Dynamics near the critical point: the hot renormalization group
in quantum field theory

I. INTRODUCTION

The experimental possibility of studying the phase transition of QCD via ultrarelativistic heavy ion collisions
in the current effort at RHIC and the forthcoming program at LHC motivates a theoretical effort to understand the
dynamical aspects of phase transitions at high temperature. QCD is conjectured to feature two phase transitions in
the critical slowing down regime, with the spatial and time directions of critical slowing down being
space \( T \sim T^c \) and \( \xi \sim (T - T^c)^{\frac{1}{4}} \), respectively. An ideal gas limit of two phase transitions in
the limit of high temperature and an \( \xi \) vanishing in the Landau-Ginzburg-Devonshire formalism with
\( \xi \sim (T - T^c)^{\frac{1}{4}} \).

The critical point we find that the depression of the condensation temperature \( T^c \) in the Landau-Ginzburg-Devonshire formalism
is \( T^c \sim T^0 \), while the critical field \( \xi \) vanishes.

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to include conservation laws[3, 4]. In the simplest case of a non-conserved order parameter, such as in a scalar field theory with discrete (ising-like) symmetry, the time dependent Landau Ginzburg equation is purely dissipative. While phenomenological, this approach has proved very successful in a variety of experimental situations and is likely to provide a suitable description of the dynamics for macroscopic, coarse-grained systems such as binary mixtures, etc.[3]. The phenomenological approach based on the time dependent Landau-Ginzburg equations which are first order in time derivatives seem to provide a suitable description of macroscopic dynamics in non-relativistic systems. However it is clear that such an approach is not justified in a relativistic quantum field theory, since the underlying equations of motion are second order and time reversal invariant.

There are important non-equilibrium consequences of slow dynamics near critical points. If the cooling time scale is much shorter than the relaxation time scale of long-wavelength fluctuations, these freeze out and undergo spinodal instabilities when the temperature falls below the critical[3, 5] during continuous (no metastability) phase transitions. These instabilities result in the formation of correlated domains that grow in time[3, 5] with a law that in general depends on the cooling rate[6].

In ultrarelativistic heavy ion collisions during the expansion of the quark gluon plasma the critical point for the chiral phase transition may be reached. If long wavelength fluctuations freeze out shortly before the transition, the ensuing instabilities may lead to distinct event by event observables[7] in the pion distribution as well as in the photon spectrum at low energies[8].

Thus an important aspect of the chiral phase transition is to establish the relaxation time scales of long-wavelength fluctuations, and whether critical slowing down and freeze out of long-wavelength fluctuations can ensue.

In the strict chiral limit with massless up and down quarks, QCD has an SU(2)_L ⊕ SU(2)_R symmetry which is spontaneously broken to SU(2)_R at the chiral phase transition, the three pions being the Goldstone bosons associated with the broken symmetry. It has been argued that the low energy theory that describes the chiral phase transition is in the same universality class as the Heisenberg ferromagnet, i.e., the O(4) linear Sigma model[9]. This argument has been used[9] to provide an assessment of the dynamical aspects of low energy QCD based on the phenomenological time-dependent Landau Ginzburg approach to dynamical critical phenomena in condensed matter[3]. While the universality arguments are appealing, a more microscopic understanding of dynamical critical phenomena in quantum field theory is needed and has begun to emerge only recently[10, 11].

In reference[10] a Wilsonian renormalization group extended to finite temperature was implemented in a scalar quartic field theory. In this approach, only one loop diagrams enter in the computation of the beta functions, and the imaginary part of the self energy, which arises first at two loops order for T ≥ T_c, is accounted for by an imaginary part in the effective quartic coupling[10]. There it is found that the relaxation rate of zero momentum fluctuations γ reveals critical slowing down in the form γ ∼ (T − T_c)ν ln[(T − T_c)] with ν ∼ 0.53 being the critical exponent for the correlation length[10].

In reference[11] the width of quasiparticles near the critical point has been studied via the large N approximation. This study revealed that at high temperature the effective coupling is driven to a (Wilson-Fischer) fixed point, a result that is in agreement with the numerical evidence presented in reference[10]. While the results in leading order in the large N limit found in[11] hinted at critical slowing down, albeit in a manner different from the numerical evidence of reference[10], they also hinted at the breakdown of the quasiparticle picture. A conclusion in[11] is that while the large N limit provides a partial resummation of the perturbative expansion, further resummation is needed to fully address the relaxation of quasiparticles.

The large N limit in static critical phenomena presents a similar situation: while it sums the series of bubbles replacing the bare vertex by the effective coupling that is driven to the fixed point in the infrared, the self-energy still features infrared logarithms that require further resummation[4]. Such a resummation is provided by the renormalization group[4].

While our motivation for studying dynamical critical phenomena near critical points is driven by the experimental program in ultrarelativistic heavy ion collisions to study the QCD phase transitions, the underlying questions are more overarching and of a truly interdisciplinary nature. In particular we mention an impressive body of work on aspects of quantum phase transitions in condensed matter systems[12] that addresses very similar questions. The work in reference[12] focuses on understanding the static, dynamical and transport properties of low dimensional systems in the quantum regime, in which the frequency and momentum of excitations is ω; p ≫ T.

Our study in this article is complementary to that program in that we focus on the dynamical aspects of long-wavelength quasiparticles with ω; p ≪ T. As discussed in[12, 13] and in detail below this is closer to the classical regime.

The goals: in this article we study the dynamical aspects of quasiparticles near the critical point in a scalar quartic field theory by implementing a renormalization group program at high temperature. While the renormalization group has been generalized to finite temperature in various formulations[14, 15], mainly to study critical phenomena associated with finite temperature phase transitions in field theory, only static aspects were studied with these
approaches.

Instead we focus on dynamical aspects, in particular the dispersion relations, and relaxation rates of long-wavelength excitations at and near the critical point. Already at the technical level one can see the difficulties: to understand dynamical aspects, in particular relaxation, a consistent treatment of absorptive parts of self-energies is required. This aspect is notoriously difficult to implement in a Wilsonian approach to Euclidean field theory [14]. Reference [10] proposes a method to circumvent this problem but a complete treatment that manifestly includes absorptive parts of self-energy contributions is still lacking in this approach. Other approaches using the Euclidean version of the renormalization group adapted to finite temperature field theory were restricted to static quantities [15] and in fact, as it will be seen in detail below, miss important phenomena that will be at the heart of the results presented here.

**Brief summary of results:** Long-wavelength phenomena at high temperature $T$ implies a dimensional reduction from the decoupling of Matsubara modes with non-zero frequency $[16, 17]$. The coupling in the dimensionally reduced theory is $\lambda T$, where $\lambda$ is the quartic coupling. By dimensional reasons, the perturbative expansion in four space-time dimensions is in terms of the dimensionless ratio $\lambda T / \mu$ with $\mu$ the typical momentum scale, which is strongly relevant in the infrared. As a result, a perturbative approach to studying long-wavelength phenomena breaks down. This is manifest in the breakdown of the quasiparticle picture in naive perturbation theory (see [11] and below).

In 5 -- $\epsilon$ space-time dimensions, the effective coupling in the high temperature, long wavelength limit is $g(\mu) = \lambda T \mu^{-\epsilon}$. We implement an $\epsilon$ expansion around five space-time dimensions and a renormalization group resummation program at high temperature with $T \gg \mu$ near the critical point, with $s, \mu$ the typical frequency and momentum scales. We analyze the high temperature behavior of the relevant graphs and find that is dominated for $\epsilon > 0$ by the zero Matsubara mode while the sum of the nonzero modes gives subdominant contributions. The effective renormalized coupling is driven to an infrared stable fixed point $g^* = O(\epsilon)$, which for small $\epsilon$ allows a consistent perturbative expansion near the fixed point.

An important feature that emerges clearly in this approach, and that has been missed in most other treatments of renormalization group at finite temperature, is the anisotropic scaling between spatial and time directions, which is manifest in a non-trivial renormalization of the speed of light. This is a consequence of the fact that in the Euclidean formulation at finite temperature time is compactified to $0 \leq \tau \leq 1 / T$, thus space and time or momentum and frequency play different roles. This results in a novel dynamical critical exponent, $z$ which determines the anisotropic scaling. The renormalization group leads to scaling in the infrared region in terms of anomalous dimensions which can be computed systematically in the $\epsilon$ expansion. In particular, to lowest order in $\epsilon$ we find for a scalar theory with discrete symmetry $z = 1 + \frac{\epsilon}{2} + O(\epsilon^2)$. All dynamical aspects, such as the relaxation rates, depend on this critical exponent, while the static aspects are completely described by the usual critical exponents.

We provide a renormalization group analysis of the quasiparticle properties near the critical point, such as their dispersion relation and width, complemented with an explicit evaluation to lowest order in the $\epsilon$ expansion. The main results obtained in this article are the following: at $T = T_c$ we find that the dispersion relation and width of quasiparticles of momentum $p$ is $\omega_p \sim p^\zeta$ and $\Gamma_p \sim (1 - \frac{\zeta}{2}) \omega_p$ respectively with a vanishing group velocity of quasiparticles in the long wavelength limit highlighting the collective nature of the quasiparticle excitations. For $T > T_c$ but $|T - T_c| \ll T$ we find $\omega_p \sim \xi^{\frac{1}{\beta}} [1 + (p\xi)^2]^\frac{1}{2\beta}$ and $\Gamma_p \sim (1 - \frac{1}{2\beta}) \omega_p$, with $\xi$ the finite temperature correlation length $\xi \propto [T - T_c]^{-\frac{\beta}{\zeta}}$. In the case of $O(N)$ symmetry we find to lowest order in the epsilon expansion that the dynamical exponent $z = 1 + \epsilon \frac{N^2}{N + 2} + O(\epsilon^2)$.

Critical slowing down emerges near the critical point and in the $\epsilon$ expansion $\Gamma_p / \omega_p \ll 1$ confirming the quasiparticle picture.

We discuss some relevant cases of threshold singularities in which the usual (Breit-Wigner) parametrization of the quasiparticle propagator is not available since the real part of the inverse Green’s function vanishes at the quasiparticle frequency with an anomalous power law.

In section II we introduce the model and discuss the breakdown of naive perturbation theory. In section III we introduce the $\epsilon$ expansion and analyze the static case. In section IV the renormalization group aspects and the anisotropic scaling is analyzed in detail. Section V presents the renormalization group in the effective, dimensionally reduced theory both at and near the critical point. This section contains the bulk of our results which are summarized in section VI. Our conclusions and a discussion of potential implications are presented in section VII. The high temperature behavior of the relevant diagrams is computed in the appendices.

**II. THE THEORY AND THE Necessity FOR RESUMMATION**

The low energy sector of QCD with two massless (up and down) quarks is conjectured to be in the same universality class as the $O(4)$ Heisenberg ferromagnet [9] described by the $O(4)$ linear sigma model. Furthermore, since we are
interested in describing the dynamical aspects associated with critical slowing down and freeze out of long-wavelength fluctuations just before the chiral phase transition, we focus on $T \Rightarrow T^\dagger$.

While our motivation for studying critical slowing down stems from the experimental program in ultrarelativistic heavy ion collisions, the questions are of a fundamental nature.

To understand the dynamical aspects near the critical point, we focus on the simpler case of a single scalar field theory, and we will recover the case of $O(N)$ symmetry at the end of the discussion. We thus focus on the theory described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2} m^2 \Phi^2 - \frac{\lambda R}{4!} \Phi^4$$  \hspace{1cm} (II.1)

where the subscripts in the mass and coupling refer to bare quantities. The case of $N$ components in the unbroken phase $T \geq T_c$ differs from the single scalar field by combinatoric factors that change the critical exponents quantitatively. These factors will be at the end of the calculation to obtain an estimate of the critical exponents for the $O(N)$ theory in section VI.

We are interested in obtaining the relaxation properties of long wavelength excitations near the critical temperature, which in this scalar theory $T_c \sim |m_R|/\sqrt{\lambda_R}$ with $m_R$; $\lambda_R$ being the renormalized mass and coupling. Thus the regime of interest for this work is $p, \omega \ll T \sim T_c$ with $p, \omega$ being the momentum and frequency of the long wavelength excitation. As it will become clear below it is convenient to work in the Matsubara representation of finite temperature field theory, which is more amenable to the implementation of the (Euclidean) renormalization group.

In the Matsubara formulation Euclidean time $\tau$ is compactified in the interval $0 \leq \tau \leq \beta = 1/T$ whereas space is infinite, bosonic fields are periodic in Euclidean time and can be expanded as $[18, 19, 20]$

$$\Phi(\vec{p}, \tau) = \frac{1}{\sqrt{\beta V}} \sum_{\omega_n = 2\pi nT}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \phi(\vec{p}, \omega_n) e^{-i\omega_n \tau + i\vec{p} \cdot \vec{x}}$$  \hspace{1cm} (II.2)

$$\omega_n = 2\pi nT ; \quad n = 0, \pm 1, \pm 2 \ldots$$  \hspace{1cm} (II.3)

Thus we see that while the spatial momentum is a continuum variable, the Matsubara frequencies are discrete as a consequence of the compactification of Euclidean time. This feature of Euclidean field theory at finite temperature will be seen to lead to anisotropic rescaling between space and time and therefore, as it will be clear below, new dynamical critical exponents. Anticipating anisotropic rescaling, we then introduce the bare speed of propagation $v_\parallel$ of excitations in the medium by writing the Euclidean Lagrangian in the form

$$\mathcal{L}_E = \frac{1}{2} \frac{(\partial_\mu \Phi)^2}{v_\parallel^2} + \frac{1}{2} (\nabla \Phi)^2 + \frac{1}{2} M^2(T) \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 + \frac{1}{2} \delta m^2(T) \Phi^2 + \frac{\delta \lambda}{4!} \Phi^4$$  \hspace{1cm} (II.4)

where we have introduced the effective renormalized temperature dependent mass $M(T)$ and the counterterms, in particular the mass counterterm $\delta m^2(T) = -M^2(T) - m^2_\parallel$ is adjusted order by order in perturbation theory so that the inverse two point function obeys

$$\Gamma^{(2)}(\vec{p} = 0; \omega_0 = 0) = M^2(T)$$  \hspace{1cm} (II.5)

The critical point is defined with (the inverse susceptibility) $M^2(T) \equiv 0$. We will begin our study by focusing our attention on the critical theory for which $M(T) = 0$. We will later consider the theory near the critical point but in a regime in which $M(T) \ll T \sim T_c$. Thus, the general regime to be studied is $\vec{p}, \omega, M(T) \ll T \sim T_c$.

A. Infrared behavior of the critical theory: static limit in three space dimensions

In order to highlight the nature of the infrared behavior when $\vec{p}, \omega \ll T$ we focus first on the critical theory in the static limit, when the Matsubara frequencies of all external legs in the n-point functions vanish. For the purpose of understanding the nature of the infrared physics in the static limit we will set $v_\parallel = 1$ in this and next section, and we will recover this variable, the speed of light, when we study the dynamics in section IV.

