disagree on whether the bing was caused by particle emission or absorption. Such a detector is called an Unruh detector and may be modeled by a coupling of the scalar field \(\phi(x(\tau))\) along the worldline \(x(\tau)\) of the observer to some operator \(m(\tau)\) acting on the internal detector states

\[
g \int_{-\infty}^{\infty} d\tau \ m(\tau)\phi(x(\tau)),
\]

where \(g\) is the strength of the coupling and \(\tau\) is the proper time along the observer’s worldline.
Let $H$ denote the detector Hamiltonian, with energy eigenstates $|E_j\rangle$,

$$H|E_j\rangle = E_j|E_j\rangle,$$

and let $m_{ij}$ be the matrix elements of the operator $m(\tau)$ at $\tau = 0$:

$$m_{ij} \equiv \langle E_i| m(0)|E_j\rangle.$$  

(50)

We will calculate the transition amplitude from a state $|0\rangle|E_i\rangle$ in the tensor product of the scalar field and detector Hilbert spaces to the state $\langle E_j|\langle\beta|$, where $\langle\beta|$ is any state of the scalar field. To first order in perturbation theory for small coupling $g$, the desired amplitude is

$$g \int_{-\infty}^{\infty} d\tau \langle E_j|\langle\beta|m(\tau)\phi(x(\tau))|0\rangle|E_i\rangle.$$  

(51)

Using

$$m(\tau) = e^{iH\tau}m(0)e^{-iH\tau},$$  

(52)

this can be written as

$$gm_{ji} \int_{-\infty}^{\infty} d\tau \ e^{i(E_j-E_i)\tau} \langle\beta|\phi(x(\tau))|0\rangle.$$  

(53)

Since we are only interested in the probability for the detector to make the transition from $E_i$ to $E_j$, we should square the amplitude (53) and sum over the final state $|\beta\rangle$ of the scalar field, which will not be measured. Using $\sum_\beta |\beta\rangle\langle\beta| = 1$, we find the probability

$$P(E_i \rightarrow E_j) = g^2|m_{ij}|^2 \int_{-\infty}^{\infty} d\tau \ d\tau' \ e^{-i(E_j-E_i)(\tau'-\tau)}G(x(\tau'), x(\tau)).$$

(54)

where $G(x(\tau'), x(\tau))$ is the Green function (33). The Green function is a function only of the geodesic distance $P(x(\tau), x(\tau'))$, and if we consider for simplicity an observer sitting on the south pole, then $P$ is given in static coordinates by $P = \cosh(\tau - \tau')$. Therefore everything inside the integral (54) depends only on $\tau - \tau'$ and we get an infinite factor from integrating over $\tau + \tau'$. We can divide out this factor and discuss the transition probability per unit proper time along the detector worldline, which is then given by

$$\dot{P}(E_i \rightarrow E_j) = g^2|m_{ij}|^2 \int_{-\infty}^{\infty} d\tau \ e^{-i(E_j-E_i)\tau}G(\cosh \tau).$$

(55)

The first hint that (55) has something to do with a thermal response is that the function $G$ is periodic in imaginary time under $\tau \rightarrow \tau + 2\pi i$, and Green functions which are periodic in imaginary time are thermal Green functions.

To investigate the nature of a thermal state, let us suppose it were true (as will be demonstrated shortly) that

$$\dot{P}(E_i \rightarrow E_j) = \dot{P}(E_j \rightarrow E_i)e^{-\beta(E_j-E_i)},$$

(56)

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and that the energy levels of the detector were thermally populated, so that

\[ N_i = N e^{-\beta E_i}, \quad (57) \]

where \( N \) is some normalization factor. Then it is clear that the total transition rate \( R \) from \( E_i \) to \( E_j \) is the same as from \( E_j \) to \( E_i \):

\[ R(E_i \rightarrow E_j) = N e^{-\beta E_j} \dot{P}(E_i \rightarrow E_j) = R(E_j \rightarrow E_i), \quad (58) \]

which is the principle of detailed balance in a thermal ensemble. In other words, if the transition probabilities are related by (56) and the population of the states is thermal as in (57), then there is no change in the probability distribution for the energy levels with time. So (56) describes the transition probabilities of a system in a thermal bath of particles at temperature \( T = 1/\beta \).

