A Geometric Unification of Dualities

F. Cachazo\textsuperscript{1}, B. Fiol\textsuperscript{2}, K. Intriligator\textsuperscript{3}, S. Katz\textsuperscript{4,5} and C. Vafa\textsuperscript{1}

\textsuperscript{1} Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA
\textsuperscript{2} Department of Particle Physics, The Weizmann Institute of Science, Rehovot, 76100, Israel
\textsuperscript{3} UCSD Physics Department, 9500 Gilman Drive, La Jolla, CA 92039
\textsuperscript{4} Departments of Mathematics and Physics
University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA
\textsuperscript{5} Department of Mathematics, Oklahoma State University, Stillwater, OK 74078 USA

Abstract

We study the dynamics of a large class of $\mathcal{N} = 1$ quiver theories, geometrically realized by type IIB D-brane probes wrapping cycles of local Calabi-Yau threefolds. These include $\mathcal{N} = 2$ (affine) A-D-E quiver theories deformed by superpotential terms, as well as chiral $\mathcal{N} = 1$ quiver theories obtained in the presence of vanishing 4-cycles inside a Calabi-Yau. We consider the various possible geometric transitions of the 3-fold and show that they correspond to Seiberg-like dualities (represented by Weyl reflections in the A-D-E case or ‘mutations’ of bundles in the case of vanishing 4-cycles) or large $N$ dualities involving gaugino condensates (generalized
conifold transitions). Also duality cascades are naturally realized in these classes of theories, and are related to the affine Weyl group symmetry in the A-D-E case.
1. Introduction

A deeper understanding of string theory on background geometries with some vanishing cycles have played a key role in various aspects of string dualities. An early example of this was in the context of the physical interpretation of the conifold [1] singularity and its possible transitions [2]. Geometric transitions have also played an important role in deriving field theoretic dualities from string theory. In particular by considering spacetime filling D-branes wrapped around cycles of Calabi-Yau 3-folds, Seiberg’s duality was derived in this way in the context of type IIB [3] and type IIA [4] string theories.

Geometric transitions have also played a key role in large $N$ dualities. The AdS/CFT correspondence [5] can be viewed as an example of such a transition [6], where before the transition (small ‘t Hooft parameter) there are D-branes wrapped around cycles, and after the geometric transition (large ‘t Hooft parameter) these cycles which supported the D-branes have disappeared, and have been replaced with flux through a dual cycle. The large $N$ duality of Chern-Simons with topological strings [6] is an example of this kind.

The geometric transition duality was embedded in type IIA superstring [7], with D6 branes wrapping an $S^3$ on one side of the transition and fluxes through a dual $S^2$ on the other side; this leads to a large $N$ duality for $\mathcal{N} = 1$ Yang-Mills theory in 4 dimensions. This duality was lifted up to M-theory [8,9] where it was interpreted as a purely geometric transition. Since, as argued in [9] and further elaborated in [10,11] the transition in quantum geometry is smooth in the M-theory lift, this leads to a derivation of the geometric transition duality.

The type IIB mirror of these large $N$ dualities has also been studied [12,13] (see also [14]). One aim of this paper is to generalize these constructions and show that the Seiberg-like dualities and large $N$ dualities/gaugino condensation can be viewed in a unified way as geometric transitions in the same setup. We consider a wide variety of 4d, $\mathcal{N} = 1$ supersymmetric gauge theories, which can be constructed via branes which partially wrap cycles of a (non-compact) Calabi-Yau 3-fold $X$. In type IIB, one can consider general combinations of $D3, D5, D7$ branes, wrapped over various cycles and filling the 4 dimensional spacetime. This generically leads to a theory with gauge group $\prod_i U(N_i)$ with some matter in the bifundamental representations, and some superpotential terms (depending on the complex structure of $X$). Changing the Kahler parameters of the underlying CY 3-fold translates to changing the coupling constants of the gauge theory (and sometimes also to FI terms).

We find, as in [3,4] that changing the Kahler parameters of $X$ (or, in the type IIA mirror with wrapped D6 branes, changing the complex parameters) changes the description
of the gauge theory. As we pass through transitions in the 3-fold geometry, corresponding to blowing up different Kahler classes, we find dual gauge theory descriptions of the same underlying theory. These are transitions where some 2- or 4-cycles shrink and other 2- or 4-cycles grow. On the other hand, for a class of these theories which eventually confine, with gaugino condensates, we find transitions of the type where the 2-cycle or 4-cycle has shrunk, disappeared, and instead a number of finite size $S^3$'s have emerged, with fluxes through them. The description in terms of the blown up $S^3$'s is better at large $N$ (in the IR), where the size of the $S^3$'s, which corresponds to the gaugino condensation, is large.

By using the holographic picture, and following the geometric transitions, we can smoothly follow the field theory dualities and dynamics along the renormalization group flow. In the UV, which corresponds to far distance to the geometry, we have a description which is best given in terms of finite size 2-cycles and 4-cycles. This is the weak coupling limit. The renormalization group flow to the IR corresponds in the geometry to going towards the tip of the cone (or more precisely towards the “tips” of the cone). In doing so, the description changes: some 2-cycles or 4-cycles shrink, and others emerge, corresponding to Seiberg-like dualities in the field theory. Eventually the gauge theory flows to e.g. a RG fixed point, a free-magnetic phase, or confinement with gaugino condensation. This is seen by following the geometry towards the tips of the cone. E.g. deep in the IR, or in the very large $N$ limit, the description might be best in terms of the blown up $S^3$'s; this is where the gauge theory confines and gaugino condensation has taken place.

In this way we have a unified geometric picture, where both kinds of dualities can be seen in the same RG trajectory, depending on where in the geometry we are. This unification sharpens the picture of Seiberg duality given in [3,4] (a similar comment applies to the brane construction of [15]): Rather than just seeing that two gauge theories are connected by changing the moduli of the theory, which by itself is not a complete derivation of duality\(^1\) we can use the geometry to follow the RG trajectory, and see which description is best, at which scale, as we flow to the IR.

We consider two classes of local 3-folds in type IIB. One type (i) involves certain Calabi-Yau threefolds which only has compact 2-cycles and no compact 4-cycles. The

\(^1\) For example, by similar changes of the moduli one can relate $\mathcal{N} = 2 \, U(N_c)$, with $N_f$ flavors, to $\mathcal{N} = 2 \, U(N_f - N_c)$ with $N_f$ flavors. But here this duality misses part of the story. The original $U(N_c)$ theory does indeed contain the free-magnetic $U(N_f - N_c)$ theory in its spectrum, but this description is only good on part of the Higgs branch, and it also must be augmented with an extra $U(1)^{2N_c - N_f}$ where this Higgs branch part intersects the Coulomb branch [16].
other type (ii) involves Calabi-Yau’s which have compact 2- and 4-cycles. For type (i) we consider $X$ to have the geometry of an A-D-E 2-fold geometry fibered over a plane, with some blown up 2-cycles $S^2_i$’s in one to one correspondence with the simple roots of A-D-E.

We can wrap D5 branes over these 2-cycles, which fill the directions transverse to $X$. In addition, one could also include $N_0$ additional D3 branes transverse to $X$. The 3-folds $X$ which we consider can thus be labelled (up to deformations) as $X(k, G)$ with $G$ the A-D-E group and $k$ an integer which labels the data about how the holomorphic 2-cycles of the A-D-E are fibered over the plane [17].

For $N_0 = 0$, the gauge groups obtained via wrapping various numbers $N_i$ D5 branes over the various $S^2_i$ of $X(k, G)$ are quiver gauge theories with gauge group $\prod_{i=1}^r U(N_i)$, with the quiver diagram the $G$ Dynkin diagram and $r = \text{rank}(G)$, and the matter in hypermultiplets dictated by the links of the Dynkin diagram. The theory arises from the corresponding $\mathcal{N} = 2$ quiver theory, broken to $\mathcal{N} = 1$ by the additional superpotentials for the adjoint superfields $\phi_i$ in the $\mathcal{N} = 2$ $U(N_i)$ vector multiplet

$$W_i = \frac{g_i}{k+1} \text{Tr} \phi_i^{k+1} + \text{lower order}. \quad (1.1)$$

The precise form of the superpotential is dictated by the fibration data. Adding $N_0$ D3 branes, the quiver gauge theory becomes $\prod_{i=0}^r U(\tilde{N}_i)$, based on the affine $\tilde{G}$ Dynkin diagram, with

$$\tilde{N}_i = N_0 d_i + N_i, \quad (1.2)$$

for $i \neq 0$ with $d_i$ the Dynkin indices. We also set $\tilde{N}_0 = N_0$.