1. The scattering amplitude in $D = 3$

We consider first the $2 \rightarrow 2$ scattering amplitude, or four point function, to one-loop order in three spatial dimensions. The full expression is given by

$$\Gamma^{(4)}(\vec{p}_1, s_1, \vec{p}_2, s_2, \vec{p}_3, s_3, \vec{p}_4, s_4) = -\lambda_\parallel + \lambda_\parallel^2 [H(\vec{p}_1 + \vec{p}_2, s_1 + s_2) + H(\vec{p}_1 + \vec{p}_3, s_1 + s_3)$$
\begin{align}
\frac{1}{2} \sum_{n \in \mathbb{Z}} \int \frac{d^4 q}{(2\pi)^3} \frac{1}{\left[(q^2 + (2\pi T n)^2)\left[(q + \tilde{p}_i)^2 + (2\pi T)^2 (n + m)^2\right]\right]},
\end{align}

where \( s_i = 2\pi T m_i \) with \( 1 \leq i \leq 4 \) and \( m_i \in \mathbb{Z} \).

\begin{align}
H(p, s) \equiv \frac{T}{2} \sum_{n \in \mathbb{Z}} \int \frac{d^4 q}{(2\pi)^3} \frac{1}{\left[(q^2 + (2\pi T n)^2)\left[(q + \tilde{p}_i)^2 + (2\pi T)^2 (n + m)^2\right]\right]},
\end{align}

\( s \equiv 2\pi T m \). Since the external momentum \( p \ll T \) it is clear from the above expression that the dominant infrared behavior of \( H(p, 0) \) is determined by the zero Matsubara frequency in the sum. As it will be explicitly shown below the contribution from the non-zero Matsubara frequencies will introduce a renormalization of the bare coupling which in the limit \( T \gg p \) is independent of the external momentum (this will be seen explicitly in the next section). Keeping only the zero internal Matsubara frequency and carrying out the three dimensional integral explicitly we find

\begin{align}
H_0(p, 0) = \frac{T}{16p}
\end{align}

Thus, defining the effective coupling constant at the symmetric point \( \tilde{p}_i = \tilde{P}_i \) where

\begin{align}
\tilde{P}_i \cdot \tilde{P}_j = (4 \delta_{ij} - 1) \frac{\mu^2}{4},
\end{align}

in the static limit one finds that in the infrared limit \( \mu / T \ll 1 \)

\begin{align}
\lambda_{\text{eff}}(\mu) = \lambda_0 \left[ 1 - \frac{3\lambda_0 T}{16\mu} \right] + \mathcal{O}(\lambda_0^3)
\end{align}

Two important features transpire from this expression: i) the factor \( T / \mu \) can be explained by dimensional arguments; in the Matsubara formulation for each loop there is a factor \( T \) from the sum over internal Matsubara frequencies. The infrared behavior for \( \mu \ll T \) is obtained by considering only the zero internal Matsubara frequency in the loop. This integral has only one scale and since \( \lambda \) is dimensionless in three spatial dimensions the one-loop contribution must be proportional to \( T / \mu \). A similar argument shows that for a diagram with \( m \) internal loops, and transferred momentum scale \( \mu \) there will be a power \( T^m \) from the Matsubara sums, the infrared behavior is obtained by the contribution with all the internal Matsubara frequencies equal to zero, which by dimensional power counting must be of the form \( (T / \mu)^m \). Therefore a diagram with \( m \)-internal loops will contribute to the scattering amplitude with \( \lambda(T / \mu)^m \). In taking only the zero internal Matsubara frequency we are assuming that the internal loop momenta are cut-off at a scale \( < T \).

Thus, at the critical point the most important infrared behavior is that of the dimensionally reduced three dimensional theory\cite{16, 17}. The reason for this dimension reduction is clear: at finite temperature \( T \) the Euclidean time is compactified to a cylinder of radius \( L = 1 / T \) for transferred momenta \( \mu \) the spatial resolution is on distances \( d \sim 1 / \mu \). Therefore for \( \mu \ll T \rightarrow d \gg L \) thus the compactification radius is effectively zero insofar as the long distance (infrared) physics is concerned.

We will study below the contribution from the non-zero Matsubara frequencies. ii) for a transferred momentum scale \( \mu \) perturbation theory breaks down for \( \mu \ll \lambda T \) since the contribution from higher orders is of the form \( \lambda(T / \mu)^m \). This suggests that a resummation scheme is needed to study the infrared limit. This situation is similar to that in critical phenomena where infrared divergences must be summed and the renormalization group provides a consistent and systematic resummation procedure. We can then obtain a hint of how to implement the renormalization group in finite temperature field theory in the limit when \( T \) is much larger than any other scale (masses, momenta and frequency) by realizing that, from the argument presented above, the perturbative expansion is actually in terms of the dimensionless coupling \( g_0 = \lambda T / \mu \). Therefore from (II.10) we can write

\begin{align}
g_{\text{eff}}(\mu) = g_0 \left[ 1 - \frac{3}{16} g_0 \right] + \mathcal{O}(g_0^3)
\end{align}

We can improve the scattering amplitude via the renormalization group by considering the RG-\( \beta \) function

\begin{align}
\beta_\gamma = \mu \frac{\partial g_{\text{eff}}(\mu)}{\partial \mu} \lambda_{\text{eff}} = -g_{\text{eff}} + \frac{3}{8} g_{\text{eff}}^2 + \mathcal{O}(g_{\text{eff}}^3)
\end{align}

The first term (with the minus sign) just displays the scaling dimension (for fixed \( \lambda T \)) of the effective coupling, that this dimension is \(-1\) is a consequence of the dimensional reduction since \( \lambda T \) is the effective dimensionful coupling of the three dimensional theory.
We thus see that the renormalization group improved coupling runs to the infrared fixed point

\[ g^* = \frac{8}{3} \]  

as the momentum scale \( \mu \to 0 \). Comparing with the renormalization group beta function of critical phenomena [4, 21, 22, 23, 24, 25] we see that this is the Wilson-Fischer fixed point in three dimensions, again revealing the dimensional reduction of the low energy theory. The resummation of the effective coupling and the fixed point structure can also be understood in the large \( N \) limit [11]. As described in [11] the large \( N \) limit can be obtained by replacing the interaction in the Lagrangian density for \([26]\)

\[ \mathcal{L}_{\text{int}} = \frac{\lambda}{2N} \left( \Phi \cdot \tilde{\Phi} \right)^2 \]  

(II.14)

with \( \Phi = (\phi_1, \ldots, \phi_N) \), and the form of the quartic coupling has been chosen for consistency with the notation of reference [11]. The leading order in the large \( N \) limit for the scattering amplitude is obtained by summing the geometric series of one-loop bubbles in the s-channel (only this channel out of the three contribute to leading order in the large \( N \) limit), each one proportional to \( N \) which is the number of fields in the loop. As a result one finds that the effective scattering amplitude at a momentum transfer \( \mu \) is given by [11]

\[ \tilde{\lambda}_{\text{eff}}(\mu) = \frac{\lambda}{1 + \frac{\mu^2}{\Lambda^2}} \]  

(II.15)

Thus, introducing the dimensionless effective coupling \( \tilde{\lambda}_{\text{eff}}(\mu) = \tilde{\lambda}_{\text{eff}}(\mu)(T/\mu) \) one finds that

\[ \lim_{\mu \to 0} \tilde{\lambda}_{\text{eff}}(\mu) = 4 \]  

(II.16)

i.e, the effective coupling constant goes to the three dimensional fixed point [11].

2. The two point function in \( D = 3 \):

The two point function in the static limit is given by

\[ \Gamma^{(2)}(p, 0) = p^2 + \delta m^2(T) - \Sigma(p, 0) + \mathcal{O}(\lambda^3) \]  

(II.17)

where \( \Sigma(p, 0) \) stands for the two loop sunset diagram at zero external Matsubara frequency and the counterterm \( \delta m^2(T) \) will cancel the momentum independent, but temperature dependent parts of the self-energy. The two loop self energy for external momentum \( \vec{p} \) and Matsubara frequency \( \omega_m = 2\pi T m \) is given by

\[ \Sigma(p, \omega_m) = \frac{\lambda^2 T^2}{6} \sum_{l, j = 2} \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{[q^2 + \omega_m^2][k^2 + \omega_m^2][\vec{p}^2 + \vec{k}^2 + \vec{q}^2 + (\omega_l + \omega_j + \omega_m)^2]} \]  

(II.18)

with \( \omega_j = 2\pi j T \). The static limit is obtained by setting \( \omega_m = 0 \) (\( m = 0 \)). In this limit the dominant contribution in the infrared for \( T \gg p \) arises from the term \( l = j = 0 \) in the sum. The terms \( l \neq 0 \), \( j \neq 0 \) for which we can take \( p = 0 \) (since \( p \ll T \) will be cancelled by the counterterm. A straightforward calculation leads to

\[ \Gamma^{(2)}(p, 0)|_R = p^2 \left[ 1 + \frac{\lambda^2 T^2}{12(4\pi)^2} \ln \left( \frac{p^2}{\mu^2} \right) + \mathcal{O}(\lambda^3) \right] \]  

(II.19)

where \( \mu \) is a renormalization scale. This expression clearly reveals the effective coupling \( \lambda T/\mu \) which becomes very large in the limit \( p \ll \Lambda T \). Clearly we need to implement a resummation scheme that will effectively replace the bare dimensionless coupling constant by the effective coupling that goes to a fixed point in the long-wavelength limit, and also ensure that at this fixed point the effective coupling is small so that perturbation theory near this fixed point is reliable. This is precisely what the renormalization group combined with the \( \epsilon \)-expansion achieves in critical phenomena [4, 21, 22, 23, 24, 25].
B. Dynamics in $D = 3$

The two-loop contribution to the self-energy for $\omega \neq 0$ is obtained from a dispersive representation of the self-energy in terms of the spectral density

$$\Sigma(p, \omega) = \int \frac{d^D \nu}{(2\pi)^D} \rho(p, \nu) \frac{\omega - \nu}{\omega - \nu + i0^+}. \quad \text{(II.20)}$$

The spectral density $\rho(p, \nu)$ has been obtained in reference [11] in the high temperature limit in $D = 3$. Using the expression given in [11] for the spectral density at two loops in the high temperature limit, and after some lengthy but straightforward algebra, we find

$$\rho(p, \nu) = \frac{\pi}{12} \left( \frac{\lambda T}{4\pi} \right)^2 \left[ \Theta(|\nu| - p) + \frac{|\nu|}{p} \Theta(p - |\nu|) \right]. \quad \text{(II.21)}$$

Carrying out the dispersive integral (II.20) and subtracting off the terms that are independent of $p, \omega$ that are absorbed by the counterterm we find

$$\Sigma(p, \omega) = -\frac{1}{12} \left( \frac{\lambda T}{4\pi} \right)^2 \left[ \ln \left( \frac{p^2 - (\omega + i0^+)^2}{\mu^2} \right) - \frac{\omega}{p} \ln \left( \frac{\omega + i0^+ - p}{\omega + i0^+ + p} \right) \right]. \quad \text{(II.22)}$$

Clearly, the static limit $\omega \to 0$ of the self-energy coincides with (II.19). The two point function is therefore given by

$$\Gamma^{(2)}(p, \omega) \approx p^2 - \omega^2 + \frac{1}{12} \left( \frac{\lambda T}{4\pi} \right)^2 \left[ \ln \left( \frac{p^2 - \omega^2}{\mu^2} \right) - \frac{\omega}{p} \ln \left( \frac{\omega - p}{\omega + p} \right) \right] \quad \text{(II.23)}$$

with $\rho(p, \omega)$ given by (II.21).

There are several features of this expression that are noteworthy:

- It is clear that for $\lambda T \gg p, \omega$ the two-loop contribution is much larger than the tree level term $p^2 - \omega^2$, this already signals the breakdown of perturbation theory in the high temperature regime when $\lambda T \gg p, \omega$ in the dynamical case.

- Consider the real part of the two point function as $\omega \to p$ i.e., near the mass shell.

$$\text{Re} \Gamma^{(2)}(p, \omega \approx p) \approx 2p^2 \left\{ 1 - \frac{\omega}{p} \right\} \left[ 1 + \frac{1}{24} \left( \frac{\lambda T}{4\pi} \right)^2 \ln \left( \frac{\omega - p}{\mu} \right) \right] + \frac{1}{12} \left( \frac{\lambda T}{4\pi} \right)^2 \ln \left( \frac{2p}{\mu} \right) \quad \text{(II.24)}$$

This expression reveals that $\omega = p$ is not the position of the mass shell of the (quasi-)particle. The coefficient of $(1 - \omega/p)$ hints at wave function renormalization but the fact that the two point function does not vanish at this point prevents such identification. Furthermore, we see that the term that does not vanish at $\omega = p$ hints at a momentum dependent shift of the position of the pole, i.e., a correction to the dispersion relation. However for $\lambda T/p \gg 1$ both contributions are non-perturbatively large and the analysis is untrustworthy.

- Now consider the width of the quasi-particle,

$$\gamma_p = -\frac{\text{Im} \Sigma(p, \omega \approx p)}{2p} = \frac{\pi p}{12} \left( \frac{\lambda T}{4\pi} \right)^2 \quad \text{(II.25)}$$

so that $\gamma_p/p \gg 1$ for $\lambda T/p \gg 1$. This signals the breakdown of the quasiparticle picture.

A similar analysis reveals the breakdown of perturbation theory away from near the critical point with $|T - T_c| \ll T$. The imaginary part of the two-loops self-energy at $\vec{p} = 0$ and in terms of the temperature dependent mass $m_R(T)$ can be obtained straightforwardly in three spatial dimensions in the limit $T \gg m(T)$. It is found to be

$$\text{Im} \Sigma^{(2)}(\vec{p} = 0, \omega = m_R(T)) \propto \lambda^2 T^2 \quad \text{(II.26)}$$

consequently, at two-loops order the width of the zero momentum quasiparticle in three spatial dimensions is given by

$$\Gamma \propto \frac{\lambda^2 T^2}{m_R(T)} \gg m_R(T) \quad \text{(II.27)}$$
This behavior is different from that of gauge theories at high temperature where low order fermion or gauge boson loops are infrared safe and determined by the hard thermal loop contributions[28]. This is so because for fermions there is no zero Matsubara frequency while in the case of gauge bosons, the vertices are momentum dependent. While the second term inside the bracket in expression (II.23) is determined by Landau damping is infrared finite and is similar to the leading contribution in the hard thermal loop program[19, 28], the first term arises from the three particle cut. The dependence of this term on the renormalization scale µ arises from the subtraction of the mass term at the critical point and reveals the infrared behavior. Furthermore, in the hard thermal loop program[19, 28] one finds thermal masses of order gT with g the gauge coupling, and widths of order g2T (up to logarithms) so that Γp/ωp ≪ 1 in the weak coupling limit, while in the scalar theory under consideration (II.27) suggests that Γp/ωp ≫ 1 in naive perturbation theory.

We note at this stage that the high temperature limit p, ω ≪ T of the self-energy calculated from the spectral representation (II.20) can be directly obtained by computing Σ(p, ωm) in the Matsubara representation given by (II.18) by setting the internal Matsubara frequencies ωk = ωm = 0 and analytically continuing ωm → −ω + i0+.

We highlight this observation since it will be the basis of further analysis in what follows: the high temperature limit of the self energy p, ω ≪ T can be obtained by setting the internal Matsubara frequencies to zero and analytically continuing in the external Matsubara frequency, i.e.,

\[ Σ(p, ω)|_{p, ω ≪ T} ≡ \frac{λ^2 T^2}{6} \int \frac{d^3 q}{(2π)^3} \frac{d^3 k}{(2π)^3} \frac{1}{q^2 + k^2 + q^2 + ω_m^2} \]  

(II.28)

We provide one and two loops examples of this statement in appendices A and B and formal proof of this statement to one loop order in appendix C.

The result for the width of the quasiparticle at two loops was anticipated in references[11, 13]. This width is infrared classical since the product λT is independent of h(11). This result for the damping rate of long wavelength quasiparticles in the critical theory is in striking contrast with that for T >> Tc which has been studied in detail in[29, 30]. For T >> Tc the thermal mass is mT ≈ λT[29] while the two loop contribution to the imaginary part of the self energy for T >> Tc, p is still ∝ λ3T2. Thus, the damping rate of long wavelength excitations is η ∝ λ3/2T ≪ 1/2T in the weak coupling limit. Therefore for T >> Tc long wavelength excitations are true weakly coupled quasiparticles with narrow widths.

III. THE ε-EXPANSION: STATIC CASE

The analysis of the previous section points out that naive perturbation theory at finite temperature breaks down at the critical point for momenta ≪ λT. The reason for this breakdown, as revealed by the analysis of the previous section, is the following: in four space-time dimensions the quartic coupling is dimensionless, however each loop diagram in the perturbative expansion has a factor T from the sum over the Matsubara frequencies. After performing the renormalization of the mass including the finite temperature corrections and setting the theory at (or near) the critical point, the effective expansion parameter for long-wavelength quasiparticles in the critical theory is λT which has dimensions of momentum. If a given diagram has a momentum transfer scale µ the effective dimensionless expansion parameter is therefore λT/µ which becomes very large for µ ≪ λT, i.e., the effective coupling is strongly relevant in the infrared. The analysis based on the RG beta function (II.12,H.13) suggests that the effective coupling g = λT/µ is driven to the three dimensional (Wilson-Fischer) fixed point in the infrared, obviously a consequence of the dimensional reduction in the high temperature limit. This is confirmed by the large N resummation of the scattering amplitude (II.15,H.16).