Let us now show that (56) holds for the transition probabilities calculated in (55). The integrand in (55) has singularities in the complex \( \tau \)-plane at \( \tau = 2\pi in \) for any integer \( n \). Consider integrating the function \( e^{-i(E_j - E_i)\tau} G(\cosh \tau) \) around the contour shown in figure 7.

![Figure 7: The integrand in (55) has singularities in the complex \( \tau \)-plane at \( \tau = 2\pi in \) for any integer \( n \). This figure shows the contour \( C \) used in the integral (59). The dotted lines signify the closure of the contour at infinity.](image)

Since the total integral around this contour is zero, we have

\[ \int_{-\infty}^{\infty} d\tau \ e^{-i(E_j - E_i)\tau} G(\cosh \tau) + \int_{-\infty+i\beta}^{\infty-i\beta} d\tau \ e^{-i(E_j - E_i)\tau} G(\cosh \tau) = 0, \quad (59) \]

where \( \beta = 1/2\pi \). The contour in figure 7 corresponds to the pole prescription for the Wightman function as discussed in section 3.1. Now redefining the variable of integration in the second integral as \( \tau' = -\tau - i\beta \) we get precisely the desired relation (56).

Although we performed this calculation only for an observer stationary at the south pole, all timelike geodesics in de Sitter space are related to each other by the \( \text{SO}(d,1) \) de Sitter isometry group. Since the Green function used in this calculation is de Sitter invariant, the result for the temperature is the same for
any observer moving along a timelike geodesic. We conclude that any geodesic observer in de Sitter space will feel that she/he is in a thermal bath of particles at a temperature

\[ T_{dS} = \frac{1}{2\pi \ell}, \tag{60} \]

where we have restored the factor of the de Sitter radius \( \ell \) by dimensional analysis.

### 3.3 Entropy

In this subsection we will associate an entropy to de Sitter space. We will restrict our attention to \( dS_3 \), where the analysis simplifies considerably.

For the case of black holes one can use similar methods as those in the previous section to calculate the temperature \( T_{BH} \) of the black hole. The black hole entropy \( S_{BH} \) can then be found by integrating the thermodynamic relation

\[ \frac{dS_{BH}}{dE_{BH}} = \frac{1}{T_{BH}}, \tag{61} \]

where \( E_{BH} \) is the energy or mass of the black hole. So if you know the value of the temperature just for one value of \( E_{BH} \) you will not be able to get the entropy, but if you know it as a function of the black hole mass then you can simply integrate (61) to find the entropy. The constant of integration is determined by requiring that a black hole of zero mass has zero entropy.

So for de Sitter space one would expect to use the relation

\[ \frac{dS_{dS}}{dE_{dS}} = \frac{1}{T_{dS}} \tag{62} \]

to find the entropy \( S_{dS} \). The problem in de Sitter space is that once the coupling constant of the theory is chosen there is just one de Sitter solution, whereas in the black hole case there is a whole one parameter family of solutions labeled by the mass of the black hole, for fixed coupling constant. In other words, what is \( E_{dS} \) in (62)? One might try to vary the cosmological constant, but that is rather unphysical as it is the coupling constant. One would be going from one theory to another instead of from one configuration in the theory to another configuration in the same theory.

Let us instead follow Gibbons and Hawking [3] and use the one parameter family of Schwarzschild-de Sitter solutions to see how the temperature varies as a function of the parameter \( E \) labeling the mass of the black hole.

**Exercise 3.** The \( SdS_3 \) solution in static coordinates is

\[ ds^2 = -(1 - 8GE - r^2)dt^2 + \frac{dr^2}{(1 - 8GE - r^2)} + r^2 d\phi^2. \tag{63} \]

Find a Green function for \( SdS_3 \) by analytic continuation from the smooth Euclidean solution. Show that this Green function is periodic in imaginary time with periodicity

\[ \tau \rightarrow \tau + \frac{2\pi i}{\sqrt{1 - 8GE}}. \tag{64} \]
From the exercise and the discussion in the previous section we conclude that the temperature associated with the Schwarzschild-de Sitter solution is

\[ T_{\text{SdS}} = \frac{\sqrt{1 - 8GE}}{2\pi}. \]  

(65)

Using the formula

\[ \frac{dS_{\text{SdS}}}{dE} = \frac{1}{T_{\text{SdS}}}, \]  