The inequivalent blowups for $\mathcal{N} = 1$ A-D-E quiver theories are given by the action of the Weyl group. As we will discuss, a Weyl reflection on a node is related to a Seiberg-like duality on the corresponding gauge group. A similar statement applies to the affine case. The duality cascade of [12], for example, corresponds to the affine $\tilde{A}_1$ case of $X(k=1, G = \tilde{A}_1)$. This will be generalized here to the arbitrary affine case. The generalized duality cascade is related to the affine Weyl group, which is the semi-direct product of the Weyl group and translation by the root lattice; the translation is responsible for the cascading reduction of the D3 branes as we flow to the IR.

For the type (ii) case, with compact 4-cycles in addition to the two-cycles, we consider local threefolds which have a toric realizations, as in the examples studied in [18]. We can

---

2 More generally we can consider one $k$ for each simple root of A-D-E, but this can also be obtained, by deformations, from the case we consider.
then consider wrapping general classes of D3,D5 and D7 branes. In this case, it is more convenient to use the mirror IIA picture of the manifold and branes, as it does not suffer from quantum corrections. Using the appropriate mirror symmetry in the context of branes [19], we write down the corresponding quiver theory, as well as the corresponding Seiberg-like dualities. The dualities involve changes of the classical parameters in the type IIA mirror. We specialize to the Calabi-Yau threefolds involving delPezzo and their transitions. Certain aspects of this case have been noted recently in [20,21].

The organization of this paper is as follows: In section 2 we give an overview of the $\mathcal{N} = 1$ A-D-E quiver theories and the results we will find for them in this paper. In section 3 we give the description of classical aspects of the A-D-E quiver gauge theories under consideration. In section 4 we discuss some aspects of the quantum dynamics of the gauge couplings and their running. In section 5 we discuss gaugino condensation in the non-affine A-D-E $\mathcal{N} = 1$ quiver theories. In section 6 we consider the geometric engineering of these theories and their large $N$ dual, involving the leading quantum corrections and the geometric realization of gaugino condensates. In section 7 we discuss Seiberg-like dualities for the A-D-E quiver theories anticipated from geometry. In section 8 we discuss the gauge theoretic interpretation of these dualities. In section 9 we consider the gauge theory dynamics of the $A_2$ quiver in more detail, as a typical situation where the Seiberg-like duality is relevant. In section 10 we discuss dynamical aspects of the affine quiver theory and its relation to the non-affine case. We also note the connection of RG cascades in this class of theories with affine Weyl reflection. In section 11 we discuss examples of $\mathcal{N} = 1$ superconformal A-D-E quiver theories. In section 12 we setup the geometric engineering of the type (ii) local threefolds, as well as dualities predicted by geometry. In section 13 we specialize to a class of examples and illustrate how the gaugino condensation takes place in these chiral theories and what geometric transition they correspond to.

2. Basic structure of the type (i) $\mathcal{N} = 1$ quiver theories and their large $N$ duals

The class of type (i) theories which we consider are fibrations of a A-D-E twofold geometry over a plane. The corresponding field theory is that of an $\mathcal{N} = 2$ A-D-E or affine $\hat{A}$-$\hat{D}$-$\hat{E}$ quiver theory, deformed to $\mathcal{N} = 1$ by superpotential terms $W_i(\phi_i)$, with $\phi_i$ the adjoint field in the $\mathcal{N} = 2$ $U(N_i)$ vector multiplet. The choice of $W_i$’s are encoded in the fibration data. For simplicity we consider the case where all the superpotentials are polynomials of degree $k + 1$. 

5
The case \( X(k = 1, G = A_1) \), for example, corresponds to the small resolution of the conifold, in which the \( S^2 \) is blown up. Wrapping \( N \) D5 branes on the \( S^2 \) leads to \( \mathcal{N} = 1 \) \( U(N) \) pure Yang-Mills. It was argued in [7] that for large \( N \), or in the IR, the theory is better described by the geometric conifold transition: \( X \to \tilde{X} \), where \( \tilde{X} \) is the deformed conifold, with its blown up \( S^3 \) having RR flux. The generalization to \( X(k, A_1) \) for arbitrary \( k \) was discussed in [13]: the worldvolume theory is \( \mathcal{N} = 2 \) \( U(N) \) gauge theory, broken to \( \mathcal{N} = 1 \) by a superpotential as in (1.1). Geometrically this means that instead of having holomorphic \( S^2 \)'s over the whole plane (corresponding to vev of \( \Phi \)) they only appear at \( k \) points. Let us label these \( S^2 \)'s by \( S^2_p \) where \( p = 1, ..., k \). One can distribute the \( N \) D5 branes by wrapping them on any of the \( S^2_p \)'s, leading to a Higgsing \( U(N) \to \prod_{p=1}^{k} U(M_p) \).

The geometric transition duality of [13] involves \( X(k, A_1) \to \tilde{X}(k, A_1) \) in which every \( S^2_p \) is blown down and replaced with a blown up \( S^3_p \) having RR flux. The geometric transition duality yields a new field theory duality, in which the original \( U(N) \) theory is dual to a \( \mathcal{N} = 2 \) \( U(1)^k \) theory, which is broken to \( \mathcal{N} = 1 \) by a particular superpotential (which can be regarded as electric and magnetic FI terms). This duality was shown to be a powerful tool for obtaining exact results about these supersymmetric field theories [13].

The case of \( N_0 \) D3 branes transverse to \( X(k = 1, G) \), without wrapped D5s, was discussed in [22]. The gauge group is as in (1.2), with all \( N_i = 0 \), and these theories flow to \( \mathcal{N} = 1 \) superconformal field theories. These superconformal field theories have a holographic dual description in terms of IIB string theory on \( AdS_5 \times M_5(1, G) \) which was discussed in [22], generalizing the work [23] corresponding to \( G = A_1 \). We can now add wrapped D5s (sometimes referred to as adding fractional D3 branes), which breaks the conformal invariance. As will be discussed, this theory undergoes a RG cascade generalizing that of [12], which is the case coming from \( X(k = 1, G = A_1) \).

The geometry of the general \( X(k, G) \) and the classical gauge theories associated with arbitrary wrapped D5s, and arbitrary transverse D3s, was obtained in [17]. It was shown there that the basic aspects of the geometry and geometric transition duality matches with what one expects for the field theory in terms of gaugino condensates. One major aim of the present work is to analyze the dynamics of these gauge theories in detail, and verify that the geometry properly predicts the correct gauge theory dynamics. We will see that the associated field theory dualities are geometrically realized via two different possible geometric dual operations:

(A) \( : \quad S^2_i \to \sum_j A_{ij} S^2_j \)

(B) \( : \quad S^2_p \to S^3_p \).

\[ 2.1 \]
The operation \((B)\) is the geometric transition duality, which occurs when a \(U(N)\) gauge theory confines, with gaugino condensation. The size of the \(S^3_p\) is related to the gaugino condensate \([7,13]\).

The operations \((A)\) on the other hand, correspond to Seiberg-type dualities \([24]\) which, from essentially the same viewpoint, was discussed in \([3,4]\) (see also the related work \([25,26]\)). As we will discuss, these are related to the \(G\) Weyl group for \(N_0 = 0\) or, for general \(N_0\), to the \(\hat{G}\) affine Weyl group. Each Seiberg-like duality corresponds to a Weyl reflection about a simple root, with the reflections about each of the simple roots generating the full Weyl group. In particular, all of the \(A_{ij}\) in (2.1) are given by Weyl reflections about each simple root \(\vec{e}_{i_0}\) as

\[
\vec{e}_i \rightarrow \vec{e}_i' = \vec{e}_i - (\vec{e}_i \cdot \vec{e}_{i_0})\vec{e}_{i_0} \equiv \sum_j A_{ij} \vec{e}_j
\]

including the affine root \(\vec{e}_0\) and its Weyl translation in the \(\hat{G}\) case). The rank of the gauge group is determined by D-brane charge conservation (as in \([3,4]\)):

\[
\sum N_i \vec{e}_i = \sum N'_i \vec{e}_i'
\]

which implies that

\[
N'_i = (A^{-1})_{ij} N_j
\]

The Weyl symmetry \((A)\) acts on the \(U(N_i)\) coupling constants and on the superpotentials as

\[
g_i^{-2} \rightarrow \sum_j A_{ij} g_j^{-2}, \quad W_i(\phi_i) \rightarrow \sum_j A_{ij} W_j(\phi_i).
\]

The action of (2.3) on \(g_i^{-2}\) follows from the fact that this is identified with the quantum volume of \(S^2_i\). When \(S^2_i\) shrinks the \(1/g_i^2 \rightarrow 0\) (i.e. the theory is strongly coupled) and if we continue it past that it become negative. However we know that another \(S^2\) has emerged whose volume is \(-1/g_i^2\) which now is positive. This is the dual gauge theory. From this point of view the duality can be viewed as an attempt to make the \(1/g_i^2\)'s positive. In the field theory, dimensional transmutation can occur, with the running \(g_i\) written in terms of dynamical scales \(\Lambda_i\); the action of (2.3) on \(g_i^{-2}\) then becomes a statement about matching the dynamical scales \(\Lambda_i\) of the dual theories. The duality is inherited from that duality of the corresponding \(N = 2\) theory with \(W_i = 0\), which in the field theory setup was noted in \([16]\)(see also \([27]\)), corresponding to \(U(N_c)\) theory with \(N_f\) flavors getting related
to $U(N_f - N_c)$ with $N_f$ flavors. Breaking to $\mathcal{N} = 1$ by $W \sim \text{Tr}\phi^2$ was considered in [16] and the case of more general $W \sim \text{Tr}\phi^{k+1}$ was considered e.g. in [15] via NS brane constructions.