If the value of the coupling at the fixed point is η ≪ 1 then a perturbative expansion near the fixed point would be reliable, however the value of the coupling at the fixed point is g2 ~ O(1) which of course is a consequence of the fact that for fixed λT the effective coupling scales with dimension of inverse momentum in the infrared. This situation is the same as in critical phenomena for theories that are superrenormalizable, in which the infrared divergences are severe.

The remedy in critical phenomena is to study the perturbative series via the ε-expansion wherein the value of the coupling at the fixed point is O(ε)[4, 21, 22, 23, 25] and sum the perturbative series via the renormalization group. We now implement this program in the high temperature limit.

In finite space-time dimensions the quartic coupling λ has canonical dimension of inverse momentum, therefore the product λT that occurs in the perturbative expansion in the dimensionally reduced low energy theory is dimensionless. Then in a perturbative expansion at (or very near) the critical point we expect that infrared divergences will manifest in the form of logarithms of the momentum scale in the loop. This implies that the effective coupling is
marginal. Considering the theory in $4 - \epsilon$ spatial dimensions and one Euclidean (compactified) time dimension the effective coupling of the dimensionally reduced theory, $\lambda T^\epsilon$, has dimensions of $\mu^\epsilon$ with $\mu$ being a momentum scale. Therefore the effective dimensionless coupling for diagrams with a transferred momentum scale $p$ is $g(p) = \lambda T^\epsilon \mu^{-\epsilon}$. Thus, for fixed $T$ the scaling dimension of this effective coupling is $-\epsilon$, hence we expect a non-trivial fixed point at which the coupling $g^* \sim O(\epsilon)$.

Therefore for $\epsilon \ll 1$ we can perform a systematic perturbative expansion near the fixed point. This is the spirit of the expansion in critical phenomena which, when combined with a resummation of the perturbative series via the renormalization group has provided a spectacular quantitative and qualitative understanding of critical phenomena[4, 21, 22, 23, 24, 25].

While dimensional regularization and the $\epsilon$ expansion have been used to study the dimensionally reduced high temperature theory insofar as thermodynamic quantities are concerned, i.e. static phenomena[15, 16, 17], we emphasize that our focus is to study dynamics at and near the critical point, which is fundamentally different from the studies of static phenomena in these references.

As a prelude to the study of the dynamics, we now revisit the scattering amplitude at one loop level and the self-energy at two loops level in $4 - \epsilon$ spatial dimensions at high temperature in the static limit. The one loop self-energy is momentum independent and is absorbed in the definition of the thermal mass[29] which is set to zero at the critical point. The are two main purposes of this exercise: the first is to quantify the role of the higher Matsubara modes and secondly to obtain a guide for the infrared running of the coupling constant.

A. Scattering amplitude:

The one loop contribution in the static limit, i.e. when the external Matsubara frequencies are zero is given by

$$H(p, 0) = \frac{T}{2} \sum_{n \in \mathbb{Z}} \int \frac{d^d q}{(4\pi)^d} \frac{1}{[q^2 + (2\pi T n)^2]^{1/2} + (2\pi T n)^2} ; \quad d = 4 - \epsilon . \quad \text{(III.1)}$$

In the high temperature limit the nonzero Matsubara terms give subdominant contributions. This property can be argued in different manners. For example, the $l \neq 0$ terms in eq. (III.1) can be interpreted as Feynman diagrams in $d = 4 - \epsilon$ dimensions with mass $(2\pi T l)^2$. Such contributions are negligible for $p \ll T$[31, 32].

We find for high temperatures (see Appendix A),

$$H(p, 0)'' \sim p^\epsilon \; H_{\text{ass}}(p, 0) = T^{-1-\epsilon} \frac{\Gamma \left( 1 + \frac{\epsilon}{2} \right) \zeta(2 + \epsilon)}{\pi^\frac{\epsilon}{2} (2\pi T n)^2} \left[ 1 + O \left( \frac{p^2}{T^2} \right) \right] \quad \text{III.2}$$

where $H_{\text{ass}}(p)$ stands for the $l = 0$ contribution to the sum (III.1) plus the dominant high temperature limit of the sum over the $l \neq 0$ terms, which is obtained by setting $p = 0$. Separating the $l = 0$ mode and setting $p = 0$ in the contribution from the sum over $l \neq 0$ we find

$$H_{\text{ass}}(p, 0) = \frac{T \pi^{\epsilon}}{2(4\pi)^\frac{\epsilon}{2}} \Gamma \left( \frac{\epsilon}{2} \right) \left[ \frac{\zeta(1 - \frac{\epsilon}{2})}{\zeta(2 - \frac{\epsilon}{2})} + 2 \frac{4\pi^2 T^2}{p^2} \right]^{-\frac{\epsilon}{2}} \zeta(\epsilon) \quad \text{III.3}$$

with $\zeta$ Riemann’s zeta function which has the following properties[33]

$$\zeta(0) = -\frac{1}{2} ; \quad \lim_{\epsilon \to 0} \zeta(\epsilon) = \frac{1}{\epsilon - 1} + \gamma \quad \text{III.4}$$

where $\gamma = 0.577216 \ldots$ stands for the Euler-Mascheroni constant.

Near four space dimensions $\epsilon \to 0$ we find

$$H_{\text{ass}}(p, 0) = -\frac{T}{2(4\pi)^2} \left[ \log \left( \frac{p^2}{4T^2} \right) - 2 \right] + O(\epsilon) \quad \text{III.5}$$

There is no pole in $\epsilon$ and the argument of the logarithm reveals that $T$ acts as an ultraviolet cutoff. The reason that there is no pole in $\epsilon$ is that the poles in $\epsilon$ should be independent of temperature and should be those of the zero temperature theory. However, in dimensional regularization one loop integrals have no poles in odd space-time dimensions[34]. On the other hand near three space dimensions $\epsilon \to 1$ we find

$$H_{\text{ass}}(p, 0) = \frac{T}{16} \left[ 1 - \frac{1}{\pi^2} \frac{p}{T} \ln \frac{T}{\mu} + \frac{1}{(4\pi)^2} \frac{1}{(\epsilon - 1)} \right] \quad \text{III.6}$$
where the pole term at $\epsilon = 1$ corresponds to the usual coupling constant renormalization. This divergent term is temperature-independent, as expected, the $\ln(T)$ is reminiscent of an upper momentum cutoff for the high temperature limit. The first term is precisely what we obtained in equation (II.8) from setting the internal Matsubara frequency to zero, i.e. the result of the dimensionally reduced theory. After subtracting the pole near three space dimensions, the first term gives the leading infrared contribution in the limit $T/p \gg 1$ whereas the logarithm is subleading.

This expression coincides with that given in references [16, 17]. In these references the four dimensional high temperature static theory was studied and a systematic analysis of Feynman diagrams in the dimensionally reduced theory (three dimensions) was performed. The $\epsilon - 1$ in our expressions should be mapped onto the $2\epsilon$ for comparison with the results in these references.

For $\epsilon > 0$ we can neglect the terms of the form $(T/p)^{\epsilon-\epsilon}$ in eqn. (III.3) in the limit $T/p \gg 1$. And for $1 \gg \epsilon > 0$ we find that the static scattering amplitude at a symmetric point (II.9) in the limit in which the temperature is much larger than the external momentum scales is given by

$$\Gamma^{(4)}(p_{f} = \tilde{p}_{i}, 0) = -\lambda_{eff}(\mu, T) = -\lambda \left[ 1 - \frac{3 \lambda T \mu^{-\epsilon}}{(4\pi)^{d/2}} \right]$$  \hspace{1cm} (III.7)

the factor $T$ arising from the Matsubara sum is such that $\lambda T$ has dimensions of (momentum)$^{\epsilon}$ so that in $5 - \epsilon$ space time dimensions $\lambda T \mu^{-\epsilon}$ is dimensionless. Thus, introducing the dimensionless renormalized coupling

$$g_{R}(\mu) = \frac{\lambda_{eff}(\mu, T) T^{\epsilon \epsilon}}{(4\pi)^{d/2}} \hspace{1cm} d = 5 - \epsilon$$  \hspace{1cm} (III.8)

we find

$$\beta_{g} = \mu \frac{\partial g_{R}}{\partial \mu_{T}} \parallel - \epsilon g + g^{2} + O(g^{3})$$  \hspace{1cm} (III.9)

Therefore this effective coupling in the infrared limit is driven to a non-trivial fixed point

$$g^{*} = \frac{\epsilon}{3}$$  \hspace{1cm} (III.10)

Hence for $\epsilon \ll 1$ the fixed point theory can be studied perturbatively. This of course is the basis of the $\epsilon$ expansion in critical phenomena [4, 21, 22, 23, 24, 25] and will be the important point upon which our analysis will hinge.

B. Two loops self-energy:

As mentioned above the one loop contribution to the self energy is momentum independent and absorbed into the definition of the thermal mass. The two loop contribution in the static limit in $d$ spatial dimensions is given by

$$\Sigma^{(2)}(k, 0) = \frac{\lambda^{2} T^{2}}{6} \sum_{l} \int \frac{d^{4}p}{(2\pi)^{d}} \frac{d^{4}q}{(2\pi)^{d}} \frac{1}{(p^{2} + \omega_{l}^{2})(q^{2} + \omega_{m}^{2})[(p + q + k)^{2} + (\omega_{l} + \omega_{m})^{2}]^{3}}$$  \hspace{1cm} (III.11)

We now introduce two Feynman parameters, separate the $l = m = 0$ term from the Matsubara sums and take $T \gg p$ in the sums with $l, m \neq 0$ to find

$$\Sigma^{(2)}(k, 0) = \frac{g^{2}}{6} \frac{\Gamma(-1 + \epsilon) \Gamma^{3}(1 - \frac{d}{2})}{\Gamma(3 - \frac{d}{2})} k^{2} \left[ \frac{k^{2}}{\mu^{2}} \right]^{-\epsilon} + g^{2} T^{2} \left[ \frac{T^{2}}{\mu^{2}} \right]^{-\epsilon} C(d)$$  \hspace{1cm} (III.12)

with $C(d)$ only depending on the dimensionality. The second term $\propto T^{4-2\epsilon}$ does not depend on the momentum and is therefore cancelled by the mass counterterm which defines the critical theory. Therefore for $\epsilon > 0$ but small we find

$$\Gamma^{(2)}(k, 0) = p^{2} - [\Sigma(k, 0) - \Sigma(0, 0)] = k^{2} \left[ 1 + \frac{g^{2}}{12 \epsilon} - \frac{g^{2}}{12} \ln \left( \frac{k^{2}}{\mu^{2}} \right) \right] + O(g^{3})$$  \hspace{1cm} (III.13)

We introduce the wave function renormalization in dimensional regularization by the usual relation

$$\Gamma_{R}^{(2)}(k, 0) = Z_{\phi} \Gamma^{(2)}(p, 0)$$  \hspace{1cm} (III.14)
and choose

\[ Z_\phi = 1 - \frac{g^2}{12\epsilon} + \mathcal{O}(g^3) \]  

(III.15)

Therefore the renormalized two point function in the static, high temperature limit is given by

\[ \Gamma^{(2)}_R(k, 0) = k^2 \left[ 1 - \frac{g^2}{12\epsilon} \ln \left( \frac{k^2}{\mu^2} \right) \right] + \mathcal{O}(g^3) \]  

(III.16)

The infrared behavior is obtained by resumming the perturbative series via the renormalization group[4, 21, 22, 23, 24, 25] which leads to the scaling form of the two point function in the infrared limit

\[ \Gamma^{(2)}_R(k) \propto k^{2-\eta} \]  

(III.17)

with

\[ \eta = \frac{g^2}{\mu^2} - \frac{g^2}{3\epsilon} + \mathcal{O}(g^3) \]  

(III.18)

This is the anomalous dimension to lowest order in \( \epsilon \) [4, 21, 22, 23, 24, 25].

C. The strategy:

The analysis of the static case above has highlighted several important features of the infrared behavior near the critical point, which determines the strategy for studying the dynamical case: a) the infrared behavior in the limit when \( T \gg p \) with \( p \) the typical momentum of the Feynman diagram is determined by the dimensionally reduced theory obtained by setting the internal Matsubara frequencies to zero. b) naive perturbation theory breaks down in three space dimensions because the dimensionless coupling is \( \lambda T/\mu \) with \( \mu \) the external momentum scale in the Feynman diagram, while a large \( N \) or renormalization group frequency resummation suggests a non-trivial infrared stable fixed point, the coupling at this fixed point is of \( \mathcal{O}(1) \). c) Just as in critical phenomena the perturbative expansion can be systematically controlled in an \( \epsilon \) expansion around four spatial dimensions corresponding to a theory dimensionally reduced from \( 5 - \epsilon \) space time dimensions. The effective dimensionless coupling of the dimensionally reduced theory (four dimensional) is \( \lambda T \) which is marginal. This combination is independent of \( \hbar \)-this can be seen by restoring powers of \( \hbar \lambda \rightarrow h \lambda; T \rightarrow T/\hbar \) - so that the low energy, dimensionally reduced theory is classical. In \( 4 - \epsilon \) spatial dimensions the effective dimensionless coupling \( g = \lambda T \mu^{-\epsilon} \) is driven to the infrared stable Wilson-Fisher fixed point of \( \mathcal{O}(1) \) by the renormalization group trajectories. Thus, the strategy to follow becomes clear: we will now study the dynamics by including the contribution from the external Matsubara frequency, focusing on the infrared behavior for \( p, \omega \ll T \) near the critical point in a systematic \( \epsilon \) expansion around four spatial dimensions.

We note that the theory in \( 5 - \epsilon \) dimension is formally non-renormalizable in the ultraviolet, however this is irrelevant for the infrared which is the region of interest here. The analysis provided above in the limit \( \epsilon \to 0 \) clearly shows that near five space-time dimensions there are no poles in dimensional regularization in one loop diagrams as expected[31]. The potential poles are replaced by \( \ln(T) \). The low-energy theory must be understood with a cutoff of \( \mathcal{O}(T) \) and the dimensionally regularized integrals in 5 space time dimensions clearly display this cutoff in the argument of logarithms. The long wavelength \( \mu/T \ll 1 \) an the \( \epsilon \to 0 \) limits do not commute: keeping the subleading terms in the high temperature limit and taking \( \epsilon \to 0 \) results in that poles in \( \epsilon \) actually translate into logarithms of the cutoff \( T \), on the other hand, keeping \( \epsilon > 0 \) and small the \( T/\mu \to \infty \) limit can be taken and the subleading high temperature corrections vanish. Clearly it is the \( \textit{latest} \) limit the one that has physical relevance, since eventually we are interested in studying the infrared behavior of the physical theory in three space dimensions. Hence in what follows we consider the long-wavelength limit for \( \epsilon > 0 \) but small and approach the physical dimensionality \( \epsilon \to 1 \) in a consistent \( \epsilon \) expansion improved via the renormalization group. This is the strategy in classical critical phenomena as well where for \( \epsilon > 0 \) and small the ultraviolet cutoff can be taken to infinity.

At this stage it is important to highlight the difference between the main focus of this work and that in references[16, 17]. The work of references[16, 17] studies the \textit{static} limit of the dimensionally reduced theory near three spatial dimensions arising from the high temperature limit of a four dimensional Euclidean theory compactified in the time direction. In contrast we here focus on studying the \textit{dynamics} in the limit when \( p, \omega \ll T \) which as emphasized by the authors of this work will be studied in an \( \epsilon \) expansion in a dimensionally reduced theory near four space dimensions.

The limit of physical interest \( \epsilon \to 1 \) must be studied by improving the perturbative expansion via the renormalization group[4, 21, 22, 23, 24, 25] and eventually by other non-perturbative resummation methods, such as Pade approximants or Borel resummation that will extend the regime of validity of the \( \epsilon \) expansion[23].
IV. DYNAMICS NEAR THE CRITICAL POINT:

We now turn to the dynamics. Our main goal is to study the feasibility of a quasiparticle description of low energy excitations at and near the critical point. Of particular interest is the dispersion relation as well as the damping rates of these excitations. This information is contained in the two point function $\Gamma^{(2)}(p, \omega_n)$ which is the inverse propagator, analytically continued to $\omega_n \rightarrow -i\omega + 0^+$. The region of interest is $p, \omega \ll T$ and if the theory is (slightly) away from the critical point $M(T) \ll T$ as well. In principle for a fixed $\omega_n$ or fixed external Matsubara frequencies in the external legs of $n-$ point functions, one must perform the sum over the internal Matsubara first and then take the analytic continuation. However, as was shown above in detail in the static case, the most infrared singular contribution arises from setting the internal Matsubara frequencies to zero. That this is also the case in the dynamics can be seen by considering a diagram with $m-$ internal lines, rerouting the external Matsubara frequency through one of the lines. All of the lines are equivalent since rerouting the external Matsubara frequency corresponds to a shift in one of the sums. The other $m-1$ contain propagators in which the internal Matsubara frequency acts as a mass of $O(2\pi l T)$. These are the superheavy modes in the description of references[16, 17, 31, 32].