(66)

and writing the result in terms of the area \( A_H \) of the de Sitter horizon at \( r_H = \sqrt{1 - 8GE} \) which is given by

\[ \sqrt{1 - 8GE} = \frac{A_H}{2\pi}, \]  

(67)

one finds that the entropy is equal to

\[ S_{\text{SdS}} = -\frac{A_H}{4G}. \]  

(68)

This differs by a minus sign from the famous formula (1)! What did we do wrong? Gibbons and Hawking suggested that to get the de Sitter entropy we should use not (66) but instead

\[ \frac{dS_{\text{SdS}}}{d(-E_{\text{dS}})} = \frac{1}{T_{\text{SdS}}}. \]  

(69)

This looks funny but in fact there is a very good reason for using this new formula.

The de Sitter entropy, although we don’t know exactly how to think about it, is supposed to correspond to the entropy of the stuff behind the horizon which we can’t observe. Now in general relativity the expression for the energy on a surface is the integral of a total derivative, which reduces to a surface integral on the boundary of the surface, and hence vanishes on any closed surface. Consider a closed surface in de Sitter space such as the one shown in figure 8. If we put something with positive energy on the south pole, then necessarily there will be some negative energy on the north pole. This can be seen quite explicitly in the Schwarzschild-de Sitter solution. With no black hole, the spacelike slice in figure 8 is an \( S^2 \), but we saw in one of the exercises that in the SdS\(_3\) solution there is a positive deficit angle at both the north and south poles. If we ascribe positive energy to the positive deficit angle at the south pole, then because the Killing vector \( \partial/\partial t \) used to define the energy changes direction across the horizon, we are forced to ascribe negative energy to the positive deficit angle at the north pole.

Therefore the northern singularity of Schwarzschild-de Sitter behind the horizon actually carries negative energy. In (66) we varied with respect to the energy at the south pole, and ended up with the wrong sign in (68), but if we
Figure 8: The energy associated to the Killing vector $\partial/\partial t$ (indicated by the arrows) along the spacelike slice $t = 0$ (solid line) must vanish. If we ascribe positive energy to a positive deficit angle at the south pole, then we must ascribe negative energy to a positive deficit angle at the south pole since the Killing vector $\partial/\partial t$ runs in the opposite direction behind the horizon.

more sensibly vary with respect to the energy at the north pole, then we should use the formula (69). Then we arrive at the entropy for Schwarzschild-de Sitter

$$S_{\text{SdS}} = A_H \frac{4H}{8G} = \frac{\pi}{2G} \sqrt{1 - 8GE}. \quad (70)$$

The integration constant has been chosen so that the entropy vanishes for the maximal energy $E = \frac{1}{8G}$ at which value the deficit angle is $2\pi$ and the space has closed up.

In conclusion we see that the area-entropy law (1) indeed applies to three dimensional Schwarzschild-de Sitter.

4 Quantum Gravity in De Sitter Space

So far we have discussed well established and understood results about classical de Sitter space and quantum field theory in a fixed de Sitter background. Now we turn to the more challenging problem of quantum gravity in de Sitter space, about which little is established or understood.

In this section we will give a pedagogical discussion of several aspects of some recent efforts in this direction [23] (which followed earlier work for example [7,12,9,22]). We will argue that quantum gravity in dS$_3$ can be described by a two dimensional conformal field theory, in the sense that correlation functions of an operator $\phi$ inserted at points $x_i$ on $I^-$ or $I^+$ are generated by a two dimensional Euclidean CFT:

$$\langle \phi(x_1) \cdots \phi(x_i) \rangle_{\text{dS}_3} \leftrightarrow \langle O_\phi(x_1) \cdots O_\phi(x_i) \rangle_{\text{S}^2}, \quad (71)$$

where $O_\phi$ is an operator in the CFT associated to the field $\phi$. Equation (71) expresses the dS/CFT correspondence. The tool which will allow us to reach this
conclusion is an analysis of the asymptotic symmetry group for gravity in dS$_3$. Parallel results pertain in arbitrary dimension, but the three dimensional case is the richest because of the infinite dimensional nature of the I$^\pm$ conformal group. The results of this section are largely contained in [23] except for the derivation of the asymptotic boundary conditions for dS$_3$, which were assumed/guessed without derivation in [23].