The two transitions $(A)$ and $(B)$ combine in a beautiful way in the geometric dual description. Geometrically, if we start far from the tip of the cone, the geometry has a description in terms of $S_i^2$’s, which change in size as we come closer to the tip of the cone (which is the geometric realization of moving towards the IR). Sometimes an $S^2$ shrinks and a dual $S^2$ grows, an $(A)$ type transition, which is interpreted as Seiberg-like duality. Sometimes an $S^2$ shrinks and an $S^3$ grows, a $(B)$ type transition, and this corresponds to the occurrence of confinement and gaugino condensation. The nice thing about this picture is that not only can we “derive” Seiberg-like dualities by connecting branes wrapping cycles of Calabi-Yau, as in [3,4], but in fact we are able to see how they occur in a dynamical sense, i.e. following the RG trajectory and seeing that they become equivalent. This picture works equally well for $G$ as well as for the affine $\hat{G}$ type quiver theories. The application of duality is particularly striking in the affine case, as one may have to undergo infinitely many applications of duality as we go from the UV to the IR. In particular the RG cascade of [12] corresponds to the $\hat{A}_1$ Weyl group. Upon flowing to the IR, one undergoes a series of $(A)$ type transitions until eventually the theory confines and undergoes the $(B)$ type transition.

Consider the $G = A-D-E$ quiver theories. Using the action (2.3) of the Weyl group on the coupling constants, one can represent the coupling constants as a $r = \text{rank}(G)$ vector $\vec{x}$ such that

$$\frac{1}{g_i^2(\vec{x})} = \vec{e}_i \cdot \vec{x} > 0. \quad (2.4)$$

The space of $\vec{x}$ satisfying this condition is a $G$ Weyl chamber, a fundamental domain for the action of the Weyl group on $\mathbb{R}^r/W$ where $W$ is the Weyl group. The Weyl chamber is a conical wedge, which has $r$ codimension one boundaries, given by $g_i^{-2} = 0$ for any $i = 1 \ldots r$. The RG flow corresponds to moving $\vec{x}$ inside the fundamental domain along a straight line, until it hits a boundary where one of the $1/g_i^2 = 0$. After this, if the ranks of the dual theories are all positive, there is a reflection off the boundary, with incident angle equal to the reflection angle; this corresponds to a Seiberg-like duality. If the rank of the dual theory is not positive, there is no reflection and this is indicative of the $(B)$ type transition in (2.1), corresponding to confinement and gaugino condensation.
The above picture also applies to the affine quiver theories, with the couplings for the non-affine nodes still labeled by \( \vec{x} \) exactly as in (2.4), for \( i = 1 \ldots r \). The only new feature is the existence of the extra affine node, \( i = 0 \), whose gauge coupling is given by

\[
g_0^{-2}(\vec{x}) = \frac{1}{g_s} - \sum_{i=1}^{r} d_i \vec{e}_i \cdot \vec{x}.
\]  

(2.5)

The \( d_i \) in (2.5) are the Dynkin indices, and the extending simple root is \( \vec{e}_0 = -\sum_{i=1}^{r} d_i \vec{e}_i \), which can be written as \( \sum_{i=0}^{r} d_i \vec{e}_i = 0 \) with \( d_0 \equiv 1 \). There is now the further restriction on the space of allowed \( \vec{x} \) that the RHS of (2.5) is also non-negative. This gives an additional codimension one boundary, cutting the Weyl chamber wedge to a finite sized box; this space of allowed \( x \) is the \( \hat{G} \) Coxeter box. It can be viewed as the fundamental domain of the affine Weyl group action on \( \mathbb{R}^r/\hat{W} \) where \( \hat{W} \) is the affine Weyl group. Equivalently it can be viewed as the fundamental domain of the Cartan torus by the Weyl group action, \( T^r/W \), noting that \( \hat{W} = W \ltimes T \) where \( T \) is the translation group of the root lattice. Note that a linear combination \( \sum_{i=0}^{r} d_i g_i^{-2}(x) = 1/g_s \) is actually independent of \( \vec{x} \). This is the gauge coupling of a diagonal \( U(N_0) \). Including the theta angles the complex version of this statement is also true: the complex gauge coupling

\[
\tau_D \equiv \sum_{i=0}^{r} d_i \tau_i = \tau_{IIB},
\]  

(2.6)

with \( \tau_{IIB} \) the IIB string coupling.

A special case of the above discussion for the \( \mathcal{N} = 1 \) affine \( \hat{G} \) quiver theories is the case \( N_i = 0 \) wrapped D5s, with \( N_0 \neq 0 \) transverse D3s. This case leads to a \( \mathcal{N} = 1 \) superconformal field theory with a \( r + 1 \) complex dimensional moduli space of gauge couplings \( \tau_i \), which are the complexification of the couplings in (2.4) and (2.5). The Weyl reflection dualities maps the theory back to itself, except for changing the coupling constants. This is part of the S-duality group of these theories. The remaining S-duality is the usual \( SL(2,\mathbb{Z}) \) action on the diagonal gauge coupling (2.6). So a fundamental domain of the moduli space of the \( \mathcal{N} = 1 \) superconformal field theories is given by (the complexification of ) (2.4) and (2.5), with \( \vec{x} \) in the \( \hat{G} \) Coxeter box, along with the \( SL(2,\mathbb{Z}) \) fundamental domain for \( \tau_D \).

This same picture holds for the special case where the deforming \( W_i(\phi_i) \) vanish, leading to \( \mathcal{N} = 2 \) rather than \( \mathcal{N} = 1 \) superconformal field theories. The Coxeter box structure for the moduli space was found in the related case of D5 branes at a \( \mathbb{C}^2/\Gamma_G \) singularity.
[28,29] in [30]. The moduli space for the $\mathcal{N} = 2$ superconformal $\hat{G}$ theories, setting $W_i = 0$, was studied in [31] for the $\hat{A}$-case and generalized to all the $\hat{A}-\hat{D}-\hat{E}$ quiver cases in [18], where the moduli space of couplings was shown to be identified with moduli of flat A-D-E connections on $T^2$. The Coxeter box can be identified with the moduli space of flat A-D-E connection on $S^1$ and the description of the moduli space along the lines discussed here was noted in [32]. Again, for both the $\mathcal{N} = 2$ and the $\mathcal{N} = 1$ superconformal theories with $W_i(\phi_i) \neq 0$, the S-duality group corresponds to $SL(2, \mathbb{Z})$ action on (2.6), along with the Weyl reflections on the $\tau_i$, as in (2.3).

3. The classical quiver gauge theories

3.1. 4d $\mathcal{N} = 1$ A-D-E quiver theories

The class of $\mathcal{N} = 1$ quiver gauge theories we consider is a deformation of $\mathcal{N} = 2$ quiver gauge theories with gauge group $\prod_i U(N_i)$, with $i$ running over the nodes of the quiver diagram, and bi-fundamental hypermultiplets for the linked nodes $i$ and $j$; these hypermultiplets can be written as $\mathcal{N} = 1$ chiral superfields $Q_{ij}$ in the $(N_i, N_j)$ and $Q_{ji}$ in the $(N_i, N_j)$ of $U(N_i) \times U(N_j)$. The quiver diagrams of interest here are the $G = A, D, E$, or affine $\hat{G} = \hat{A}, \hat{D}, \hat{E}$, Dynkin diagrams; these are the most general asymptotically free, or conformal respectively, $\mathcal{N} = 2$ quiver gauge theories [18]. We consider deformations of these theories to $\mathcal{N} = 1$ supersymmetric theories by adding a superpotential for the adjoint fields, $W_i(\phi_i)$, so the full tree-level superpotential is

$$W = \sum_i [\text{Tr} \sum_j s_{ij} Q_{ij} Q_{ji} \phi_i - \text{Tr} W_i(\phi_i)]$$

(3.1)

where $s_{ij} = -s_{ji}$ is the intersection matrix of nodes $i$ and $j$, which is zero if the nodes are not linked and $\pm 1$ if they are linked (nothing depends on the choice for the $s_{ij}$ signs). The first term in (3.1) is that of the original undeformed $\mathcal{N} = 2$ theory.