Once the internal Matsubara frequencies had been set to zero we can analytically continue the external Matsubara frequency to a continuous Euclidean variable $\omega_n \rightarrow s$ to obtain the Euclidean two point function. The dispersion relation and damping rate are obtained by further analytical continuation $s \rightarrow -i\omega + 0^+$.

As anticipated in section(II) because Euclidean time is compactified and plays a different role than the spatial dimensions, we must consider the anisotropic Lagrangian density (II.4) which includes the velocity of light multiplying the derivatives with respect to Euclidean time. If this velocity of light is simply a constant it can be reabsorbed into a trivial redefinition of the time variable. However, as it will become clear below, this velocity of light acquires a non-trivial renormalization as a consequence of the anisotropy between space and time directions at finite temperature and will run with the renormalization group. Thus, the Euclidean propagator is generalized to

$$G(k, \omega_n) = \frac{1}{k^2 + \frac{\omega_n}{v_0}}$$  \hspace{1cm} (IV.1)

A. The scattering amplitude

We begin by studying the scattering amplitude, now as a function of external momenta and frequencies. The one loop contribution is determined by the function $H(p, s)$ given by eqn. (II.7), which for $p, s \ll T$ is given by (see Appendix)

$$H(p, s) \rightarrow p^4 \cdot H_{ass}(p, s) = \frac{\Gamma(1 + \frac{\epsilon}{2}) \zeta(2 + \epsilon)}{192 \pi^{4+\frac{\epsilon}{2}} T^{1+\epsilon}} \left[ p^2 + \frac{s^2}{v_0^2} (1 - \epsilon) \right] \left[ 1 + O \left( \frac{p^2 + s^2}{T^2} \right) \right],$$  \hspace{1cm} (IV.2)

where $H_{ass}(p, s)$ stands for the $l = 0$ contribution to the sum (II.7) plus the dominant high temperature limit of the sum over the $l \neq 0$ terms which is obtained by setting $p = 0$

$$g_{inequation} H_{ass}(p, s) = \frac{T}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + s^2} \left[ \frac{1}{(q + p)^2 + s^2} \right] + \frac{\Gamma(1 + \frac{\epsilon}{2}) \zeta(\epsilon)}{8 \pi^{2+\frac{\epsilon}{2}} \epsilon} \left( \frac{3}{2} - 2 \right) T^{1-\epsilon},$$  \hspace{1cm} (IV.3)

This integral is computed in the appendix A with the result

$$H_{ass}(p, s) = -\frac{\Gamma(1 + \frac{\epsilon}{2})}{2(4\pi)^{1-\epsilon}} T \left( \frac{s^2}{v_0^2} + p^2 \right)^{-\frac{\epsilon}{2}} F \left( \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}; 2 - \frac{\epsilon}{2}; \frac{p^2}{v_0^2} \right) + \frac{\Gamma(1 + \frac{\epsilon}{2}) \zeta(\epsilon)}{8 \pi^{2+\frac{\epsilon}{2}} \epsilon} T^{1-\epsilon},$$  \hspace{1cm} (IV.4)

where $F(a, b; c; z)$ stands for the hypergeometric function. For $\epsilon > 0$ and $p, \frac{s}{v_0} \ll T$ we can neglect the second term, since it is proportional to $T^{1-\epsilon} \ll T \left( p^2 + \frac{s^2}{v_0^2} \right)^{-\frac{\epsilon}{2}}$. We note that the infrared dominant contribution can be written in
the form \( T \rho^{-\epsilon} \mathcal{F}(\frac{\mu^2}{v_\|^2}) \) the factor \( T \) thus combines with \( \frac{\mu^2}{v_\|^2} \) to give the effective coupling of dimension \( \mu^\epsilon \) in \( d = 4 - \epsilon \) spatial dimensions.

For \( \epsilon > 0 \) but small and neglecting the second term in (IV.4) can be expanded in \( \epsilon \) leading to

\[
\lambda_8 H_{s,s}(p, s) = \frac{g(\mu)}{2} \left[ \frac{2}{\epsilon} - \left( 1 + \frac{s^2}{v_\|^2 p^2} \right) \ln \left( \frac{s^2 + p^2 v_\|^2}{\mu^2 v_\|^2} \right) + \frac{s^2}{v_\|^2 p^2} \ln \left( \frac{s^2}{v_\|^2 \mu^2} \right) + \ln 4\pi + 2 - \gamma + O(\epsilon) \right] \quad \text{(IV.5)}
\]

where we introduced the dimensionless bare coupling

\[
g(\mu) = \frac{\lambda_8 T \mu^{-\epsilon}}{(4\pi)^\frac{d}{2}} \quad \text{(IV.6)}
\]

We remark that one cannot take \( \epsilon \to 0 \) in this expression since in this limit the pole is actually cancelled by the second term in (IV.4) as emphasized above, this expression must be understood for \( \epsilon > 0 \) but small so that the contributions of the form \( (T/s, T/p)^{-\epsilon} \to 0 \) for \( T \gg s, p \). Therefore, the expression above must be understood in the sense that i) \( T \gg s, p \) with fixed \( \epsilon > 0 \) and ii) \( \epsilon \ll 1 \) and the resulting expressions have a Laurent expansion for small \( \epsilon \).

### B. The self energy at two loops

Neglecting the contribution from the non-zero Matsubara frequencies which will be absorbed by the mass counter-term in the definition of the critical temperature (or \( M^2(T) \) away from the critical point) and also neglecting terms that vanish in the limit \( T \gg p, s \) the dominant contribution in the infrared to the two loops self-energy is

\[
\Sigma(p, s) = \frac{\lambda_8^2 T^2}{6} \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{q^2 k^2 \left[ (q + k + p)^2 + \frac{s^2}{v_\|^2} \right]} = \frac{\lambda_8^2 T^2}{6(4\pi)^d} \frac{\Gamma^2(\frac{d}{2} - 1) \Gamma(3 - d)}{\Gamma(d - 2)} \int_0^1 dx (1 - x)^{d - 3} x^{1 - \frac{d}{2}} \left[ x p^2 + \frac{s^2}{v_\|^2} \right]^{d - 3} \quad \text{(IV.7)}
\]

While this expression can be written in terms of hypergeometric functions, it is more convenient to expand it in \( \epsilon \) with the result that

\[
\Gamma^{(2)}(p, s) = p^2 \frac{\mu^2}{v_\|^2} \ln \left( \frac{p^2}{2} + \frac{s^2}{v_\|^2} \right) + \frac{g^2(\mu)}{6\epsilon} \left[ \frac{p^2}{2} + \frac{s^2}{v_\|^2} - \frac{p^2}{2} \ln \left( \frac{p^2}{\mu^2} \right) - \frac{s^2}{v_\|^2} \ln \left( \frac{s^2}{v_\|^2 \mu^2} \right) \right] + O(\rho^2, \epsilon^2, \epsilon^3) \quad \text{(IV.8)}
\]

with \( g(\mu) \) given by eqn. (IV.6).

### C. Renormalization

The forms of the two and four point functions immediately suggests a renormalization scheme akin to the familiar one used in critical phenomena\[21, 22, 23, 25\] with one important difference: we see from eqn. (IV.8) that the velocity of light \( v_\| \) must also be renormalized. The wave function renormalization is introduced as usual via

\[
\Gamma^{(2)}_R(p, s, v_\|) = Z_\phi \Gamma^{(2)}(p, s, v_\|) \quad \text{(IV.9)}
\]

the renormalized mass as a function of temperature is defined as

\[
\Gamma^{(2)}_R(0, 0) = M^2(T) \quad \text{(IV.10)}
\]

this definition, however, defines the inverse susceptibility or correlation length, rather than the pole mass, the critical theory is defined by \( M^2(T) = 0 \). Coupling constant and velocity of light renormalization are achieved by

\[
\lambda_\| = Z_\phi^2 Z_\lambda \lambda_8 \quad \text{(IV.11)}
\]

\[
\Gamma^{(4)}_R(p_\|, s = 0) = -\lambda_\|
\]
\[ v_R^2 = \frac{v_s^2 Z_s}{Z_\phi} \]  

(IV.12)

The renormalization conditions that determine the constants \( Z_\phi, Z_\lambda, Z_v \) are

\[
\begin{align*}
\frac{\partial \Gamma^{(2)}_R}{\partial p^2}  & \bigg|_{p^2 = \mu^2, \frac{v_s^2}{v_R^2}} = 1 \\
\frac{\partial \Gamma^{(2)}_R}{\partial s^2}  & \bigg|_{p^2 = \mu^2, \frac{v_s^2}{v_R^2}} = \frac{1}{v_R} \Gamma^{(4)}_R(p_i; \ s = 0) = -\lambda_R
\end{align*}
\]

(IV.13)

Consistently with the \( \epsilon \) expansion we choose the renormalization constants \( Z_\phi, Z_\lambda, Z_v \) in the minimal subtraction scheme to lowest order, since keeping higher powers of the coupling or \( \epsilon \) results in higher order corrections in the \( \epsilon \) expansion:

To lowest order, one loop for the 4-point function and two loops for the two point function, we find from the results (IV.5) and (IV.8)

\[
\begin{align*}
Z_\lambda & = 1 - \frac{3g(\mu)}{\epsilon} \Rightarrow g_R = g(\mu) - \frac{3g^2(\mu)}{\epsilon} ; \quad g_R = \frac{\lambda_R T \mu^{-\epsilon}}{(4\pi)^2} \\
Z_\phi & = 1 - \frac{g^2}{12\epsilon} \\
Z_v & = 1 - \frac{g^2}{3\epsilon^2}
\end{align*}
\]

(IV.14)

(IV.15)

Thus, the renormalized two point function reads

\[
\Gamma^{(2)}_R(p, s) = p^2 \left[ 1 - \frac{g^2}{12} \ln \left( \frac{p^2}{\mu^2} \right) \right] + \frac{g^2}{v_R} \left[ 1 - \frac{g^2}{3\epsilon} \ln \left( \frac{s^2}{v_R^2} \right) \right] + \mathcal{O}(g^3, g^2\epsilon)
\]

(IV.16)

V. THE RENORMALIZATION GROUP:

Before we embark on the resummation program via the renormalization group, it is important to highlight two important features

- The contributions that are dominant in the infrared in the limit \( T \gg p, s \) correspond to the terms with internal Matsubara frequencies equal to zero, the non-zero Matsubara frequencies give subleading contributions for \( \epsilon > 0 \). This in turn results in that the dependence on temperature is solely through the effective coupling \( g = \lambda T \mu^{-\epsilon} \). This can be seen from the fact that each loop has a factor \( \Gamma \) from the Matsubara sum over the internal loop frequencies as well as one power of the coupling constant \( \lambda \), by dimensional reasons the dimensionless coupling is obtained by multiplying by \( \mu^{-\epsilon} \).

- The velocity of light \( v \) always enters in the form \( s/v \) since this is the form that enters in the propagators and the renormalization conditions above.

A. The critical point:

The bare \( n \)-point functions are independent of the renormalization scale \( \mu \), and this independence leads to the renormalization group equations (we now suppress the subscript \( R \) understanding that all quantities are renormalized)

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \beta_v \frac{\partial}{\partial v} - \frac{N}{2} \gamma \right] \Gamma^{(N)}(p_1, \frac{s_1}{v}, p_2, \frac{s_2}{v}, \ldots, p_N, \frac{s_N}{v}, g, \mu) = 0
\]

(V.1)
with
\[
\beta_g = \frac{\partial g}{\partial \mu} \bigg|_{\lambda_0, T, v_0} \tag{V.2}
\]
\[
\beta_v = \frac{\partial v}{\partial \mu} \bigg|_{\lambda_0, T, v_0} \tag{V.3}
\]
\[
\gamma = \mu \frac{\partial \ln Z_0}{\partial \mu} \bigg|_{\lambda_0, T, v_0} \tag{V.4}
\]

To lowest order we find
\[
\beta_g = -\epsilon g + 3g^2 + \mathcal{O}(g^3, g^2\epsilon) \tag{V.5}
\]
\[
\beta_v = \frac{1}{2} \left[ \frac{2g^2}{3\epsilon} - \gamma \right] v + \mathcal{O}(g^3, g^2\epsilon) \tag{V.6}
\]
\[
\gamma = \frac{g^2}{6} + \mathcal{O}(g^3, g^2\epsilon) \tag{V.7}
\]

While we can write down the general solution of the RG equation (V.1) for an arbitrary \( N \)-point function, our focus is to understand the quasiparticle structure which is obtained from \( \Gamma^{(2)} \).

Since \( \Gamma^{(2)} \) has dimension two, it follows that
\[
\Gamma^{(2)}(p, \frac{s}{v}, g, \mu) = \mu^2 \Phi \left( \frac{p}{\mu}, \frac{s}{v\mu}, g \right) \tag{V.8}
\]

therefore
\[
\Gamma^{(2)}(\epsilon^4 p, \frac{\epsilon^4 s}{v}, g, \mu) = \epsilon^{2t} \Gamma^{(2)}(p, \frac{s}{v}, g, \mu \epsilon^{-t}) \tag{V.9}
\]

This scaling property then leads to
\[
\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \mu} = 2 \Gamma^{(2)}(\epsilon^4 p, \frac{\epsilon^4 s}{v}, g, \mu) = 0 \tag{V.10}
\]

which combined with the RG equation (V.1) leads to the following equation that determines the scaling properties of the two point function
\[
\left[ -\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + \beta_v \frac{\partial}{\partial v} - (\gamma - 2) \right] \Gamma^{(2)}(\epsilon^4 p, \frac{\epsilon^4 s}{v}, g, \mu) = 0 \tag{V.11}
\]

The solution of this equation is standard\[21, 22, 23\]
\[
\Gamma^{(2)}(\epsilon^4 p, \frac{\epsilon^4 s}{v}, g, \mu) = e^{\int_0^t \frac{d\tau}{4}(2-\gamma(\tau))} \Gamma^{(2)}(p, \frac{s}{v}, g(t), \mu) \tag{V.12}
\]

with
\[
\frac{\partial g(t)}{\partial t} = \beta_g(g(t)) ; \ g(0) = g_R(\mu)
\]
\[
\frac{\partial v(t)}{\partial t} = \beta_v(v(t), g(t)) ; \ v(0) = v_R(\mu)
\]
\[
\gamma(t) = \gamma(g(t), v(t)) \tag{V.13}
\]

As \( t \to -\infty \) i.e. the momentum and frequency are scaled towards the infrared we see from the RG \( \beta \) function (V.5) that coupling is driven to its fixed point
\[
\lim_{t \to -\infty} g(t) = g^* = \frac{\epsilon}{3} + \mathcal{O}(\epsilon^2) \tag{V.14}
\]
which in turn implies that

\[
\lim_{t \to -\infty} \gamma(t) = \eta = \frac{\epsilon^2}{54} + \mathcal{O}(\epsilon^3)
\]  
\[
\lim_{t \to -\infty} v(t) = v(0) \epsilon^{(z-1)\eta}
\]  

(V.15) (V.16)

where we introduced the new dynamical critical exponent

\[
z = 1 + \frac{1}{2}(\eta_k - \eta) + \mathcal{O}(\epsilon^2)
\]

and

\[
\eta_k = \frac{2g\epsilon^2}{3\epsilon} = \frac{2\epsilon}{27} + \mathcal{O}(\epsilon^2)
\]  

(V.17)

Therefore, in the asymptotic infrared limit we find that

\[
\Gamma^{(2)}\left(\epsilon^t \eta, \frac{\epsilon^t s}{v(t)}, g, \mu\right) = \epsilon^{(z-2)\eta} \Gamma^{(2)}\left(p, \frac{s}{v(0)} \epsilon^{(1-\eta)\mu}, g^*, \mu\right)
\]  

(V.18)

It is convenient to re-define \(p \epsilon^t = P\); \(s \epsilon^t = S\) to find

\[
\Gamma^{(2)}\left(P, \frac{S}{v}, g, \mu\right) = \epsilon^{(1-\eta)\mu}\Gamma^{(2)}\left(p \epsilon^{-t}, \frac{S}{v(0)} \epsilon^{-t} \epsilon^{(1-\eta)\mu}, g^*, \mu\right)
\]  

(V.19)

and finally writing \(P = \mu \epsilon^t\) and using the property (V.8) we find the scaling form in the infrared limit

\[
\Gamma^{(2)}\left(P, \frac{S}{v}, g, \mu\right) = \mu^2 \left[p \epsilon^{-t}\right]^{2-\eta} \Phi(\vartheta) ; \quad \vartheta = \left(\frac{S}{v(\mu)\epsilon^{(1-\eta)\mu}}\right)^2
\]  

(V.20)

The solution of the RG equation clearly shows that the two point function in the infrared limit is a scaling function of the ratio \(s/v(\mu)\epsilon^{1-\eta}\mu^z\) highlighting the role of the new dynamical exponents \(z\) given by eqn. (V.17) with \(\eta_k\) to lowest order given by eqn. (V.17).