4.1 Asymptotic Symmetries

Consider a simple U(1) gauge theory in flat Minkowski space. A gauge transformation which goes to zero at spatial infinity will annihilate physical states (this is just the statement that a physical state is gauge invariant), while a gauge transformation which goes to a constant at spatial infinity will act nontrivially on the states. In fact the generator will be proportional to the charge operator, by Noether's theorem.

It is useful therefore to consider the so-called asymptotic symmetry group (ASG), which is defined as the set of allowed symmetry transformations modulo the set of trivial symmetry transformations.

\[
\text{ASG} = \frac{\text{Allowed Symmetry Transformations}}{\text{Trivial Symmetry Transformations}}.
\]

Here ‘allowed’ means that the transformation is consistent with the boundary conditions that we have specified for the fields in the theory, and ‘trivial’ means that the generator of the transformation vanishes after we have implemented the constraints—for example asymptotically vanishing gauge transformations in the example of the previous paragraph. The states and correlators of the theory clearly must lie in representations of the ASG. Of course one must know the details of the theory to know which representations of the ASG actually appear, but in some cases a knowledge of the ASG already places strong constraints on the theory.

In this section we will see that the ASG of quantum gravity in dS$_3$ is the Euclidean conformal group in two dimensions. Since this group acts on I$^\pm$, this means that correlators with points on I$^\pm$ are those of a conformal field theory, and the correspondence (71) is simply an expression of diffeomorphism invariance of the theory. Although we will not learn anything about the details of this theory, the fact that the conformal group in two dimensions is infinite dimensional already strongly constrains the physics.

In quantum gravity the relevant gauge symmetry is diffeomorphism invariance, and in de Sitter space the only asymptotia are I$^\pm$. Therefore we need to consider diffeomorphisms in dS$_3$ which preserve the boundary conditions on the metric at I$^\pm$ but do not fall off so fast that they act trivially on physical states. The analogous problem for three dimensional anti-de Sitter space was solved long ago by Brown and Henneaux [55]. The result for de Sitter differs only by a few signs. However the physical interpretation in the dS$_3$ case is very different from that of AdS$_3$, and remains to be fully understood.
4.2 De Sitter Boundary Conditions and the Conformal Group

Our first task is to specify the boundary conditions appropriate for an asymptotically dS$_3$ spacetime. In general specification of the boundary conditions is part of the definition of the theory, and in principle there could be more than one choice. However if the boundary conditions are too restrictive, the theory will become trivial. For example in 4d gravity, one might try to demand that the metric fall off spatially as $\frac{1}{r^2}$. This would allow only zero energy configurations and hence the theory would be trivial. On the other hand one might try to demand that it fall off as $\frac{1}{\sqrt{r}}$. Then the energy and other symmetry generators are in general divergent, and it is unlikely any sense can be made of the theory. So the idea is to make the falloff as weak as possible while still maintaining finiteness of the generators.

Hence we need to understand the surface integrals which generate the diffeomorphisms of dS$_3$. A convenient and elegant formalism for this purpose was developed by Brown and York [56, 57] (and applied to AdS$_3$ in [58]). They showed that bulk diffeomorphisms are generated by appropriate moments of a certain stress tensor which lives on the boundary of the spacetime.$^3$ We will define an asymptotically dS$_3$ spacetime to be one for which the associated stress tensor, and hence all the symmetry generators, are finite.

The Brown-York stress tensor for dS$_3$ with $\ell = 1$ is given by

$$T_{\mu\nu} = \frac{1}{4G}[K_{\mu\nu} - (K + 1)\gamma_{\mu\nu}].$$

(73)

Here $\gamma$ is the induced metric on the boundary $\mathcal{I}^-$ and $K$ is the trace of the extrinsic curvature $K_{\mu\nu} = -\nabla_{(\mu}n_{\nu)} = -\frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu}$ with $n^\mu$ the outward-pointing unit normal. (73) vanishes identically for vacuum dS$_3$ in planar coordinates

$$ds^2 = -dt^2 + e^{-2t}dzd\bar{z}.$$  

(74)