In the non-affine case, there is no restriction on $W_i(\phi_i)$. In the affine case, however, the geometric engineering of these quiver theories [17] leads to one restriction on the superpotentials:

$$\sum_{i=0}^{r} d_i W_i(x) = 0$$

The equations of motion following from (3.1) are

$$\sum_j s_{ij} Q_{ij} Q_{ji} = \partial_i W_i(\phi_i), \quad \phi_i Q_{ij} = Q_{ij} \phi_j,$$

(3.2)
for every $Q_{ij}$. The vacua are the solutions of these equations, modulo complexified gauge transformations. We now review the vacuum structure, which was derived in [17]. For the case where the quiver diagram is a non-affine $G = A, D, E$ Dynkin diagram, there are various vacua which are given in terms of the positive roots $\rho_K \subset \Delta^+$ of $G$; here $K = 1, \ldots, R_+$, with $2R_+ + r = |G|$, and the positive roots can be expanded in terms of the simple roots $e_i$ as

$$\rho_K = \sum_{i=1}^{r} n_i^K e_i,$$  

for appropriate $n_i^K \geq 0$. This corresponds to the fact that the associated geometry has 2-cycles $S^2_K$ corresponding to the positive roots $\rho_K$.

For each $\rho_K$ there are a number of irreducible branches of the supersymmetric theory, given by the roots $x$ the equation

$$W_K'(x) \equiv \sum_{i} n_i^K W_i'(x) = 0.$$  

(3.4)

For simplicity we take all $W$’s to be polynomials of the same degree $k+1$ (the more general case can also be constructed geometrically [17]). Then, for each positive root $\rho_K$, the above equation has $k$ roots, which we label as $x = a_{(p,K)}$, with $p = 1, \ldots, k$ and $K = 1 \ldots R_+$. There is a susy vacuum for every choice of $M_{(p,K)} \geq 0$ such that

$$N_i = \sum_{K=1}^{R_+} \sum_{p=1}^{k} M_{(p,K)} n_i^K.$$  

(3.5)

In these vacua $\phi_i$ has $n_i^K M_{(p,K)}$ eigenvalues given by the root $a_{(p,K)}$ and the gauge group is Higgsed as

$$\prod_{i=1}^{r} U(N_i) \rightarrow \prod_{K=1}^{R_+} \prod_{p=1}^{k} U(M_{(p,K)}).$$  

(3.6)

For the case of affine quiver diagrams, the vacua are similarly labeled by the positive affine roots [17]. We will consider the cases where there are no pure 3-brane branches (this is the analogue of the Coulomb branch of the $\mathcal{N} = 4$). In this case the Higgs branches are also labeled by the positive roots of affine A-D-E, which are described as follows: Recall that the highest root of $G$ is $\psi = \sum_{j=1}^{r} d_j e_j$, with $e_j$ the simple roots; the extending affine root $e_0$ is $e_0 \equiv -\psi$, so $\sum_{i=0}^{r} d_i e_i = 0$, with $d_0 \equiv 1$. The extended Cartan matrix is $C_{ij} = e_i \cdot e_j$ for all $i, j = 0, \ldots, r$. For affine Lie algebras one replaces $e_i \rightarrow \hat{e}_i = (e_i, 0)$ for $i = 1, \ldots, r$ and $e_0 \rightarrow \hat{e}_0 = (-\psi, 1)$. Note that $\sum_{i=0}^{r} d_i \hat{e}_i = \delta$, with $\delta = (0, 1)$ which we
identify as the D3 brane charge direction (called the ‘imaginary direction’ for the affine algebra). The positive roots of the affine algebra are given by

\[ \hat{\rho}_K^\hat{\rho} : (\Delta, n^+), (\Delta^+, 0) \]

where \( n^+ \) is a positive integer and \( \Delta \) denotes all roots. Each such vector can be written as positive combination of positive affine roots:

\[ \hat{\rho}_K^\hat{\rho} = \sum_{i=0}^{r} n_K^i \hat{e}_i \]

For each such root, consider its projection to the root lattice which is either a positive root or its negative, given by \( \pm \sum_{i=1}^{r} n_K^i e_i \) as \( K = 1, ..., R_+ \). For each such branch we consider solutions to

\[ W'(\hat{\rho}_K^\hat{\rho}) = \pm \sum_i W'_i(x)n_K^i = 0 \]

which is exactly the equation we considered in the non-affine case (the possible minus sign does not affect the solutions to the above equation). There are \( k \) solutions for each branch, which we label with \( (p, \hat{K}) \). Choose non-negative integers \( M_{(p, \hat{K})} \) which label how many of each irreducible branch we choose. These should satisfy

\[ \sum_{\hat{K}} M_{(p, \hat{K})} n_K^i = \hat{N}_i \]

In this branch the gauge group is Higgsed to

\[ \prod_{i=0}^{r} U(\hat{N}_i) \rightarrow \prod_{\hat{K}} \prod_{p=1}^{k} U(M_{(p, \hat{K})}). \tag{3.7} \]

4. Aspects of the quantum dynamics: gauge couplings and their running

In this section we discuss some aspects of the quantum dynamics of the gauge couplings and their running as a function of scale. We will first consider the underlying \( \mathcal{N} = 2 \) quiver theory and then we will discuss aspects of the \( \mathcal{N} = 1 \) deformed theory by adding superpotential terms.
First let us ignore the superpotentials $W_i(\phi_i)$ so we have an $\mathcal{N} = 2$ quiver theory. This is also a good approximation for the dynamics of the $\mathcal{N} = 1$ quiver theory for energy scales large enough compared to the superpotential deformations (i.e. for scales $\mu$ large compared to the adjoint mass $W''_i(\mu)$).

The $\mathcal{N} = 2$ exact beta function for the coupling $\tau_i = \frac{\theta_i}{2\pi} + 4\pi i g_i^{-2}$ of $U(N_i)$ is

$$\beta_i \equiv -2\pi i \beta(\tau_i) = \sum_j C_{ij} N_j,$$  \hspace{1cm} (4.1)

with $C_{ij} = 2\delta_{ij} - |s_{ij}| = \vec{e}_i \cdot \vec{e}_j$ the Cartan matrix of the A-D-E diagram, or the extended Cartan matrix of the affine $\tilde{A}$-$\tilde{D}$-$\tilde{E}$ diagram. The sign of $\beta_i$ in (4.1) is chosen so that the theory is asymptotically free if (4.1) gives $\beta_i > 0$. Note that this can be conveniently summarized by a vector $\vec{\beta}$ whose projection on $\vec{e}_i$ gives $\beta_i$

$$\beta_i = \vec{e}_i \cdot \vec{\beta} \quad \text{with} \quad \vec{\beta} = \vec{N} \equiv \sum_i N_i \vec{e}_i.$$  \hspace{1cm} (4.2)

In the affine $\tilde{G}$ case we include the affine node $i = 0$ in (4.2). In the affine case, since $\sum_{i=0}^r d_i \vec{e}_i = 0$, the $\tilde{G}$ affine quiver theory with $\tilde{N}_i = N_0 d_i$ has $\vec{\beta} = 0$; it’s an $\mathcal{N} = 2$ superconformal field theory for any $N_0$. This corresponds to $N_0$ D3 branes and no wrapped D5 branes. There are $r + 1$ complex moduli, given by the $U(\tilde{N}_i)$ gauge couplings $\tau_i$ for $i = 0, \ldots, r$. More generally, for any $\tilde{N}_i$, the beta functions (4.1) are invariant under the shift

$$\tilde{N}_i \to \tilde{N}_i + N_0 d_i,$$  \hspace{1cm} (4.3)

for any $N_0$. Also, for any $\tilde{N}_i$, the beta function for the coupling

$$\tau_D \equiv \sum_{i=0}^r d_i \tau_i$$  \hspace{1cm} (4.4)

of a diagonally embedded $U(N)$ vanishes, as $\sum_i \beta_i d_i = 0$.

In the construction of the affine $\tilde{G}$ quiver theories [28] via D3 branes at $G$ type ALE singularities, $\tau_D$ is the IIB string coupling, while the other $\tau_i$ are given by the orbifold blowing up modes coming from the twisted sector NS or RR fields, as in [32]. Thus $\tau_D$ must have the $SL(2, \mathbb{Z})$ S-duality of IIB string theory. The other independent $\tau_i$ also exhibit S-dualities, which correspond to $G$ Weyl reflections. As already mentioned, this shows that the $r + 1$ complex dimensional moduli space of the $\tilde{G}$ quiver $\mathcal{N} = 2$ superconformal theories consists of the $SL(2, \mathbb{Z})$ fundamental domain for $\tau_D$, along with the complexification of the $\tilde{G}$ Coxeter box for the remaining linear combinations of the couplings $\tau_i$ of each $U(\tilde{N}_i)$ gauge group factor.
4.2. The $\mathcal{N} = 1$ quiver theories

Now consider the $\mathcal{N} = 1$ A-D-E quiver theories, with superpotential as in (3.1), with $W_i(\phi_i)$ as in (1.1): $W_i \sim \text{Tr} \phi_i^{k+1}$+lower order. For $k = 1$ the $\phi_i$ are massive and can be integrated out; for $k > 1$ the $\phi_i$ should be kept (unless one adds the generic lower order terms in (1.1), in which case $\phi_i$ is again massive and can be integrated out at low energies). Note that, for $k > 1$, the deformations of the superpotential appear to be irrelevant ($k = 2$ is marginally irrelevant), and thus divergent in the UV limit. In the UV one needs a cutoff to define the theory, but the IR aspects we will discuss are universal and independent of the cutoff. The deforming operators are actually “dangerously irrelevant,” much as in [33], in that they get large anomalous dimensions and they control the IR dynamics.