The emergence of the new dynamical exponent \(z\) is a consequence of the anisotropic renormalization between momentum and frequency, or space and time manifest in the renormalization of the speed of light. This novel phenomenon can be traced back to the different role played by time (compactified) and space in the Euclidean formulation at finite temperature. A similar anisotropic rescaling emerges in a different context: a Heisenberg ferromagnet with correlated impurities [33] with similar renormalization group results.

While the formal solution does not yield the function \(\Phi\), we can find it by matching to the lowest order perturbative expansion (IV.16) when the coupling is at the non-trivial fixed point. From the form of the of the perturbative renormalized two point function given by eqn. (IV.16) and assuming the exponentiation of the leading logarithms via the renormalization group near the non-trivial fixed point

\[
\Gamma^{(2)}_R(p, s; g^*) = p^2 \left[1 - \eta \ln \left(\frac{p}{\mu}\right)\right] + \frac{s^2}{v_R} \left[1 - \eta \ln \left(\frac{s}{v_R(\mu)}\right)\right] \approx p^{2-\eta} \mu^\eta + \left(\frac{s}{v_R}\right)^{2-2\eta} \mu^{2\eta}
\]  

(V.21)

which can be immediately written in the scaling form

\[
\Gamma^{(2)}_R(p, s; g^*) \sim p^{2-\eta} \mu^\eta \left[1 + \left(\frac{s}{v(\mu)\epsilon^{(1-\eta)\mu}}\right)^{2-2\eta}\right]
\]

\[
z = \frac{2 - \eta}{2 - \eta_k} \simeq 1 + \frac{1}{2}(\eta_k - \eta)
\]  

(V.22)

Clearly this form coincides with the scaling solution of the renormalization group and the perturbative expansion in the regime in which it is valid. We note, however, that in the computation of \(\eta_k\) we have neglected contributions to the renormalization of the velocity \(v\) of \(\mathcal{O}(g^2/\epsilon)\) which would appear at next order and would lead to an \(\mathcal{O}(\epsilon^2)\) contribution to \(\eta_k\), which is of the same order as \(\eta\) in \(z\). Thus consistently we must neglect the contribution of \(\eta\) to the dynamical exponent \(z\) which to lowest order is therefore

\[
z = 1 + \frac{\eta_k - \eta}{2} + \mathcal{O}(\epsilon^2) = 1 + \frac{\epsilon}{27} + \mathcal{O}(\epsilon^2) .
\]  

(V.23)
1. Quasiparticles and critical slowing down:

The quasiparticle structure of the theory is obtained from the Green’s function \( G^{-1}(p, \omega) = \Gamma^{(2)}(p, s, \omega + 0^+, \sqrt{s}) \). In particular the dispersion relation and the width of the quasiparticle are obtained from the real and imaginary parts, respectively.

While the general solution of the RG equation does not determine the scaling function \( \Phi \) in eqn. (V.20) the fact that it is only a function of the scaling ratio \( \vartheta \), allows to extract the quasiparticle structure. The analytic continuation \( s \rightarrow -i \omega + 0^+ \) leads to the analytic continuation of the scaling variable \( \vartheta \rightarrow -\varpi^2 - i \text{sign}(\varpi) \ 0^+ \) with

\[
\varpi = \frac{\omega}{v(\mu) p^{1-z} \varpi^2} \tag{V.24}
\]

Writing the scaling function \( \Phi \) analytically continued in terms of the real and imaginary parts \( \Phi(\vartheta = -\varpi^2 - i \text{sign}(\varpi) \ 0^+) = \Phi_R(\varpi) + i \Phi_I(\varpi) \), the position of the quasiparticle pole corresponds to the value of \( \varpi \) for which the real part vanishes. Call this dimensionless real number \( \varpi^* \), hence it is clear that the dispersion relation for the quasiparticles obeys

\[
\omega_p = \varpi^* \frac{v(\mu) \ p^{1-z} \ p^z}{\varpi} \tag{V.25}
\]

Furthermore, assuming that \( \Phi_R \) vanishes linearly at \( \varpi^* \) we can write the Green’s function near the position of the pole in the form

\[
G(\omega, p) \sim \frac{1}{\mu^2 \left[ \frac{\varpi}{\varpi^*} \right]^{2-\eta}} \frac{1}{\varpi - \varpi^* + i \frac{\varpi}{\varpi^*} \Phi_R(\varpi^*) + i \Phi_I(\varpi^*)} \tag{V.26}
\]

Alternatively, we can write the RG improved propagator, near the quasiparticle pole in the Breit-Wigner form

\[
G_{BW}(\omega, p) \sim \frac{\varpi_p}{\omega - \omega_p + i \Gamma_p} \tag{V.27}
\]

with the dispersion relation, residue at the quasiparticle pole and quasiparticle width given by

\[
\begin{align*}
\omega_p &= \varpi^* \frac{v(\mu) \ p^{1-z} \ p^z}{\varpi} \tag{V.28} \\
v_p &= (z) \varpi^* \frac{v(\mu) \ p^{1-z} \ p^z}{\varpi} \tag{V.29} \\
\varpi_p &= \frac{v(\mu) \ p^{1-z} \ p^z}{\mu^2 \left[ \frac{\varpi}{\varpi^*} \right]^{2-\eta} \Phi_R(\varpi^*)} \tag{V.30} \\
\Gamma_p &= \frac{\Phi_I(\varpi^*) \ v(\mu) \ p^{1-z} \ p^z}{\Phi_R(\varpi^*)} \tag{V.31}
\end{align*}
\]

The imaginary part \( \Phi_I(\varpi) \) must be proportional to the anomalous dimensions, hence perturbatively small in the \( \epsilon \) expansion (this will be seen explicitly below to lowest order).

The definite values for \( \varpi^* \), \( \Phi_R(\varpi^*) \), \( \Phi_I(\varpi^*) \) must be found by an explicit calculation. However, the above quasiparticle properties, such as the position of the pole, group velocity, residue and width are universal in the sense that they only depend on the fixed point theory. For a positive dynamical exponent \( z \) the above analysis reveals a vanishing group velocity and width for long-wavelength quasiparticles at the critical point.

Furthermore the expression for the width given by (V.31) not only displays the phenomenon of critical slowing down, i.e., the width of the quasiparticle vanishes in the long-wavelength limit, but also the validity of the quasiparticle picture, since \( \Gamma_p/\omega_p \ll 1 \) in the \( \epsilon \) expansion.

Threshold singularities: While we have assumed above that the real part of the scaling function vanishes linearly at the quasiparticle pole, this need not be the general situation. It is possible that the real part vanishes with an anomalous power law, i.e.,

\[
\Phi_R(\Omega) \sim |\Omega - \Omega_0|^{1+\chi} \quad ; \quad \chi = \chi^{(1)} \epsilon + \chi^{(2)} \epsilon^2 + \ldots \tag{V.32}
\]

In this case a quasiparticle width cannot be defined as the residue will either vanish or diverge depending on the sign of \( \chi \). It is also possible that \( \Phi_I(\Omega) \) also vanishes with an anomalous power at \( \Omega^* \). We refer to these cases as threshold
singularities and we will find below an example of this case. Another example of this situation has been found in
dense QCD as a result of the breakdown of the quark and gluon theory in the normal phase\cite{30}. Clearly only a detailed
calculation of the scaling functions can reveal whether it vanishes linearly or with an anomalous power law at \( \Omega^* \).
The set of quasiparticle properties given above (V.28-V.31) are only valid provided the real part vanishes linearly.

We can go further and find the explicit form of the scaling function by focusing on the renormalization group
improved propagator obtained in lowest order in the \( \epsilon \) expansion given by eqn. (V.22).

The analytic continuation to real frequencies of the RG improved two point function to lowest order given by (V.22)
leads to
\[
G^{-1}(p, \omega) = p^2 - \eta p^0 \left[ 1 - \left( \frac{|\omega|}{v(\mu)p^{1-z}} \right)^{2-\eta} \left( 1 + i \frac{\pi \eta p}{2} \text{sign}(\omega) \right) \right] \tag{V.33}
\]
where we have approximated \( \cos(\frac{\pi \eta p}{2}) \approx \frac{\pi \eta p}{2} \approx \pi \eta p \) to lowest order.

From this expression we see that the dispersion relation \( \omega_p \), group velocity \( v_g(p) \) width \( \Gamma(p) \) quasiparticles residue
\( \Sigma_p \) are of the form given by equations (V.28-V.31) with \( \pi^* = 1 \), and with the following explicit expressions to leading order in the \( \epsilon \) expansion
\[
|\omega_p| = v(\mu) \mu^{1-z} p^z \tag{V.34}
\]
\[
v_g(p) = z v(\mu) \mu^{1-z} p^{z-1} \tag{V.35}
\]
\[
\Gamma(p) = \frac{\pi \eta p}{4} v(\mu) \mu^{1-z} p^{z-1} \equiv \frac{\pi \eta p}{4} |\omega_p| \tag{V.36}
\]
\[
\Sigma_p = \frac{1}{2} v(\mu) \mu^{1-z} p^{z-1} \tag{V.37}
\]
with \( z \) given by (V.23) above.

Several important features of these expressions must be highlighted:

- The dispersion relation (V.25) features an \textit{anomalous dimension} given by the dynamical exponent \( z \approx 1 + \eta/2 = 1 + \epsilon/27 \). The product \( v(\mu) \mu^{1-z} \) is a renormalization group invariant as can be seen from equations (V.3) and (V.6) evaluated at the fixed point. Thus, all of the above quantities that describe the physical quasiparticle properties are manifestly renormalization group invariant.

- The group velocity (V.35) \textit{vanishes} in the long wavelength limit as a power law completely determined by
the dynamical anomalous dimension \( z \). This feature highlights the collective aspects of the long-wavelength
excitations.

- Critical slowing down: is explicitly manifest in the width \( \Gamma(p) \) since \( \Gamma(p) \to 0 \) as \( p \to 0 \). Furthermore we
also emphasize the validity of the quasiparticle picture, the ratio \( \Gamma(p)/\omega_p \approx \pi \eta p/2 \approx \mathcal{O}(\epsilon) \ll 1 \). Thus, the
quasiparticles are narrow in the sense that their width is much smaller than the position of the pole. Even
considering \( \epsilon = 1 \) corresponding to dynamical critical phenomena in three spatial dimensions \( \Gamma_p/\omega_p \sim 0.1 \)

Thus, we see that the renormalization group resummation has led to a consistent quasiparticle picture, but in terms
of a dispersion relation that features an anomalous dimension and a group velocity that vanishes in the long-wavelength
limit. Obviously these features of the quasiparticles cannot be extracted from a naive perturbative expansion.

B. Away from the critical point: \( T > T_c \)

Having studied the quasiparticle aspects at the critical point, we now turn our attention to their study slightly
away from the critical point. The critical region of interest is \([T - T_c] \ll T_c\). Critical behavior in the broken
symmetry phase near the critical point with \( T < T_c \) will be studied elsewhere with particular attention on the
critical dynamics of Goldstone bosons. In this article we restrict our attention to the normal phase near the critical
point.

In the Lagrangian density (II.4) the term \( M^2(T) \) is the \textit{exact} mass (rather the exact inverse susceptibility) determined
by the condition (II.5) and the counterterm \( \delta m^2(T) \) is adjusted consistently in perturbation theory to fulfill
this condition. However, to relate the mass to the departure away from the critical temperature it is more convenient
to re-arrange the perturbative expansion in a manner that displays explicitly the departure from \( T_c \). This is achieved
as follows. Consider the one loop contribution to the self-energy in the \textit{massless} theory in \( d \)-space dimensions:
\[
\Sigma^{(1)} = -\frac{\lambda T}{2} \sum_{m} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} = -\lambda T d^{-1} A(d) + \mathcal{O}(T^2)
\]

\textit{(V.38)}
with \( \mathcal{A}(d) \) only depends on the spatial dimensionality and z.T.t stands for zero temperature terms. It is convenient to group this contribution with the bare mass term in the Lagrangian in the form

\[
m^2(T) = -m_0^2 - \Sigma^{(1)} = \lambda T^{d-1} \mathcal{A}(d) - m_R^2(0) \approx a(T - T_c) \quad \text{for} \quad T \sim T_c = \left( \frac{m_R^2}{\lambda \mathcal{A}(d)} \right)^{\frac{1}{d-1}} \tag{V.39}
\]

where the zero temperature contributions (denoted by z.T.t in eqn. (V.38)) had been absorbed in \( m_R^2(0) \). We re-organize the perturbative expansion by re-writing the Euclidean Lagrangian in the form

\[
\mathcal{L}_E = \frac{1}{2} \left( \frac{\partial_i \Phi^2}{v_0^2} + \frac{1}{2} \Delta \phi \right)^2 + m^2(T) \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 + \frac{1}{2} \delta m^2(T) \Phi^2 + \frac{\delta \lambda}{4} \Phi^4 \tag{V.41}
\]

where now the counterterm \( \delta m^2(T) \) is simply the one loop tadpole diagram evaluated for zero mass given by (V.38), it is of \( \mathcal{O}(\lambda) \) and is included in the perturbative expansion consistently.

To one-loop order the two point function is now given by

\[
\Gamma^{(2)}(p, s) = p^2 + \frac{s^2}{v_0^2} + m^2(T) - \Sigma_S \\
\Sigma_S = (\Sigma - \delta m^2(T)) = \frac{\lambda T}{2} \sum_{m} \int \frac{d^d q}{(2\pi)^d} \frac{m^2(T)}{q^2 + \omega_m^2} \frac{m^2(T)}{q^2 + \omega_m^2 + m^2(T)} \tag{V.42}
\]

The integral above is of the typical form as those studied in the previous sections (see (III.3)). Separating the \( m = 0 \) Matsubara contribution from the \( m \neq 0 \) for which we can set \( m^2(T) \propto (T - T_c) \approx 0 \) for \( T - T_c \ll T_c \), we obtain

\[
\sum_m \int \frac{d^d q}{(2\pi)^d} \frac{m^2(T)}{q^2 + \omega_m^2} \frac{m^2(T)}{q^2 + \omega_m^2 + m^2(T)} = m^2(T) \left[ \frac{\Gamma(\frac{d}{2}) \Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(1)} \right] \left[ m^2(T) + C(d) T^{\epsilon} \right] \tag{V.43}
\]

Again, for \( \epsilon > 0 \) we can neglect the second term in the brackets in the limit \( T \gg m(T) \). Expanding in \( \epsilon \) to obtain the lowest order contribution consistently in the \( \epsilon \) expansion, we obtain the two point function to one loop order

\[
\Gamma^{(2)}(p, s) \approx p^2 + \frac{s^2}{v_0^2} + m^2(T) \left[ 1 - \frac{g(\mu)}{\epsilon} + \frac{g(\mu)}{2} \ln \left( \frac{m^2(T)}{\mu^2} \right) \right] \tag{V.44}
\]

The renormalized mass parameter \( m_R(T) \) is defined by

\[
Z_m m^2(T) = Z_m m_R^2(T) = m_R^2(T) \tag{V.45}
\]

and \( Z_m \) is fixed by the renormalization condition

\[
\Gamma^{(2)}(p = 0, s = 0, m_R^2 = \mu^2) = \mu^2 \tag{V.46}
\]

Since \( Z_m \) receives corrections at \( \mathcal{O}(g^2) \) we choose \( Z_m \) to lowest order in the \( \epsilon \) expansion to be

\[
Z_m = 1 + \frac{g(\mu)}{\epsilon} + \mathcal{O}(g^2, g\epsilon) \tag{V.47}
\]

1. Static aspects:

Before we embark on a full discussion of the dynamics away from the critical point, it proves convenient and illuminating to discuss the static aspects first. In particular since we will study the \( p = 0 \) case but \( T \neq T_c \) a relevant quantity is the inverse susceptibility \( \chi^{-1}(T) \), which is defined as

\[
\chi^{-1}(T) = M^2(T) = \Gamma^2(p = 0; s = 0) \tag{V.48}
\]

which near the critical point and the non-trivial fixed point \( g^* \) given by eqn. (V.14) is given by

\[
M^2(T \sim T_c) \approx m_R^2(T) \left[ 1 + \frac{g^*}{2} \ln \left( \frac{m_R^2(T)}{\mu^2} \right) \right] \approx \left[ m_R^2(T) \right]^{1 + \frac{g^*}{2}} \tag{V.49}
\]
where we anticipated an exponentiation of the leading logarithms via the renormalization group, which will be borne out by the renormalization group analysis below. Recalling that $m^2(T) \propto |T - T_c|$ by eqn. (V.39), we find

$$\chi^{-1}(T \sim T_c) \propto \left[ \frac{g^2 T^2}{2} \right]^y = 1 + \frac{\epsilon}{6} + \ldots$$

(V.49)

The critical exponent $\gamma$ is seen to be the correct one [4, 21, 22, 23].