For a perturbed metric $g_{\mu\nu} + h_{\mu\nu}$ we obtain the Brown-York stress tensor

$$T_{zz} = \frac{1}{4G}\left[h_{zz} - \partial_z h_{tz} + \frac{1}{2} \partial_t h_{zz}\right] + \mathcal{O}(h^2),$$

$$T_{z\bar{z}} = \frac{1}{4G}\left[\frac{1}{2}e^{-2t}h_{tt} - h_{zz} + \frac{1}{2}(\partial_z h_{tz} + \partial_{\bar{z}} h_{t\bar{z}} - \partial_t h_{z\bar{z}})\right] + \mathcal{O}(h^2).$$  

(75)

Details of this calculation are given in appendix A. Requiring the stress tensor to be finite evidently leads to the boundary conditions

$$g_{z\bar{z}} = \frac{e^{-2t}}{2} + \mathcal{O}(1),$$

$$g_{tt} = -1 + \mathcal{O}(e^{2t}),$$

$$g_{zz} = \mathcal{O}(1),$$

$^3$Brown and York mainly consider a timelike boundary, but their results can be extended to the spacelike case.
\[ g_{tz} = \mathcal{O}(1). \quad (76) \]

It is not hard to see that the most general diffeomorphism \( \zeta \) which preserves the boundary conditions (76) may be written as

\[ \zeta = U \partial_z + \frac{1}{2} U' \partial_t + \mathcal{O}(e^{2t}) + \text{complex conjugate}, \quad (77) \]

where \( U = U(z) \) is holomorphic in \( z \).\(^4\) A diffeomorphism of the form (77) acts on the Brown-York stress tensor as

\[ \delta \zeta T_{zz} = -U \partial T_{zz} - 2U' T_{zz} - \frac{1}{8G} U'''. \quad (78) \]

The first two terms are those appropriate for an operator of scaling dimension two. The third term is the familiar linearization of the anomalous Schwarzian derivative term corresponding to a central charge

\[ c = \frac{3l}{2G}, \quad (79) \]

where we have restored the power of \( \ell \).\(^5\) Note that the \( \mathcal{O}(e^{2t}) \) terms in (77) do not contribute in (78). Therefore they are trivial diffeomorphisms, in the sense described above. We conclude that the asymptotic symmetry group of \( \text{dS}_3 \) as generated by (77) is the conformal group of the Euclidean plane.

The last boundary condition (76) differs from the condition \( g_{tz} = \mathcal{O}(e^{2t}) \) assumed in [23] and obtained by analytically continuing the \( \text{AdS}_3 \) boundary conditions of Brown and Henneaux [55] from anti-de Sitter to de Sitter space. The resolution of this apparent discrepancy comes from noting that if \( g_{tz} \rightarrow f \) on the boundary where \( f = f(z, \bar{z}) \) is an arbitrary function, then applying the diffeomorphism \( \zeta = e^{2t} f \partial \bar{z} \) gives \( \delta \zeta g_{tz} = \mathcal{O}(e^{2t}) \). Therefore one can always set the component \( g_{tz} \) of the metric to be \( \mathcal{O}(e^{2t}) \) with a trivial diffeomorphism. In other words, if \( g_{tz} = \mathcal{O}(1) \), then in fact one can always choose a gauge in which \( g_{tz} = \mathcal{O}(e^{2t}) \). Exploiting this freedom one can impose the asymptotic boundary conditions

\[
\begin{align*}
g_{zz} &= \frac{e^{-2t}}{2} + \mathcal{O}(1), \\
g_{tt} &= -1 + \mathcal{O}(e^{2t}), \\
g_{zz} &= \mathcal{O}(1), \\
g_{tz} &= \mathcal{O}(e^{2t}),
\end{align*}
\]

as given in [23].

A special case of (77) is the choice

\[ U = \alpha + \beta z + \gamma z^2, \quad (81) \]

\(^4\)We allow isolated poles in \( z \). In principle this should be carefully justified (as (76) is violated very near the singularity), and we have not done so here. A parallel issue arises in \( \text{AdS}_3/\text{CFT}_2 \).

\(^5\)Parallel derivations of the central charge for \( \text{AdS} \) were given in [58, 59].
where $\alpha$, $\beta$, $\gamma$ are complex constants. In this case $U'''$ vanishes, and the dS$_3$ metric is therefore invariant. These transformations generate the SL(2, $\mathbb{C}$) global isometries of dS$_3$.