The $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superpotential deformation $W_i(\phi_i)$ does not change the 1-loop holomorphic beta functions, so they are the same as in (4.1):

$$\beta_i = \sum_j C_{ij} N_j = \vec{e}_i \cdot \vec{N},$$

which gives

$$e^{2\pi i \tau_i(\mu)} = \left( \frac{\Lambda_i}{\mu} \right)^{\beta_i}. \tag{4.5}$$

The quantity appearing in (4.5) is $e^{-S_{\text{inst}}^i}$, with $S_{\text{inst}}^i$ the action for a $U(N_i)$ instanton. The gauge coupling running (4.5) and scales $\Lambda_i$ apply above the mass scale $\Delta \sim W''$ where $\phi_i$ gets a mass (this occurs for $k = 1$ or, for higher $k$, if $W_i'(x)$ has no coinciding roots). Below the mass scale $\Delta$, the $\phi_i$ can be integrated out and the holomorphic beta functions are instead

$$\beta_i^{\text{low}} = -2\pi i \beta(\tau_i) = N_i + \sum_j C_{ij} N_j. \tag{4.6}$$

Matching the running coupling $g_i$ at the scale $\Delta$ gives that the low-energy theory has dynamical scale $\Lambda_i^{\text{low}}$ given by $(\Lambda_i^{\text{low}})^{\beta_i^{\text{low}}} = \Delta^{N_i} \Lambda_i^{\beta_i^i}$. The more general matching relations, associated with the different Higgsing branches, will be discussed in detail the following section.

We refer to the above 1-loop beta functions as the “holomorphic beta functions” since, as usual, they exactly give the running of the coefficient of the $U(N_i)$ gauge kinetic term, when the superpotentials are written in terms of the holomorphic (bare) quantities. Also of interest are the “physical beta functions,” which are the ones of relevance for analyzing RG flows and determining the existence of RG fixed points. The physical beta functions can be written in terms of the anomalous dimensions [34], which for our theories yields

$$\beta_i^{\text{phys}} \equiv -2\pi i \beta^{\text{phys}}(\tau_i) = 3N_i - N_i(1 - \gamma(\phi_i)) - \sum_{j \neq i} |\vec{e}_i \cdot \vec{e}_j| N_j (1 - \gamma(Q_{ij}))$$

$$= (1 + \frac{1}{2} \gamma(\phi_i)) \sum_j C_{ij} N_j + \frac{1}{2} \sum_{j \neq i} |\vec{e}_i \cdot \vec{e}_j| N_j \beta(\lambda_{ij}), \tag{4.7}$$
where $\lambda_{ij}$ is the coefficient of $Q_{ji} \Phi_i Q_{ij}$ in the superpotential (note that this can be scaled to one, by rescaling $Q$'s). Here $\gamma(\phi_i)$ is the anomalous dimension of $\phi_i$ and we define

$$\beta(\lambda_{ij}) \equiv \gamma(\phi_i) + 2\gamma(Q_{ij}),$$

which is (proportional to) the beta function for the $\lambda_{ij}$. The expression (4.7) is essentially the NSVZ beta function, though without the denominator factor of [34]; this is because we are using the holomorphic gauge kinetic terms, $\sim \int d^2 \theta \tau W_\alpha W^\alpha$, and the canonical matter kinetic terms.

The beta function (4.7) applies above the possible scale $\Delta$ where $\phi_i$ could be integrated out. Below such a scale, the exact beta function is

$$\beta_i = 3N_i - \sum_{j \neq i} |\vec{e}_i \cdot \vec{e}_j| N_j (1 - \gamma(Q_{ij}))$$

$$= \frac{3}{2} \sum_j C_{ij} N_j + \sum_{j \neq i} |\vec{e}_i \cdot \vec{e}_j| \tilde{\gamma}(Q_{ij}),$$

(4.9)

where we define $\tilde{\gamma}(Q_{ij}) = \gamma(Q_{ij}) + \frac{1}{2}$. Integrating out $\phi_i$ induces a quartic superpotential for the $Q_{ij}$, which would be marginal if $\gamma(Q_{ij}) = -\frac{1}{2}$, corresponding to $\tilde{\gamma}(Q_{ij}) = 0$. Indeed, in this case the beta functions (4.9) all vanish for the affine $\tilde{G}$ quiver theories with $N_i = N_0 d_i$.

5. Gaugino condensation in the non-affine $\mathcal{N} = 1$ quiver theories

Let us now consider the dynamics of the $\mathcal{N} = 1$ quiver theory taking into account the fact that at scales lower than the relevant scales for the superpotentials $W_i$ the theory gets higgsed to various branches. For simplicity let us assume that all the $W_i$’s become relevant at the scale $\Delta$. Thus for scales $\mu >> \Delta$ we effectively have an $\mathcal{N} = 2$ quiver theory with the running of the coupling constants we have noted. Let us assume that at the scale $\Delta$ all these couplings are still small, i.e. $1/g_i^2(\Delta) >> 1$, so that the classical analysis of the branches is reliable. For scales below $\Delta$ the superpotential becomes relevant and the theory is Higgsed (for generic $W_i$ to a product of pure $\mathcal{N} = 1$ theories $\prod_{p=1}^k \prod_{K=1}^{R_+} U(M_{(p,K)})$ with some additional massive fields. The $SU(M_{(p,K)})$ factor in (3.6) gets a mass gap, with gaugino condensation and confinement.
The naive low-energy superpotential associated with the $\prod SU(M_{p,K})$ gaugino condensations can be written as

$$ W_{g.c.} = \sum_{p=1}^{k} \sum_{K=1}^{R_{+}} S_{(p,K)} \left( \log \left( \frac{\Lambda_{(p,K)}^{3M_{(p,K)}}}{S_{(p,K)}^{M_{(p,K)}}} \right) + M_{(p,K)} \right), \quad (5.1) $$

where the $\Lambda_{(p,K)}$ are the scales of the low energy $U(M_{(p,K)})$ gauge groups as found via naive threshold matching, which we will discuss in what follows. $S_{(p,K)}$ is the $SU(M_{(p,K)})$ glueball field $S_{(p,K)} \sim \text{Tr}_{SU(M_{(p,K)})} W_{\alpha} W^{\alpha}$, whose expectation value is the $SU(M_{(p,K)})$ gaugino condensate. The $S_{(p,K)}$ in (5.1) are massive and can be integrated out.

In general, the naive gaugino condensation superpotential is only a leading approximation to a more non-trivial exact result. This is seen, among other examples, in the analysis of [13], where the geometric transition duality emerged as a powerful tool to obtain the exact superpotential. Though $W_{g.c.}$ is not exact, it does exactly give the non-trivial monodromies of the superpotential. Moreover it should be a good approximation in the case where the $\mathcal{N} = 2$ gauge couplings are weak at the scale $\Delta$ where the Higgsing takes place.

The coupling constant of the $U(M_{(p,K)})$ theory at the scale $\Delta$ where Higgsing takes place satisfies

$$ g_{(p,K)}^{-2}(\Delta) = \sum_{i} n_{K}^{i} g_{i}^{-2}(\Delta). $$

We still need to match the running gauge couplings $g_{(p,K)}$ across the thresholds of various massive matter fields in order to relate the low energy scales $\Lambda_{(p,K)}$ to the high energy scales $\Lambda_{i}$ of the original quiver theory. This is what we will now do.

As discussed earlier, the possible eigenvalues of the adjoints $\phi_{i}$ are the solutions $a_{K}^{P}$, with $p = 1 \ldots k$ and $K = 1 \ldots R_{+}$, of (3.4). E.g. for the case $k = 1$, with $W_{i} = \frac{1}{2} m_{i} \text{Tr} \phi_{i}^{2}$, we have

$$ a_{K} = \frac{\sum_{i} n_{K}^{i} m_{i} a_{i}}{\sum_{i} n_{K}^{i} m_{i}}. \quad (5.2) $$

In general, $\phi_{i}$ can have $M_{(p,K)} n_{K}^{i}$ eigenvalues equal to $a_{(p,K)}$, with $N_{i} = \sum_{(p,K)} n_{K}^{i} M_{(p,K)}$,

$$ \phi_{i} \rightarrow \oplus_{(p,K)} a_{(p,K)} \left( 1_{M_{(p,K)}} \right)^{n_{K}^{i}}, \quad (5.3) $$

which breaks

$$ U(N_{i}) \rightarrow \prod_{p=1}^{k} \prod_{K=1}^{R_{+}} U(M_{(p,K)})^{n_{K}^{i}}. \quad (5.4) $$
Under this breaking the $Q_{ij}$ decompose as

$$Q_{ij} \rightarrow \oplus_{(p,K)} \oplus_{(q,L)} (M_{(p,K)}, M_{(q,L)})^{n^i_K n^j_L}, \quad (5.5)$$

with the bifundamental in (5.5) of mass $a_{(p,K)} - a_{(q,L)}$.