Just as in the case of the theory at the critical point studied above, we now study the dynamics of the theory in an $\epsilon$ expansion and implement a resummation of the leading infrared divergences via the renormalization group.

2. Dynamics away from the critical point

As argued above the leading infrared behavior is obtained by setting the internal Matsubara frequencies to zero in the two loops self energy. In $d = 4 - \epsilon$ spatial dimensions, the self-energy at two loops is

$$\Sigma^{(2)}(\mu; s; m(T)) = \frac{\lambda^2 T^2}{6} \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{\delta}{[\mu^2 + m^2(T)][(q + k + p)^2 + \frac{\beta}{v_0^2} + m^2(T)]}$$

(V.50)

The loop integrals are evaluated by introducing two Feynman parameters leading to

$$\Sigma^{(2)}(\mu; s; m(T)) = g^2(\mu) \Gamma(-1 + \epsilon) \mu^{2 \epsilon} \int_0^1 dx \int_0^1 dy \left[ x(1-x) \frac{1}{\epsilon} y^{\epsilon - 1} \right] \left[ x(1-y) y(1-y) \mu^2 + \right.

+ (1-y)x(1-x) m^2(T) + y m^2(T) + x \frac{s^2}{v_0^2} \right]^{1-\epsilon}$$

(V.51)

It is convenient to separate the static contribution from the dynamical part by writing

$$\Sigma^{(2)}(\mu; s; m(T)) = \tilde{\Sigma}^{(2)}(\mu; 0; m(T)) + \tilde{\Sigma}^{(2)}(\mu; s; m(T))$$

(V.52)

The static contribution $\tilde{\Sigma}^{(2)}(\mu; 0; m(T))$ leads to wave function renormalization a renormalization of the mass and to $O(\epsilon^2)$ corrections to the static anomalous dimensions, which will be neglected to leading order in the $\epsilon$ expansion. The second, dynamical contribution is obtained consistently in an $\epsilon$ expansion: the regions of the integrals in the Feynman parameters that lead to inverse powers of $\epsilon$ in an $\epsilon$ expansion are $x \sim 0, 1$ and $y \sim 0$. The contributions of these regions can be isolated by partial integration, and after some straightforward algebra we find

$$\tilde{\Sigma}^{(2)}(\mu; s; m(T)) = g^2(\mu) \ln \left[ 1 + \frac{s^2}{v_0^2} \right] + 2 \frac{s^2}{v_0^2} \ln \left( \frac{m^2(T)}{\mu^2} \right) + 2 \left( \frac{m^2(T)}{\mu^2} + \frac{s^2}{v_0^2} \right) \ln \left( 1 + \frac{s^2}{v_0^2} \right) + O(\epsilon^2, \epsilon)$$

(V.53)

Thus, putting together the one loop contribution found above and the two loop contribution found above, the two point function at zero spatial momentum but away from the critical point is found to be

$$\Gamma^{(2)}(\mu; s; m(T)) = \mu^2 \left[ 1 + \frac{g^2(\mu)}{12} + O(\epsilon^2) \right] + \frac{s^2}{v_0^2} + m^2(T) \left[ 1 - \frac{g^2(\mu)}{\epsilon} + \frac{g^2(\mu)}{2} \ln \left( \frac{m^2(T)}{\mu^2} \right) \right] + \frac{g^2(\mu)}{6\epsilon} \frac{s^2}{v_0^2} - \frac{s^2}{v_0^2} \ln \left( \frac{m^2(T)}{\mu^2} \right) - 2 \left( m^2(T) + \frac{s^2}{v_0^2} \right) \ln \left( 1 + \frac{s^2}{v_0^2} \right) + O(g^2, g^2 \epsilon)$$

(V.54)

where we have neglected logarithmic corrections that will exponentiate to anomalous dimensions of $O(\epsilon)$ for momentum dependent terms. We have only displayed the contribution that will be cancelled by wave function renormalization just as in the critical case.

Obviously the $m^2(T) = 0$ limit coincides with the two point function at the critical point (IV.8) to leading order $O(\epsilon)$. In the above expression we have not included the two loop contribution to the static $s = 0$ part, since it will lead to an $O(\epsilon^2)$ correction to the critical exponent for the correlation length (inverse susceptibility).

The renormalization conditions for the two point function away from the critical point are now summarized as follows:
\[
\Gamma^{(2)}_R(p, s; m_R(T)) = Z_\phi \Gamma^{(2)}(p, s, v_0; m(T))
\]
\[
v_R^2 = v_0^2 \frac{Z_\rho}{Z_\phi} ; \quad Z_\phi m^2(T) = m_R^2(T) Z_m
\]
\[
\frac{\partial \Gamma^{(2)}_R}{\partial p^2} \bigg|_{p^2 - \mu^2, \frac{v_R^2}{v_0^2} - \mu^2} = 1 ; \quad \frac{\partial \Gamma^{(2)}_R}{\partial s^2} \bigg|_{p^2 - \mu^2, \frac{v_R^2}{v_0^2} - \mu^2} = \frac{1}{v_R^2}
\]
\[
\Gamma^{(2)}_R(p = 0, s = 0; m_R^2(T; \mu^2)) = \mathcal{O}(v, \epsilon^2)
\]
\[
\Gamma^{(2)}_R(p; m_R(T)) = p^2 + m^2(T) \left[ 1 + \frac{g(\mu)}{2} \ln \left( \frac{m_R^2(T)}{\mu^2} \right) \right] + \frac{s^2}{v_R^2} \left[ 1 - \frac{g^2(\mu)}{3\epsilon} \ln \left( \frac{m_R^2(T)}{\mu^2} \right) \right] - \frac{g^2(\mu)}{\epsilon} \left( \frac{m_R^2(T)}{v_0^2} + \frac{s^2}{v_R^2} \right) \ln \left( 1 + \frac{s^2}{v_R^2} m_R^2(T) \right) + \mathcal{O}(g^2, g^2\epsilon)
\]
\[
\tilde{m} = m_R^2(T) / \mu^2
\]
\[
\beta_n = \mu \frac{\partial \tilde{m}}{\partial \mu} \bigg|_{m_R(T), \lambda_0} = (2n - 2) \tilde{m}^2
\]
\[
\gamma_n = g + \mathcal{O}(g^2, \epsilon)
\]

The new ingredient as compared to the critical case (V.1) is the dependence on \( \tilde{m} \). Following the same steps as for the critical case, we now find that the solution of the renormalization group equation for the two point function obeys
\[
\Gamma^{(2)} \left( \epsilon^p, \frac{\epsilon^s}{v}, g, \tilde{m}, \mu \right) = e^{\int_0^t \frac{dt}{\epsilon(t)} (2n - 2) \tilde{m}^2(t)} \Gamma^{(2)} \left( p, \frac{s}{v(t)}, g(t), \tilde{m}^2(t), \mu \right)
\]

with \( \tilde{m}(t) \) the solution of the differential equation
\[
\frac{\partial \tilde{m}^2(t)}{\partial t} = \beta_n(g(t), v(t), \tilde{m}(t))
\]

with the initial condition
\[
\tilde{m}^2(0) = \frac{m_R^2(T, \mu)}{\mu^2} \propto |T - T_c(\mu)|
\]

In the infrared the coupling is driven to the non-trivial fixed point \( g^* = \epsilon/3 \) and
\[
\tilde{m}^2(t) \to \tilde{m}^2(0) e^{(1/\gamma_m - 2)\epsilon} \quad : \quad \gamma_m = \frac{\epsilon}{3}
\]
Just as in the solution of the renormalization group equation at criticality near the fixed point (V.18, V.19), introducing \( p e^t \equiv P; \, s e^t \equiv S \) we now find
\[
\Gamma^{(2)} \left( P, \frac{S}{v}, g, \mu \right) = e^{t(2-\eta)} \Gamma^{(2)} \left( P e^{-t}, \frac{S}{v(0)} e^{-tr^{(2-\eta)}}, g^*; \, \tilde{m}_1^2(0) e^{(\gamma_s - 2)\eta} \right) \quad (V.66)
\]
Following the analysis of the critical case, and the scaling property (V.8) and writing \( P = \mu e^t \) we find the following scaling form
\[
\Gamma^{(2)} \left( P, \frac{S}{v}, g, \mu \right) = \mu^2 \left[ \frac{P}{\mu} \right]^{2-\eta} \Psi \left( \frac{S}{v(0)\mu^{1-\eta}}, P \xi \right) \quad (V.67)
\]
\( \xi \) is therefore identified with the correlation length\([4, 21, 22, 23]\)
\[
\xi \sim |T - T_c|^{-\nu} \quad ; \quad \nu = \frac{1}{2} - \frac{\gamma_s}{2} \sim \frac{1}{2} + \frac{\epsilon}{12} + \ldots \quad (V.68)
\]
It is important to note at this stage that the correlation length \( \xi \) is a renormalization group invariant, as can be easily checked by using (V.58) with the renormalization group beta function (V.59).
To study the limit of zero spatial momentum it is more convenient to rewrite the above scaling solution in the following form
\[
\Gamma^{(2)} \left( P, \frac{S}{v}, g, \mu \right) = \mu^2 \left[ \frac{m_R^2(T, \mu)}{\mu^2} \right]^{\frac{\nu}{\gamma_s}} \Psi \left( \frac{S}{v(0)\mu^{1-\eta}}, P \xi \right) \quad (V.69)
\]
From the definition of the inverse susceptibility \( M^2(T) \equiv \chi^{-1}(T) \equiv \Gamma^{(2)}(p = 0, s = 0) \) we find the known result\([4, 21, 22, 23, 25]\)
\[
\chi^{-1}(T) \propto |T - T_c|^{-\gamma} \quad ; \quad \gamma = \frac{2 - \eta}{2} \quad (V.70)
\]
Furthermore the two-point function is a function of two renormalization group invariant, dimensionless scaling variables
\[
\Gamma^{(2)}(p, s, m_R^2(T, \mu)) = \mu^2 \left[ \frac{m_R^2(T, \mu)}{\mu^2} \right]^{\frac{\nu}{\gamma_s}} \Psi \left( \varphi, \delta \right) \quad (V.71)
\]
with
\[
\varphi = \left[ \frac{s}{v(0)\mu} \right]^2 \left[ \frac{m_R^2(T)}{\mu^2} \right]^{\frac{\nu}{\gamma_s}} \equiv \left[ \frac{s}{v(0)\mu^{1-\eta}} \right]^2 \quad (V.72)
\]
\[
\delta = \frac{p^2}{\mu^2} \left[ \frac{m_R^2(T, \mu)}{\mu^2} \right]^{\frac{\nu}{\gamma_s}} \equiv (p \xi)^2 \quad (V.73)
\]
The renormalization condition (V.56) determines that \( \Psi(0, 0) = 1 \).
We can now follow the arguments provided in the previous subsection for the critical case. Under the analytic continuation \( s^2 \rightarrow -\omega^2 - i \text{sign}(\omega) \, 0^+ \)
\[
\varphi \rightarrow -\Omega^2 - i \text{sign}(\Omega) \, 0^+ \quad ; \quad \Omega = \frac{\omega \xi^\gamma}{v(0)\mu^{1-\eta}} \quad (V.74)
\]
\[
\Psi(\varphi = -\Omega^2 - i \text{sign}(\Omega) \, 0^+, \delta) = \Psi_R(\Omega, \delta) + i \, \Psi_I(\Omega, \delta) \quad (V.75)
\]
The position of the quasiparticle pole in the two point Green’s function corresponds to the value of \( \Omega = \Omega^*(\delta) \) for which \( \Psi_R(\Omega^*(\delta), \delta) = 0 \). This condition determines the dispersion relation of the quasiparticle and is given by
\[
\omega_p = \Omega^*(\delta) \, v(\mu) \, \mu^{1-\eta} \xi^{-\eta} \quad (V.76)
\]
this expression emphasizes that the dispersion relation depends on \( p \) through the scaling variable \( \delta = (p \xi)^2 \).

Assuming that near the quasiparticle pole \( \Psi_R \) vanishes linearly the Green’s function can be approximated by

\[
G(p, \omega, m_R(T)) \sim \frac{(\mu \xi)^{2-\eta}}{\mu^2} \frac{1}{(\Omega - \Omega^*) \Psi_R(\Omega^*, \delta) + i \Psi_I(\Omega^*, \delta)}
\]

where \( \Psi_R(\Omega^*, \delta) = \partial \Psi_R(\Omega, \delta) / \partial \Omega \vert_{\Omega=\Omega^*} \). Near the quasiparticle pole we can further write the above expression in the Breit-Wigner form

\[
G_{BW}(p, \omega, m_R(T)) \sim \frac{Z_p}{\omega - \omega_p + i \Gamma_p}
\]

with

\[
\omega_p = \Omega^* \frac{v(\mu) \mu^{1-\frac{z}{2}}}{\xi^2}
\]

\[
Z_p = \frac{\Psi_R(\Omega^*)}{\Psi_R(\Omega^*)} \frac{v(\mu) \mu^{1-\frac{z}{2}}}{(\mu \xi)^{2-\eta}}
\]

\[
\Gamma_p = \frac{\omega_p \Psi_I(\Omega^*)}{\Psi_I(\Omega^*)} \frac{\Omega^*(\delta) \Psi_I(\Omega^*)}{\Omega^*(\delta) \Psi_I(\Omega^*)} \sim |T - T_c|^\nu
\]

where we have suppressed the dependence on the scaling variable \( \delta \) in the arguments of the real and imaginary parts to avoid cluttering of notation. Furthermore, we have made explicit the combination of static and dynamic critical exponents using the expression given in eqn. (V.68) for the static critical exponent \( \nu \) and the dependence on the momentum is implicit through the dependence on the scaling variable \( \delta \) of \( \Omega^* \) as well as the explicit dependence of the real and imaginary parts.

Again, the imaginary part must be proportional to the anomalous dimensions, hence perturbatively small in the \( \epsilon \) expansion. Therefore the expression for the width (V.81) reveals both critical slowing down since \( \Gamma_p \sim |T - T_c|^\nu \) vanishing at \( T = T_c \), and the validity of the quasiparticle picture since \( \Gamma_p / \omega_p \ll 1 \) in the \( \epsilon \) expansion.

At this point we recognize a fundamental difference with the Wilsonian results of reference[10]. While in ref. [10] the width was found to be proportional to \( |T - T_c|^\nu \) up to logarithms, we see from (V.81) that the quasiparticle width actually involves the new dynamical anomalous exponent \( z \). The difference can be traced to the fact that the Wilsonian approach advocated in ref.[10] does not include two loop diagrams which are necessary to reveal the anisotropic renormalization through the renormalization of the speed of light and are directly responsible for the new dynamical anomalous exponent \( z \).