Where do conformal transformations come from? Recall that a conformal transformation in two dimensions is a combination of an ordinary diffeomorphism and a Weyl transformation. In two dimensions a diffeomorphism acts as

$$g_{zz} \rightarrow \frac{dz'}{dz} \frac{d\bar{z}'}{d\bar{z}} g_{z'\bar{z}'} ,$$

and a Weyl transformation acts as

$$g_{zz} \rightarrow e^{2\phi} g_{zz} .$$

A conformal transformation is just an ordinary diffeomorphism (82) followed by a Weyl transformation (83) with $\phi$ chosen so that $g_{zz} \rightarrow g_{z'z'}$ under the combined transformation.

Now if we look at what the diffeomorphism $\zeta$ defined in (77) does, we see that the first term $U \partial_z$ generates a holomorphic diffeomorphism of the plane. Now the form of the metric (74) makes it clear that this can be compensated by a shift in $t$, which from the point of view of the $z$-plane is a Weyl transformation. This accounts for the second term $\frac{1}{2} U \partial_t$ in $\zeta$. So a diffeomorphism in dS$_3$ splits into a tangential piece, which acts like an ordinary diffeomorphism of the complex plane, and a normal piece, which acts like a Weyl transformation. A three dimensional diffeomorphism is thereby equivalent to a two dimensional conformal transformation.

Since $U(z)$ was arbitrary, we conclude that the asymptotic symmetry group of gravity in dS$_3$ is the conformal group of the complex plane. The isometry group is the SL(2, $\mathbb{C}$) subgroup of the asymptotic symmetry group. In particular, the ASG is infinite dimensional, a fact which highly constrains quantum gravity on dS$_3$. This is particular to the three dimensional case, since in higher dimensional de Sitter space the ASG is the same as the isometry group SO($d, 1$).

We conclude these lectures with a last

Exercise 4.

(a) Find an example of string theory on de Sitter space.
(b) Find the dual conformal field theory.

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A Calculation of the Brown-York Stress Tensor

We wish to calculate the Brown-York stress tensor (73) for a metric which is a small perturbation of $dS_3$. We write the metric in planar coordinates $(t, z, \bar{z})$ as

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + e^{-2t}dzd\bar{z} + h_{\mu\nu}dx^\mu dx^\nu, \]

and we will always drop terms of order $O(h^2)$. We can put (84) into the form

\[ ds^2 = -Ndt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \]

where the lapse and shift functions are given by

\[ N = 1 - \frac{1}{2}h_{tt}, \quad N^z = 2e^{2t}h_{t\bar{z}}, \quad N^{\bar{z}} = 2e^{2t}h_{tz}, \]

and the induced metric on the boundary $\mathcal{I}^-$ is

\[ \gamma_{zz} = h_{zz}, \quad \gamma_{z\bar{z}} = \frac{1}{2}e^{-2t} + h_{z\bar{z}}, \quad \gamma_{\bar{z}\bar{z}} = h_{\bar{z}\bar{z}}. \]

The outward pointing unit normal vector to the boundary is

\[ n^\mu = \frac{1}{N}(-1, N^z, N^{\bar{z}}) = \left(-1 - \frac{1}{2}h_{tt}, 2e^{2t}h_{t\bar{z}}, 2e^{2t}h_{tz}\right). \]

Upon lowering the indices, we have

\[ n_\mu = \left(1 - \frac{1}{2}h_{tt}, 0, 0\right) \]

and we use the formula $K_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ to obtain

\[ K_{zz} = -\partial_z h_{zz} + \frac{1}{2}\partial_t h_{zz}, \]
\[ K_{z\bar{z}} = -\frac{1}{2}e^{-2t}(1 + \frac{1}{2}h_{tt}) - \frac{1}{2}(\partial_z h_{t\bar{z}} + \partial_\bar{z} h_{tz} - \partial_t h_{z\bar{z}}). \]

The trace is

\[ K = g^{\mu\nu}K_{\mu\nu} = \gamma^{ij}K_{ij} = -2 - h_{tt} + 4e^{2t}h_{z\bar{z}} - 2e^{2t}(\partial_z h_{t\bar{z}} + \partial_\bar{z} h_{tz} - \partial_t h_{z\bar{z}}). \]

Plugging (90) and (91) into (73) gives the desired result (75).

References