There is additional Higgsing, besides that of (5.3), due to non-zero expectation values of components of the $Q_{ij}$; this Higgsing breaks

$$\prod_i U(M_{(p,K)})^{n^i_K} \rightarrow U(M_{(p,K)}) \quad (5.6)$$

which is a diagonally embedded subgroup. The $Q_{ij}$ expectation values can be seen by plugging into the equations of motion (3.2),

$$\sum_j s_{ij} Q_{ij} Q_{ji} = W_i'(\phi_i), \quad (5.7)$$

which we should evaluate for the above eigenvalues $a_{(p,K)}$ of $\phi_i$. In the end, the unbroken gauge group is

$$\prod_{i=1}^r U(N_i) \rightarrow \prod_{p=1}^k \prod_{K=1}^{R_+} U(M_{(p,K)}), \quad \text{with} \quad N_i = \sum_{p=1}^k \sum_{K=1}^{R_+} n^i_K M_{(p,K)}. \quad (5.8)$$

The naive superpotential (5.1) arises from gaugino condensation in the unbroken gauge group factors of (5.8).

The original high energy $U(N_i)$ theory, with its adjoint included, has beta functions as in (4.2): $\beta_i = \vec{e}_i \cdot \vec{N}$, with $\vec{N} \equiv \sum_i N_i \vec{e}_i$. Using (5.8) we can also express these in terms of the ranks $M_{(p,K)}$ of the low-energy gauge group:

$$\beta_i = \vec{e}_i \cdot \vec{N} = \vec{e}_i \cdot \vec{M} \quad \text{where} \quad \vec{M} \equiv \sum_{p=1}^k \sum_{K=1}^{R_+} M_{(p,K)} \vec{\rho}_K. \quad (5.9)$$

Thus we can write the $U(N_i)$ instanton factors, in terms of the dynamical scales $\Lambda_i$ of the original high energy theory, as in (4.5):

$$\Lambda_i^{\sum_j C_{ij} N_j} = \Lambda_i^{\vec{N} \cdot \vec{e}_i} = \Lambda_i^{\vec{M} \cdot \vec{e}_i}. \quad (5.10)$$
We determine the scales $\Lambda_{(p,K)}$ of the low energy $U(M_{(p,K)})$ theory in (5.8) by naive threshold matching relations at the scales of all massive $U(M_{(p,K)})$ matter and W-boson fields. The result we thus obtain is

$$\Lambda_{(p,K)}^{3M_{(p,K)}} = [W'_{K}(a_{(p,K)})]^{M_{(p,K)}} \prod_{(q,L)\neq(p,K)} (a_{(p,K)} - a_{(q,L)})^{-\bar{\rho}_{K} \cdot \bar{\rho}_{L} M_{(q,L)}} \prod_{i=1}^{r} \left( \Lambda_{i}^{\bar{M} \cdot \bar{e}_{i}} \right)^{n_{i}^{K}},$$

(5.11)

where we define $W''_{K}(a_{(p,K)}) \equiv \sum_{i=1}^{r} n_{i}^{K} W''_{i}(a_{(p,K)})$. Note that the RHS of (5.11) properly has mass dimension $M_{(p,K)}(1 + \bar{\rho}_{K} \cdot \bar{\rho}_{K}) = 3 M_{(p,K)}$ (since all positive roots of the simply laced $ADE$ satisfy $\bar{\rho}_{K} \cdot \bar{\rho}_{K} = \sum_{i,j=1}^{r} n_{i}^{K} n_{j}^{K} C_{ij} = 2$).

To see how (5.11) is obtained, write the exponent of $(a_{(p,K)} - a_{(q,L)})$ in (5.11) as $M_{(q,L)} \sum_{i,j=1}^{r} n_{i}^{K} n_{j}^{L} C_{ij}$; the terms involving $C_{ii} = 2$ are associated with $W$ boson threshold matching, whereas the $C_{ij} = -1$ terms are associated with threshold matching for matter fields coming from components of $Q_{ij}$. The products of $\Lambda_{i}^{\bar{M} \cdot \bar{e}_{i}}$, with exponent $n_{i}^{K}$, appearing $\Lambda_{i}$ in (5.11) results from the fact that $U(M_{(p,K)})$ arises as the diagonal subgroup, as in (5.6), so the $U(M_{(p,K)})$ gauge coupling is

$$g_{K}^{-2} = \sum_{i} n_{i}^{K} g_{i}^{-2}$$

(5.12)

at the scale of the Higgsing (5.6), and using (4.5).

We can now plug (5.11) into (5.1) to get the final expression

$$W_{g.c} = \sum_{(p,K)} S_{(p,K)} \left( M_{(p,K)} + M_{(p,K)} \log \left( \frac{m_{(p,K)} \prod_{i} \Lambda_{i}^{n_{i}^{K} n_{i}^{L} C_{ij}}}{S_{(p,K)}} \right) \right)$$

$$+ \sum_{(p,K)} \sum_{(q,L)\neq(p,K)} \sum_{i,j=1}^{r} S_{(p,K)} M_{(q,L)} n_{i}^{L} n_{j}^{K} C_{ij} \log \left( \frac{\Lambda_{i}}{a_{(p,K)} - a_{(q,L)}} \right).$$

(5.13)

We stress again that this is only an approximation, valid in the regime where the gauge couplings are weak at the scale determined by the superpotentials. Nevertheless, the non-trivial monodromies of (5.13) are expected to be exact, as the additional quantum corrections are single valued.

6. Geometric Construction

In this section we will study the geometric realization of the $N = 1$ A-D-E quiver theories, and connect with the field theoretic analysis of these theories presented in the
previous sections. The geometric description allows the formulation of large N duals via transitions of the form $S^2 \to S^3$, which we interpret as the field theory developing gaugino condensates. The dynamics of the gauge theory can be mapped to geometric language in a beautiful way. In particular, we show that the running of the gauge couplings is imposed upon us by the log divergences in the periods of the holomorphic three form on non-compact 3-cycles. The superpotential obtained in (5.13), from naive integrating in, is shown to be the leading order approximation in a weak coupling expansion of the exact superpotential given in terms of geometric periods. This leading order approximation can be obtained from the geometry via a monodromy analysis in the form of the Picard-Lefschetz formula.

6.1. Review of Geometric Engineering of $\mathcal{N} = 1$ A-D-E Quiver theories

The geometric engineering of $\mathcal{N} = 1$ A-D-E quiver theories is done in two steps. The first is to consider Type IIB on an ALE space with a blown up A-D-E singularity. Wrapping D5-branes around different non-trivial 2-cycles will give rise to $\mathcal{N} = 2$ gauge theories on the world volume. Likewise, adding D3-branes transverse to the ALE space will give $\mathcal{N} = 2$ affine A-D-E quiver theories on their worldvolumes. The second step is to realize that these ALE spaces can also be made nonsingular by adding relevant deformations. These deformations can then vary over the complex plane transverse to the ALE space, D3 and D5 branes. This fibration induces a superpotential in the theories, breaking the supersymmetry down to $\mathcal{N} = 1$.

A-D-E singularities in dimension 2

Blown down singular ALE spaces can be viewed as hypersurfaces $f(x, y, z) = 0$ of $\mathbb{C}^3$:

\[
\begin{align*}
G = A_r : & \quad f = x^2 + y^2 + z^{r+1} \\
G = D_r : & \quad f = x^2 + y^2 z + z^{r-1} \\
G = E_6 : & \quad f = x^2 + y^3 + z^4 \\
G = E_7 : & \quad f = x^2 + y^3 + yz^3 \\
G = E_8 : & \quad f = x^2 + y^3 + z^5
\end{align*}
\]

These spaces can be made smooth by adding relevant deformations of the form,

\[
\sum_{i=1}^{r} P_{c_i(G)}(t_1, \ldots, t_r) R_{C_2(G) - c_i(G)}(y, z),
\]

19
where the subscripts are the degrees of the polynomials under the scaling where \( t_i \)'s have degree one and \( f(x, y, z) \) has degree \( C_2(G) \), the dual Coxeter number of \( G \) \((c_i(G) \) are the degrees of the Casimirs of \( G \)). Notice that there are \( r = \text{rank}(G) \) deformation parameters \( t_i \)'s. For generic \( t_i \)'s, there are \( r \) independent classes of non-vanishing \( S^2 \)'s and their intersection can be chosen to correspond to the \( G \) Dynkin diagram. The holomorphic volumes of the \( S^2_i \)'s are denoted by,
\[
\alpha_i = \int_{S^2_i} \frac{dydz}{x} \quad \text{for} \quad i = 1, \ldots, r
\]
The \( \alpha_i \) are in 1-1 correspondence with the simple roots of \( G \), and are linearly related to the \( t_i \).