We emphasize that the above Breit-Wigner form as well as the quasiparticle properties rely on the assumption that the real part of the scaling function vanishes linearly near the quasiparticle pole. As emphasized before in the critical case this need not be the general situation, and anomalous power laws can lead to threshold singularities as discussed above.

While the solution of the renormalization group leads to a scaling form of the two point correlation function, it does not explicitly specify the scaling function \( \Psi \). However, we can obtain the function \( \Psi \) by matching the leading logarithms to those of the perturbative expression (V.57) evaluated at the fixed point \( g^* = \epsilon / 3 \) to lowest order in the \( \epsilon \) expansion. Matching the leading logarithms and assuming their exponentiation via the renormalization group it is straightforward to see that the two point function is given by

\[
\Gamma^{(2)}(p, s, m_R(T, \mu)) \sim \mu^2 \left[ \frac{m_R^2(T, \mu)}{\mu^2} \right]^{\frac{\gamma_p}{2}} \left\{ \delta + [1 + \varphi]^{2-\frac{z}{2}} \right\}
\]

where we have used the lowest order results in the \( \epsilon \) expansion

\[
\gamma_p = \frac{\epsilon}{3} \; ; \; z = 1 + \frac{\epsilon}{2\eta} \; ; \; \eta = \mathcal{O}(\epsilon)
\]

and kept consistently the lowest \( \mathcal{O}(\epsilon) \) in the exponentiation of the leading logarithms leading to (V.82). Thus, we obtain the lowest order result for the scaling function

\[
\Psi(\varphi, \delta) = \delta + [1 + \varphi]^{2-\frac{z}{2}}
\]

We can now obtain an explicit form of the real and imaginary parts of the scaling function that enter in the quasiparticle parameters. This is achieved by performing the analytic continuation (V.74) which leads to

\[
\Psi_R(\Omega) + i \Psi_I(\Omega) = \delta + [1 - \Omega^2 - i \text{sign}(\Omega) \Theta(0^+)]^{2-\frac{z}{2}}
\]

\[
= \delta - [\Omega^2 - 1]^{2-\frac{z}{2}} \left[ 1 + i \pi (z - 1) \text{sign}(\Omega) \Theta(\Omega^2 - 1) \right]
\]
For $p = 0$ i.e. $\delta = 0$ we see that both the real and imaginary part of the scaling function vanish at $\Omega^* = 1$ with an anomalous power law providing an explicit example of the case of threshold singularities mentioned above.

For $p \neq 0$ and $T \neq T_c$ we find a quasiparticle pole at

$$\Omega^* = 1 + \delta \frac{1}{2\pi \epsilon} \sim 1 + (p \xi)^2$$  \hspace{1cm} (V.86)

where we have approximated the anomalous dimension by its leading order in $\epsilon$ using $z = 1 + \epsilon/27$. From this expression for $\Omega^*$ we obtain the dispersion relation for quasiparticles

$$\omega_p^2 = v^2 (\mu)^2 \left\{ \frac{m^2_R(T, \mu)}{\mu^2} \right\}^{2z} + \left\{ \frac{p^2}{\mu^2} \right\}^{2z}$$  \hspace{1cm} (V.87)

with $\nu$ given by eqn. (V.68), in particular we find that the frequency of zero momentum quasiparticles $\omega_{p=0} \propto |T - T_c|^z$. Obviously at $T = T_c$ $(m_R(T) = 0)$ the dispersion relation coincides with that of the critical case given by eqn. (V.25). For $p \neq 0$, i.e. $\delta \neq 0$ the real part of the scaling function vanishes linearly and the Breit-Wigner approximation (V.78) near the quasiparticle pole is valid and the relations (V.79-V.81) describe the properties of the quasiparticles.

To lowest order in the $\epsilon$ expansion we find, using (V.23) that

$$\frac{\Psi_p(\Omega^*)}{\Omega^* (\delta)} \Psi_p(\Omega^*) = \frac{\pi \eta_\epsilon}{4} - \frac{(p \xi)^{2z}}{1 + (p \xi)^{2z}}$$  \hspace{1cm} (V.88)

for $p = 0$, i.e. $\delta = 0$ this ratio vanishes and $\Psi_p(\Omega^*) \propto |\Omega^* - 1|^{1-z^2}$ diverges displaying the phenomenon of threshold singularity with a residual divergence $Z_p$.

For $p \neq 0$ but $T \rightarrow T_c$ ($\delta \rightarrow \infty$) this ratio equals that of the critical case (see eqn. (V.36)).

For $T \neq T_c; p \neq 0$ we finally find the width of the long wavelength quasiparticles to be given to lowest order in the $\epsilon$ expansion by

$$\Gamma_p \sim \frac{\pi \eta_\epsilon}{4} \frac{(p \xi)^{2z}}{1 + (p \xi)^{2z}} \left\{ \frac{v(\mu)}{\mu^{1-z}} \right\}^{1+z} \left[ 1 + (p \xi)^{2z} \right]^{1/2}$$  \hspace{1cm} (V.89)

with the following behavior to lowest order in the $\epsilon$ expansion

$$\Gamma_p \sim \frac{\pi \eta_\epsilon}{4} \frac{v(\mu)}{\mu^{1-z}} \left\{ \begin{array}{ll} \frac{p^2}{\xi^2} & \text{for } p \text{ fixed, } T \rightarrow T_c \\ \frac{p^2}{\xi^2} & \text{for } \xi \text{ fixed, } p \rightarrow 0 \end{array} \right. \hspace{1cm} (V.90)$$

Thus, critical slowing down emerges in both limits, furthermore the validity of the quasiparticle picture is warranted in the $\epsilon$ expansion, since $\eta_\epsilon \simeq 2(z-1) = 2\epsilon/27 + \mathcal{O}(\epsilon^2) \ll 1$.

VI. SUMMARY OF RESULTS:

Critical phenomena, both static and dynamic in quantum field theory at finite temperature results in dimensional reduction since momenta and frequencies are $p, \omega \ll T$ and the correlation length is $\xi \gg 1/T$. The infrared physics is dominated by the contribution of the zero Matsubara frequency in internal loops, which in turn results in an effective coupling $\lambda T$ in the perturbative expansion. Naive perturbation theory at high temperature breaks down in four spacetime dimensions because of the strong infrared behavior of loop diagrams near the critical point for long-wavelength phenomena.

We propose an implementation of the renormalization group to study dynamical critical phenomena which hinges upon two main ingredients:

- The leading infrared behavior near the critical point is determined by keeping only the zero Matsubara internal frequency in the loops. To control the infrared consistently we implement an expansion in $\epsilon$ in $\delta - \epsilon$ space-time dimensions. Dimensional reduction for long-wavelength phenomena near the critical point results in that the perturbative expansion is in terms of $g(\mu) \propto \lambda T \mu^{-\epsilon}$ where $\mu$ is the scale of external momenta and frequencies in the diagram. The renormalized effective coupling is driven to a fixed point in the infrared which is of $\mathcal{O}(\epsilon)$. Therefore long-wavelength phenomena can be studied in perturbation theory around this fixed point for $\epsilon \ll 1$. The perturbative expansion is improved by implementing a renormalization group resummation which reveals dynamical scaling phenomena with anomalous dimensions. Eventually the limit of physical interest $\epsilon \rightarrow 1$ must be studied by further Borel and/or Padé resummations.
The second important ingredient is the \textit{anisotropic} scaling between space and time. While space is infinite, at finite temperature in the Euclidean formulation the time direction is compactified to the interval \([0, 1/T]\). We introduce a new parameter, the effective speed of light in the medium, which is renormalized and runs with the renormalization transformations. The infrared renormalization of the speed of light results in a new \textit{dynamical} anomalous exponent which determines the dispersion relation and all the quasiparticle properties. The \(\epsilon\) expansion combined with the renormalization group leads to a consistent quasiparticle description of long-wavelength excitations near the critical point.

The critical exponents, both static and dynamic are summarized below for the critical case \(T = T_c\) as well as for \(T \neq T_c\) in the symmetric phase with \(T \to T_c^+\).

### Table 1: Quasiparticles at \(T = T_c\)

| \(\omega_p \propto\) | \(p^\beta\) |
| \(v_p \propto\) | \(p^{\beta - 1}\) |
| \(Z_p \propto\) | \(p^{\beta + \eta - 2}\) |
| \(\Gamma_p \propto\) | \(\eta_b p^\beta\) |
| \(\eta =\) | \(\frac{m_p(T_c)}{\mu^2} + O(\epsilon^2)\) (static) |
| \(\eta_b =\) | \(\frac{m_p(T_c)}{\mu^2} + O(\epsilon^2)\) (dynamic) |
| \(z =\) | \(1 + \frac{1}{2}(\eta_b - \eta) = 1 + \frac{m_p(T_c)}{\mu^2} + O(\epsilon^2)\) (dynamic) |

\[(VI.1)\]

### Table 2: Quasiparticles at \(T > T_c\)

| \(\omega_p^2 \propto\) | \(\frac{m_p^2(T_c, \mu)}{\mu^2} + \frac{\epsilon^2}{\mu^2}\) |
| \(\Gamma_p \propto\) | \(\eta_b \omega_p \frac{m_p(T_c)}{\mu^2} + \frac{\epsilon^2}{\mu^2}\) |
| \(\xi \propto\) | \(\frac{|T - T_c|^{-\nu}}{|T - T_c|}\) |
| \(m_p(T) \propto\) | \(\frac{|T - T_c|^{\nu}}{|T - T_c|}\) |
| \(\nu =\) | \(\eta + \frac{m_p(T_c)}{\mu^2} + O(\epsilon^2)\) (static) |
| \(\eta_b =\) | \(\frac{m_p(T_c)}{\mu^2} + O(\epsilon^2)\) (dynamic) |
| \(z =\) | \(1 + \frac{1}{2}(\eta_b - \eta) = 1 + \frac{m_p(T_c)}{\mu^2} + O(\epsilon^2)\) (dynamic) |

\[(VI.2)\]

The new dynamical exponent \(z\) is missed by the Wilsonian approach advocated in reference\[10\] since two loops diagrams are completely neglected in that approach and anisotropic rescaling of frequency and momenta becomes manifest at two loop order and beyond.

#### Critical exponents for \(O(N)\) symmetry:

At this stage we can generalize our results to the case of a scalar theory with \(O(N)\) symmetry at or slightly above the critical point. While the static critical exponents for the \(O(N)\) case are available in the literature\[4, 21, 22, 23, 24\], the dynamical critical exponent to lowest order in \(\epsilon\) can be obtained simply by recognizing that the symmetry factors corresponding to the \(O(N)\) theory multiply the two loop expression for the self-energy by an overall factor. From the expression \((IV.8)\) we see that the coefficient of \(s^2/\nu^2\) is a factor \(4/\nu\) times the coefficient of \(p^3\), which immediately leads to the result

\[\eta_b = \frac{4}{\nu} \eta (VI.3)\]

Since for the \(O(N)\) theory \(\eta = \epsilon^2 (N + 2)/(2(N + 8)) + O(\epsilon^2)\) we find to lowest order in \(\epsilon\)

\[\eta_b = \epsilon^2 \frac{2(N + 2)}{(N + 8)^2} + O\left(\epsilon^3\right).

In summary the static and dynamic critical exponents to lowest order in the \(\epsilon\) expansion for the \(O(N)\) theory are given by

### Table 3: Critical exponents for \(O(N)\),

| \(\nu =\) | \(\eta + \frac{N + 2}{2(N + 8)^2} \epsilon + O(\epsilon^2)\) (static) |
| \(\eta =\) | \(\epsilon^2 \frac{N + 2}{2(N + 8)^2} + O(\epsilon^3)\) (static) |
| \(\eta_b =\) | \(\epsilon^2 \frac{2(N + 2)}{(N + 8)^2} + O(\epsilon^3)\) (dynamic) |
| \(z =\) | \(1 + \epsilon^2 \frac{N + 2}{(N + 8)^2} + O(\epsilon^3)\) (dynamic) |

\[(VI.4)\]
VII. CONCLUSIONS, DISCUSSION AND IMPLICATIONS

We have studied the dynamical aspects of long-wavelength (collective) excitations at and near the critical point in scalar quantum field theories at high temperatures. After recognizing that naive perturbation theory breaks down at high temperature in the long-wavelength limit, we introduced an $\epsilon$ expansion around $5 - \epsilon$ space time dimensions combined with the renormalization group at high temperature to resum the perturbative series.

The effective long-wavelength theory at high temperature is described by a non-trivial fixed point at which the correlation functions feature scaling behavior. The anisotropy between spatial and time coordinates in Euclidean space-time at finite temperature leads to consider the renormalization of the speed of light, which, in turn leads to a new dynamical exponent $\nu$. All dynamical quantities, such as the dispersion relation and widths of long-wavelength quasiparticle (collective) excitations depend on this new dynamical exponent, as well as the static exponents.

Our results are summarized in the tables in the previous section.

Two very important aspects emerge from this treatment: i) critical slowing down, i.e., the relaxation rate of the quasiparticle vanishes in the long-wavelength limit or at the critical point with definite anomalously dimensions determined by the new dynamical exponent $\nu$; and ii) the quasiparticle picture, i.e., narrow widths $\Gamma_\mu \ll \omega_\mu$ is valid. The group velocity of quasiparticles vanishes at the critical point in the long-wavelength limit revealing the collective aspects of these excitations.

As mentioned in the introduction these phenomena have phenomenological implications for the chiral phase transition in the Quark Gluon plasma with potential observational consequences if long-wavelength pion fluctuations freeze out at the chiral phase transition. An important aspect revealed by this program is that the effective coupling $\lambda_{eff} T^2 \mu^{-\nu}$ is driven to the Wilson-Fischer fixed point in the infrared, this in turn means that in this limit $\lambda_{eff} \rightarrow 0$. This may be important in the linear Sigma model description of low energy QCD near the critical point and may give rise to interesting phenomenological consequences.

In this article we focused our attention on the approach to the critical temperature from above, therefore our results regarding the dynamical exponent $\nu$ are valid in the symmetric phase. An important question that we are currently addressing is the relaxation of pions slightly below $T_c$. Since the scattering amplitude of pions (at zero temperature) vanishes in the long-wavelength limit we expect novel behavior of critical slowing down for pion fluctuations below the critical temperature.

We expect to report on our findings on these and other related issues soon.

While we have provided a quantitative implementation of the program of the $\epsilon$ expansion with the resummation via the renormalization group, the physical limit $\epsilon \rightarrow 1$ requires higher order calculations with Borel or Pade resummations much in the same way as in static critical phenomena. We have studied the dynamical aspects to lowest order in the $\epsilon$ expansion but clearly a formal proof of the consistency of the $\epsilon$ expansion to higher orders, just as in usual critical phenomena, must be explored.

While clearly such programs are beyond the scope and goals of this article, we here provided the first steps of the program whose potential phenomenological implications as well as intrinsic interest in finite temperature quantum field theory warrant further study.

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APPENDIX A: ONE-LOOP DIAGRAM AT HIGH TEMPERATURE

We derive in this appendix the behaviour of the one-loop diagram $H(p, s)$ contributing to the four points function for high temperatures $T \gg p, s$. Setting $v = 1$ to avoid cluttering of notation (the velocity of light is not relevant for the discussion in this section), we have from eq. (II.7)

$$H(p, s) = \frac{T}{2} \sum_{i \in \mathbb{Z}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 + (2\pi T l)^2][(q + \vec{p})^2 + (2\pi T)^2 (n + l)^2]},$$

(A.A1)
where $d = 4 - \epsilon$ is the number of spatial dimensions. The denominators in eq. (A.A1) can be combined using Feynman parameters with the result

$$H(p, s) = \frac{T}{2} \sum_{l \in \mathbb{Z}} \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \frac{1}{[q^2 + At(x, p, s)]^2} = \frac{T}{2(4\pi)^{\frac{d-2}{2}}} \Gamma \left( \frac{d}{2} \right) \int_0^1 dx \frac{A_0(x, p, s)}{[1 - (1 - x)p^2 + s]^2}^{\frac{d}{2} - 2},$$

(A.A2)

where we integrated over the spatial momenta and

$$A_t(x, p, s) = x(1 - x)(p^2 + s^2) + (\omega_t + xs)^2; \quad \omega_t = 2\pi Tt; \quad s = 2\pi Tn.$$

(A.A3)

We single out now the contribution from the $l \neq 0$ mode and study the behaviour of the sum over $l \neq 0$ for large $T \gg p, s$.

Let us first evaluate the $l = 0$ term in the sum (A.A2).

$$\frac{T}{2(4\pi)^{\frac{d-2}{2}}} \Gamma \left( \frac{d}{2} \right) \int_0^1 dx \frac{A_0(x, p, s)}{[1 - (1 - x)p^2 + s]^2}^{\frac{d}{2} - 2} =$$

$$= \frac{T}{2(4\pi)^{\frac{d-2}{2}}} \Gamma \left( \frac{d}{2} \right) \int_0^1 dx \frac{1}{[1 - (1 - x)p^2 + s]^2} + \frac{1}{[1 - (1 - x)p^2 + s]} \mu^{2-\epsilon} F \left( \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}, 2 - \frac{\epsilon}{2}, \frac{p^2 + s^2}{p^2 + s} \right),$$

(A.A4)

where $F(a, b; c; z)$ stands for the hypergeometric function [33].

We have for the $l \neq 0$ terms in the high temperature limit,

$$[A_t(x, p, s)]^{\frac{d}{2} - 2} F_{T \gg p, s} (2\pi T|l|^{-\epsilon} - \text{sign}(l) \frac{\epsilon x}{2(2\pi T)^{1+\epsilon}} - \frac{\epsilon x}{2(2\pi T)^{1+\epsilon}} (1 - x)p^2 + s^2[1 - (2 + \epsilon)x]) + O \left( T^{-3-\epsilon} \right).$$

(A.A5)

The sum over $l \neq 0$ then yields in the high temperature limit,

$$\frac{T}{2(4\pi)^{\frac{d-2}{2}}} \Gamma \left( \frac{d}{2} \right) \int_0^1 dx \sum_{l \neq 0} [A_t(x, p, s)]^{\frac{d}{2} - 2} F_{T \gg p, s}$$

$$= \frac{(2\pi T)^{1-\epsilon} \Gamma \left( \frac{d}{2} \right)}{2(4\pi)^{\frac{d-2}{2}}} \left\{ \frac{\zeta(\epsilon)}{\pi} - \frac{\epsilon \zeta(2 + \epsilon)}{12 \pi (2\pi T)^2} [p^2 - s^2(1 + 2\epsilon)] + O \left( T^{-3-\epsilon} \right) \right\},$$

(A.A6)

We see that only the first term in the r.h.s. is important for $0 < \epsilon < 1$ and high temperature. This term is the dominant high temperature limit of the sum of nonzero Matsubara modes. We then have,

$$H(p, s) \underset{T \gg p, s}{\cong} H_{\text{ss}}(p, s) - \frac{C(2 + \epsilon)}{384 \pi^{1+\epsilon} T^{1+\epsilon}} [p^2 - s^2(2 - \epsilon)] + O \left( \frac{p^2, s^2}{T^2} \right),$$

where

$$H_{\text{ss}}(p, s) \equiv \frac{T}{2} \mu^{2-\epsilon} \Gamma \left( \frac{d}{2} - 1 \right) \frac{s^2 + p^2}{|\mu|^2}^{\frac{d}{2} - 2} F \left( \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}, 2 - \frac{\epsilon}{2}, \frac{p^2 + s^2}{p^2 + s} \right) +$$

$$\frac{\mu^{2-\epsilon} C(\epsilon)}{8 \pi^{1+\epsilon} \epsilon} T^{1-\epsilon}.$$  

(A.A7)

**APPENDIX B: TWO LOOP DIAGRAM AT HIGH TEMPERATURE**

The renormalized two points function is given by (all quantities are renormalized below)
\[
\Gamma^{(2)}(p, s) = Z_\phi \frac{p^2}{v^2} s^2 - \Sigma^{(2)}(p, s) + \mathcal{O}(\lambda^3)
\]  
(B.B1)

where \( \Sigma^{(2)}(p, s) \) is the two-loops self-energy. Using the renormalization conditions (IV.13) we find for the wave function and velocity of light renormalizations,

\[
Z_\phi = 1 + \frac{\partial \Sigma^{(2)}(p, s)}{\partial p^2} \bigg|_{p = \mu, s = \mu v} + \mathcal{O}(\lambda^3),
\]

\[
Z_v = 1 + v^2 \frac{\partial \Sigma^{(2)}(p, s)}{\partial s^2} \bigg|_{p = \mu, s = \mu v} + \mathcal{O}(\lambda^3).
\]  
(B.B2)

The two loops contribution to the self energy \( \Sigma^{(2)}(p, s) \) is given by

\[
\Sigma^{(2)}(p, s) = \frac{\lambda^2 T^2}{6} \int_{t, j \in \mathbb{Z}} d^4 \xi \frac{d^4 \xi}{(2\pi)^{4-\epsilon}} \frac{1}{\left[ (\bar{p} + k + q)^2 + \frac{(\omega_1 + \omega_2 + s)^2}{v^2} \right]^2}.
\]  
(B.B3)

where \( \omega_j = 2\pi j T; s = 2\pi T n \).

We combine the propagators in eq.(B.B3) using Feynman parameters and integrate over the momenta with the result

\[
\Sigma^{(2)}(p, s) = \frac{\lambda^2 T^2}{6} \frac{\Gamma(\epsilon - 1)}{(4\pi)^{\epsilon}} \sum_{t, j \in \mathbb{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2}-1}}{[1 - x + x \xi(1 - \xi)]^{2-\epsilon}} \left\{ \frac{x(1 - x)\xi(1 - \xi)}{1 - x + x \xi(1 - \xi)} p^2 + \left( \frac{2\pi T}{v} \right)^2 \left[ j^2 x^{\xi} + l^2 x(1 - \xi) + (j + l + n)^2 \right] \right\}^{1-\epsilon}
\]  
(B.B4)

Using the definition (IV.6) for the dimensionless coupling, to order \( g^3 \) we find from eqs. (B.B2) and (B.B4)

\[
Z_\phi = 1 - \frac{g^2}{6} \left( \frac{v \mu}{2\pi T} \right)^{2\epsilon} \Gamma(\epsilon) \sum_{t, j \in \mathbb{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2}-1}(1 - x)\xi(1 - \xi)}{[1 - x + x \xi(1 - \xi)]^{2-\epsilon}} \left\{ \frac{x(1 - x)\xi(1 - \xi)}{1 - x + x \xi(1 - \xi)} \left( \frac{v \mu}{2\pi T} \right)^2 \left[ j^2 x^{\xi} + l^2 x(1 - \xi) + (j + l + \frac{\mu}{2\pi T})^2 \right] (1 - x) \right\}^{1-\epsilon},
\]

\[
Z_v = 1 - \frac{g^2}{3} \left( \frac{2\pi T}{v} \right)^{1-2\epsilon} \Gamma(\epsilon) \sum_{t, j \in \mathbb{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2}-1}(1 - x) \left( j + l + \frac{\mu}{2\pi T} \right) (1 - x) \right\}^{1-\epsilon},
\]  
(B.B5)

We find from the definition of the anomalous dimension (V.4) and (B.B5) for \( \gamma(g, \frac{T}{\mu}, v) \),

\[
\gamma(g, \frac{T}{\mu}, v) = g^2 \left( \frac{v \mu}{2\pi T} \right)^{2\epsilon} \frac{\Gamma(\epsilon + 1)}{3} \sum_{t, j \in \mathbb{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2}}(1 - x)^2 \xi(1 - \xi)}{[1 - x + x \xi(1 - \xi)]^{2-\epsilon}} \times \left( 1 - x + 2 x \xi(1 - \xi) \left[ 1 + \frac{\pi T}{v \mu} (j + l) \right] \right) \times \left\{ \frac{x(1 - x)\xi(1 - \xi)}{1 - x + x \xi(1 - \xi)} \left( \frac{2\pi T}{v} \right)^2 \left[ j^2 x^{\xi} + l^2 x(1 - \xi) + (j + l + \frac{v \mu}{2\pi T})^2 \right] (1 - x) \right\}^{1-\epsilon} + \mathcal{O}(g^3).
\]  
(B.B6)

We split the expression for \( \gamma(g, \frac{T}{\mu}, v) \) as follows,

\[
\gamma(g, \frac{T}{\mu}, v) = \gamma_1(g, v) + \gamma_2 g(g, \frac{T}{\mu}, v),
\]
where $\gamma_0(g, v)$ is the contribution from the zero Matsubara mode in eq.(B.B6)

$$\gamma_0(g, v) = g^2 \frac{\Gamma(\epsilon + 1)}{3} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2}} (1-x)^{1-\epsilon} \xi (1-\xi)}{[1-x + x\xi(1-\xi)]^{\frac{3-\epsilon}{2}}} + O(g^3)$$

and $\gamma_{nz}(g, \frac{T}{\mu}, v)$ stands for the contribution of the non-zero Matsubara modes.

For $T \gg \mu$, we see from eq.(B.B6) that $\gamma_{nz}(g, \frac{T}{\mu}, v)$ decreases as $\left(\frac{T}{T_H}\right)^{-2-\epsilon}$. [Notice that the coefficient of $\left(\frac{T}{T_H}\right)^{-1-\epsilon}$ vanishes by symmetry when summing over $j + l$.]

Therefore, $\gamma_0(g, v)$ dominates for $T \gg \mu$. $\gamma_0(g, v)$ can be easily computed for small $\epsilon > 0$ with the result,

$$\gamma_0(g, v) = g^2 \frac{\Gamma(1+\epsilon)}{3} \int_0^1 dx \int_0^1 d\xi \frac{(1-x)\xi (1-\xi)}{[1-x + x\xi(1-\xi)]^{3-\frac{\epsilon}{2}}} + O(g^2, g^3)$$

$$= g^2 + O(\epsilon g^2, g^3) \quad \text{(B.B7)}$$

Therefore,

$$\gamma(g, \frac{T}{\mu}, v) = \frac{g^2}{\epsilon} + O(\epsilon g^2, g^3, \frac{\mu^2}{T^2}) \quad \text{(B.B8)}$$

To the lowest non-trivial order in $g$, that is $g^2$ (two-loops), we find for the function $\beta_\epsilon(g, \frac{T}{\mu}, v)$

$$\beta_\epsilon(g, \frac{T}{\mu}, v) = \frac{g^2}{\epsilon} \gamma(g, \frac{T}{\mu}, v) \left(\frac{\partial}{\partial \mu} - 2\epsilon\right) \log Z_v$$

where the derivatives are now at constant (bare) $g$. Using eq.(B.B5) for $\log Z_v$ yields

$$W \equiv \left(\frac{\partial}{\partial \mu} - 2\epsilon\right) \log Z_v = g^2 \frac{\Gamma(1+\epsilon)}{3} \sum_{i,j \in \mathbb{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2}} (1-x)^{1-\epsilon}}{[1-x + x\xi(1-\xi)]^{\frac{3-\epsilon}{2}}} Q_{ij}(x, \xi)^{-\epsilon}$$

$$\left\{ \frac{x\xi(1-\xi)}{1-x + x\xi(1-\xi)} + 1 \right\} \frac{(1-x)\xi (1-\xi)}{\pi Q_{ij}(x, \xi)} + \left[ 1 + \frac{2\epsilon (1-x)}{Q_{ij}(x, \xi)} \right] (j+l) \frac{T}{\mu} \quad \text{(B.B10)}$$

where,

$$Q_{ij}(x, \xi) \equiv \frac{x(1-x)\xi(1-\xi)}{1-x + x\xi(1-\xi)} + \left(\frac{2\pi T}{\mu}\right)^2 \left[ j^2 x + l^2 x(1-x) \right] + (1-x) \left[ \frac{2\pi T}{\mu} (j+l) + 1 \right]^2$$

We find in the high temperature limit $T \gg \mu$ that this expression is dominated by its zero mode contribution $W_0$ (corresponding to $j = l = 0$),

$$W_0 = g^2 \frac{\Gamma(1+\epsilon)}{3} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2}} (1-x)^{1-\epsilon}}{[1-x + x\xi(1-\xi)]^{\frac{3-\epsilon}{2}}} \left[ \frac{x\xi(1-\xi)}{1-x + x\xi(1-\xi)} + 1 \right]^{-\epsilon}$$

which turns out to be $T$-independent.

The sum of non-zero terms gives a subdominant contribution for $T \gg \mu$ and $\epsilon$ strictly positive. We find from eq.(B.B10) after calculation,

$$W_{nz} \overset{\mu \gg T, \epsilon \ll 1}{\approx} g^2 \left(\frac{4\pi T}{2\mu}\right)^{2\epsilon} \frac{\Gamma(1+\epsilon)}{3} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2} - 1}(1-x)^{1-\epsilon}}{[1-x + x\xi(1-\xi)]^{\frac{3-\epsilon}{2}}}$$

$$\times \sum_{i,j \in \mathbb{Z}} \left[ j^2 (1-x + x\xi) + l^2 x - 2jl x \xi \right]^{\frac{\epsilon}{2} - \epsilon} + O\left(\frac{\mu^2}{T^2}, g^3\right) \quad \text{(B.B11)}$$

For $0 < \epsilon \ll 1$ and for $T \gg \mu$, $W_0$ and therefore $W$ are dominated by the pole of $W_0$ at $\epsilon = 0$. That is

$$W \overset{T \gg \mu, \epsilon \ll 1}{\approx} g^2 \frac{\Gamma(1+\epsilon)}{3} \int_0^1 dx \int_0^1 d\xi \frac{x^{\frac{\epsilon}{2} - 1}(1-x)^{1-\epsilon}}{[1-x + x\xi(1-\xi)]^{\frac{3-\epsilon}{2}}}$$

$$+ O\left(\frac{\mu^2}{T^2}, g^3\right) \quad \text{(B.B12)}$$

where we used that $x^{\frac{\epsilon}{2} - 1} = \frac{d^2}{dx^2} (\ln x^\epsilon)$. Therefore, we find for $\beta_\epsilon(g, \frac{T}{\mu}, v)$ from eqs.(B.B9), (B.B10) and (B.B12),

$$\beta_\epsilon(g, \frac{T}{\mu}, v) \overset{T \gg \mu, \epsilon \ll 1}{\approx} g^2 \frac{\Gamma(1+\epsilon)}{3} \left[ \frac{1}{\epsilon} + O(\epsilon^2) \right] + O\left(\frac{\mu^2}{T^2}\right)^{2\epsilon}$$
APPENDIX C: FORMAL PROOF.

The formal proof to one loop order begins with the expression (A.41) from appendix A above. We now use the identity [18, 19]

\[ I = T \sum_{l=\infty}^{\infty} \left[ A_l(x, p, s) \right]^{-\frac{1}{l+1} + \frac{1}{2}} = \int_C \frac{dk_0}{4\pi i} \left[ A(k_0, x, p, s) \right]^{-\frac{1}{2}} \coth \left[ \frac{k_0}{2T} \right]^2 \quad (C.1) \]

with \( A(k_0, x, p, s) = A_0(x, p, s; w_l = -ik_0) \) and the contour \( C \) is displayed in figure (1) below. The function \( A(k_0, x, p, s) \) has a cut running parallel to the real axis which for \( p \neq 0 \) or in the massive case for arbitrary \( p \) begins away from the imaginary axis and the contour \( C \). The contour can now be deformed to wrap around the cut and the analytic continuation \( s \to -i\omega + 0^+ \) can be performed. For \( p, \omega \ll T \) the infrared behavior is dominated by \( k_0 \ll T \) for which \( \coth \left[ k_0/2T \right] \sim 2T/k_0 \) and the resulting expression features a pole at \( k_0 = 0 \) while the cut begins away from the origin. The cut can be deformed again to circle the origin and the integral is simply the residue at the pole \( k_0 = 0 \). Therefore the infrared dominant term is given by \( I_{tr} = \left[ A_{tr}(x, p, s = -i\omega + 0^+) \right]^{-\frac{1}{2}} \), result which coincides with the analysis in terms of the Matsubara sums provided in appendix A.

![Contour in the complex k₀ plane](image)

FIG. 1: Contour in the complex \( k_0 \) plane

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[26] Note that the quartic self-coupling in ref. [11] has a different normalization that differs from the one used here by a factor of 12.
[27] It is given by $\rho_n$ right before equation (3.8) in [11].