The \( \alpha_i \) are in 1-1 correspondence with the simple roots of \( G \), and are linearly related to the \( t_i \).

The deformed ALE space is simple to write in the \( A \) and \( D \) cases, namely,
\[
A_r : \quad x^2 + y^2 + \prod_{i=1}^{r+1} (z + t_i) \quad \sum_{i=1}^{r+1} t_i = 0 \tag{6.1}
\]
\[
D_r : \quad x^2 + y^2z + \prod_{i=1}^{r} (z + t_i^2) - \prod_{i=1}^{r} t_i^2 + 2 \prod_{i=1}^{r} t_i y \tag{6.2}
\]
and the holomorphic volumes are given by,
\[
A_r : \quad \alpha_i = t_i - t_{i+1} \quad i = 1, \ldots, r \tag{6.3}
\]
\[
D_r : \quad \alpha_i = t_i - t_{i+1} \quad i = 1, \ldots, r - 1 \quad \text{and} \quad \alpha_r = t_{r-1} + t_r \tag{6.4}
\]
The corresponding equations for \( E_6, E_7 \) and \( E_8 \) deformations in terms of \( t \)'s (which are chosen to be linearly related to \( \alpha \)'s) are more complicated and we refer the reader to [35].

**Fibration**

We want to obtain a Calabi-Yau 3-fold by fibering the ALE space described above over a complex plane whose coordinate we denote by \( t \). This fibration is implemented by allowing the \( t_i \)'s to be polynomials in \( t \). Therefore, the holomorphic volumes \( \alpha_i \) will also be functions of \( t \). Wrapping \( N_i \) D5 branes around the \( S^2_i \) fiber, but with world volume transverse to the \( t \)-plane, will induce a classical superpotential in the gauge theory satisfying,
\[
W'_i(t) = \alpha_i(t)
\]
where \( t \) corresponds to \( \langle \Phi_i \rangle \), the expectation value of the adjoint of the \( U(N_i) \).
Notice that without the superpotentials, i.e., with a trivial fibration, the normal bundle over each \( S^2 \) is \( O(-2) \oplus O(0) \). However, with the introduction of superpotentials, the geometry will have points where a given cycle can have zero holomorphic volume. These points, which are singular in the geometry, can be blown up, giving rise to \( S^2 \)'s with normal bundle \( O(-1) \oplus O(-1) \). If the degree of \( W'_i(t) \) is \( k \), there will be \( k \) points in the \( t \)-plane for each positive root \( \vec{\rho}_K \) of \( G \) where the holomorphic volume vanishes:

\[
\alpha(\vec{\rho}_K) = \sum_{i=1}^{r} n^i_K \alpha_i(t) = \sum_{i=1}^{r} n^i_K W'_i(t) = 0. \tag{6.5}
\]

These are the supersymmetric vacua corresponding to the Higgsing (3.6) where we wrap \( M_{p,K} \) D5 branes around the cycle at the \( p \)-th solution of (6.5). Let us rewrite (3.6),

\[
\prod_i U(N_i) \rightarrow \prod_{K=1}^{R_+} \prod_{p=1}^{k} U(M_{(p,K)}).
\]

Clearly, the charge conservation condition is (3.5),

\[
N_i = \sum_{K=1}^{R_+} \sum_{p=1}^{k} M_{(p,K)} n^i_K.
\]

6.2. Large \( N \) duality

In the IR limit of the gauge theory we are left with pure \( \mathcal{N} = 1 \) SYM with gauge group \( \prod_{K=1}^{R_+} \prod_{p=1}^{k} U(M_{(p,K)}) \). This theory is expected to have gaugino condensation in each factor of the gauge group, as discussed in section 5. As in [12,36,7,13], the proposal is that the geometry realizes this process by geometric transitions of the form \( S^2_{(p,K)} \rightarrow S^3_{(p,K)} \).

It is important to notice that all these are conifold-like transitions since the \( S^2_{(p,K)} \)'s being blown down have normal bundle \( O(-1) \oplus O(-1) \).

The number of singular points after blowing down all \( S^2_{(p,K)} \)'s is \( kR_+ \), where \( k \) is the degree of \( W'_i(t) \)'s. The large \( N \) dual is therefore achieved by deforming the complex structure of the singular Calabi-Yau 3-fold,

\[
x^2 + F(y,z,t_1(t),\ldots,t_r(t)) = 0 \tag{6.6}
\]

by normalizable deformations (including the log normalizable). These normalizable deformations correspond to dynamical fields, as opposed to fixed parameters [37]. The dynamical fields which they correspond to are precisely the \( SU(M_{(p,K)}) \) glueball fields
$S_{(p,K)} \sim \text{Tr}_{SU(M_{(p,K)})} W_\alpha W^\alpha$. In [17], it was shown that the total number of these normalizable deformations is exactly $kR_+$, the expected number of $S^3$’s. This matches the natural idea that the $kR_+$ gaugino condensates are independent, dynamical, and control the sizes of the $S^3$, parameterizing the deformation of the geometry.

The normalizable deformations can be easily found by noting that (6.6) has the form

$$f(x, y, z) + at^{kC_2(G)} + \ldots = 0. \quad (6.7)$$

Charges can be assigned to $x$, $y$, $z$, and $t$ such that the above equation has charge 1. In particular, $t$ will always have charge $1/kC_2(G)$. Thinking about (6.7) as the superpotential of a Landau-Ginzburg theory, the central charge is given by,

$$\hat{c} = (1 - 2Q(x)) + (1 - 2Q(y)) + (1 - 2Q(z)) + (1 - 2Q(t)) = \frac{2}{kC_2(G)} (k(C_2(G) - 1) - 1). \quad (6.8)$$

The normalizable deformations are those monomials $t^\beta y^\delta z^\gamma$ with charge $Q(t^\beta y^\delta z^\gamma) < \frac{\hat{c}}{2}$; we also include the log normalizable deformations, with charge $Q(t^\beta y^\delta z^\gamma) = \frac{\hat{c}}{2}$ [37].

**Periods and Superpotential**

The geometry after the deformation is smooth and contains $kR_+$ non-trivial $S^3$’s. These 3-cycles form a natural basis for $A_{(p,K)}$ cycles in the Calabi-Yau geometry and we define $kR_+$ non compact cycles $B_{(p,K)}$ dual to the $A_{(p,K)}$’s producing a symplectic pairing. An important role in the sequel is played by the periods of the holomorphic three form $\Omega$ over $A_{(p,K)}$’s and $B_{(p,K)}$’s. We denote the periods,

$$\int_{A_{(p,K)}} \Omega \equiv S_{(p,K)} \quad \int_{B_{(p,K)}} \Omega \equiv \Pi_{(p,K)} = \frac{\partial F}{\partial S_{(p,K)}}, \quad (6.9)$$

where $\Lambda_0$ is a cutoff needed to regulate the divergent $B_{(p,K)}$ integrals. The $kR_+$ periods $S_{(p,K)}$ are determined by (6.9) in terms of the coefficients of the $kR_+$ normalizable deformations. One can then invert these relations, to write the coefficients of the normalizable deformations in terms of the $S_{(p,K)}$.

After the transition the D branes have disappeared and have been replaced by fluxes on the $S^3$’s of a suitable 3-form $H$. This leads to a superpotential [38,39],

$$W = \int H \wedge \Omega = \sum_{p=1}^{k} \sum_{K=1}^{R_+} \left( \int_{A_{(p,K)}} H \int_{B_{(p,K)}} \Omega - \int_{B_{(p,K)}} H \int_{A_{(p,K)}} \Omega \right). \quad (6.10)$$
This thus gives for the full effective superpotential

$$-rac{1}{2\pi i}W = \sum_{p=1}^{k} \sum_{K=1}^{R_p} \left( M_{(p,K)} \Pi_{(p,K)} + \frac{\alpha_K}{2\pi i} S_{(p,K)} \right)$$

(6.11)

where $\alpha_K$’s are related to the bare coupling constants of the original $U(N_i)$’s with $i = 1, \ldots, r$ of the quiver theory. The precise correspondence will be given below when we show that the logarithmic dependence on $\Lambda_0$ of $\Pi$’s can be absorbed in the $\alpha$’s, rendering the superpotential finite (up to irrelevant constant terms) as we send the cut off $\Lambda_0$ to infinity.

6.3. Dynamics of the theory

It was shown in [13] for the $G = A_1$ quiver theory case that (6.11) is the exact effective superpotential of the $X(k, A_1)$ theory, and that the superpotential obtained from naive integrating in is the leading order approximation of (6.11) in a weak coupling expansion. Here will see that the same is true for the general class of A-D-E quiver theories we have geometrically engineered in this section.

Renormalization of gauge couplings

The superpotential (6.11) contains the periods of $\Omega$ over non-compact cycles. These periods are divergent and need a cut off $\Lambda_0$ to be well defined. These are long distance (IR) divergences and therefore we expect them to be related to short distance (UV) divergences in the field theory. This was the case for $X(k, A_1)$ [13], where the renormalization of the gauge coupling constant in field theory was forced upon us in the geometric set up by the IR divergence. This is also true for the general A-D-E cases as we now proceed to show.

The periods of $\Omega$ can be computed using the fact that the Calabi-Yau under consideration is an ALE fibration over a complex plane. The three cycles in this geometry project to lines in the $t$-plane where, over each point, there is an $S^2$. Compact $S^3$ cycles are those for which the projection is a line segment and the holomorphic volume of the $S^2$ vanishes at each end. Non-compact cycles on the other hand are semi- infinite lines in the limit when $\Lambda_0$ is infinite. The periods can be computed as integrals of the holomorphic volume of a given $S^2$ over the path in the $t$-plane, i.e.,

$$\int_{B_{(p,K)}}^{\Lambda_0} \Omega = \int_{C_{(p,K)}} \tilde{\alpha}(\tilde{\rho}_K) dt,$$
where $C_i$ is an appropriate contour and $\tilde{\alpha}(\tilde{\rho}_K)$ is the volume (6.5) after the deformations are introduced, so we should have $\tilde{\alpha}(\tilde{\rho}_K) \rightarrow \alpha(\tilde{\rho}_K)$ when the deformations are turned off.

Let us expand $\tilde{\alpha}$ in a Laurent expansion in $t$,

$$\tilde{\alpha}(\tilde{\rho}_K) = \sum_{m=-\infty}^{\infty} \sigma_m t^m$$

The charge of the LHS can be seen to be $kQ(t)$ by setting all deformations to zero. This implies that

$$Q(\sigma_m) = (k - m)Q(t) = \frac{k - m}{kC_2(G)}.$$

Our aim is to find the possible dependence of $\sigma_m$ on the deformation parameters. Recall that deformation parameters $d_{\beta\delta\gamma}$ are the coefficients of the allowed monomials $t^\beta y^\delta z^\gamma$. In the following we will suppress the subscripts since only the charge will be important. The charge of deformation parameters is therefore,

$$Q(d) = 1 - Q(\text{monomial}) \geq 1 - \frac{k + 1}{2} = \frac{k + 1}{kC_2(G)}$$

where equality holds for the log normalizable deformations.

Finally, imposing the condition that $\tilde{\alpha}_i(t) \rightarrow \alpha_i(t)$ upon turning off the deformations, $d \rightarrow 0$, implies that

$$\sigma_m = 0 \quad \text{for} \quad m > k$$

$$\sum_{m=0}^{k} \sigma_m = \alpha(\tilde{\rho}_K)$$

$$\sigma_{-1} = \sum_{i=1}^{r} g_i d^\log_i \quad \text{where} \quad d^\log_i \text{ are log normalizable},$$

$g_i$ are classical superpotential parameters and $\sigma_m$ for $m \leq -2$ depend on normalizable as well as log normalizable deformations.

The conclusion is then that the $\Lambda_0$ dependence of the non-compact periods is

$$\Pi_{(p,K)} = \int_{\Lambda_0}^{\Lambda_0} \tilde{\alpha}(\tilde{\rho}_K) dt = \sum_{i=1}^{r} n_i W_i(\Lambda_0) + \sigma_{-1} \log(\Lambda_0) + O\left(\frac{1}{\Lambda_0}\right) + \ldots.$$

The first term on the RHS is an irrelevant constant, which is independent of the deformation parameters. So the only dangerous divergence is the log one, with coefficient $\sigma_{-1}$. The only parameters in the superpotential (6.11) which can be renormalized to absorb
these log divergences are the $\alpha_K$'s. It is non-trivial for this to be possible, as the $\alpha_K$'s are the coefficients of very special functions of $S_{(p,K)}$'s; so we need to show that the $\sigma_{-1}$'s conspire to give this same $S_{(p,K)}$ dependence.

Let us choose the orientation of all the contours for computing the compact periods to be counter-clockwise and for the non-compact dual periods to go from $\Lambda_0$ on the lower sheet to $\Lambda_0$ on the upper sheet crossing the branch cuts defined to be between the two points that split when the deformation is tuned on. The notion of upper sheet and lower sheet refers to the fact that for each $S_3$ we have a double point on the $t$ plane and the fibered geometry has naturally related to a double covering.

Now we only have to remember that at each point on the $t$-plane we have a fiber with a basis of two cycles intersecting according to the Cartan matrix of the corresponding A-D-E root system. Let us pick one of the non-compact periods $\Pi_{(p,K)}$ and keep track of how it changes as we change $\Lambda_0 \rightarrow e^{i\theta} \Lambda_0$ with $\theta \in \{0, 2\pi\}$.

Using the Picard-Lefschetz formula \[40\], the cycle corresponding to the positive root $\vec{\rho}_K$ will change as the contour crosses the vanishing cycles \[3\] according to their intersection. This can be made very precise by denoting $\vec{\rho}_L$ the class of the compact cycle $\vec{\rho}_K$ is crossing. The change in the period is then,

$$\Delta \Pi_{(p,K)} = (\vec{\rho}_K \cdot \vec{\rho}_L) S_{(m,L)}$$

where $m = 1, \ldots, k$ refers to the particular solution of $\sum_{i=1}^{r} n^i W'_i(t) = 0$ which corresponds to the cycle which we are crossing. Now we can write the total change in the non-compact period as $\Lambda_0$ goes around as

$$\Delta \Pi_{(p,K)} = \sum_{L \in \Delta^+} (\vec{\rho}_K \cdot \vec{\rho}_L) \sum_{m=1}^{k} S_{(m,L)}.$$  

This implies that $\Pi_{(p,K)}$ has a logarithmic dependence on $\Lambda_0$ as expected:

$$\Pi_{(p,K)} = \frac{1}{2\pi i} \left( \sum_{L \in \Delta^+} (\vec{\rho}_K \cdot \vec{\rho}_L) \sum_{m=1}^{k} S_{(m,L)} \right) \log(\Lambda_0) + \ldots \quad (6.12)$$

where $\ldots$ are the cut-off single valued pieces.

---

\[3\] Vanishing cycles in Picard-Lefschetz formula refer to cycles that can shrink by changing the complex structure. In our case by setting to zero the deformations.
Recall that the second term in (6.11) was obtained by the identification $S_{(p,K) \leftrightarrow \text{Tr}(W_0^2)}$ the SU($M(p,K)$) glueball field. Therefore, $\alpha_K$ is also identified with the bare coupling of the corresponding gauge factor $\frac{8\pi^2}{(g_{YM(K)})^2}$. This implies that only $r$ of all $\alpha_K$’s are linearly independent. Let us choose as basis $\alpha_i$ with $i \in \Delta^0$, the set of simple roots. The other $\alpha_K$’s corresponding to positive roots $\vec{\rho}_K = \sum_{p=1}^{r} n^i_K \vec{e}_i$ are given by,

$$
\alpha_K = \sum_{i=1}^{r} n^i_K \alpha_i. \tag{6.13}
$$

Clearly, each $\alpha_i$ has to have a logarithmic dependence on $\Lambda_0$. In order to have a dimensionally sensible expression we need to include new parameters $\Lambda_i$, which will be identified with the dynamically generated scales of the high energy $\prod U(N_i)$ theory. Let us assume the simplest ansatz for the basis,

$$
\alpha_i = -\frac{8\pi^2}{(g_{YM(i)})^2} = \beta_i \log \left( \frac{\Lambda_i}{\Lambda_0} \right) \quad \text{with} \quad i = 1, \ldots, r \tag{6.14}
$$

where $\beta_i$ are yet to be determined. This is the same phenomenon as dimensional transmutation in field theoretic language.

Let us collect the possibly log-divergent pieces of the superpotential (6.11), using the result from (6.12):

$$
-W_{\text{divg}} = \sum_{L \in \Delta^+} \sum_{m=1}^{k} S_{(m,L)} \left( \sum_{p=1}^{k} \sum_{K \in \Delta^+} M_{(p,K)} (\vec{\rho}_K \cdot \vec{\rho}_L) \log(\Lambda_0) + \alpha_L \right).
$$

The $\alpha_L$ appearing in the above must cancel these divergences term by term in $L$, requiring that

$$
\alpha_L = -\sum_{p=1}^{k} \sum_{K \in \Delta^+} M_{(p,K)} (\vec{\rho}_K \cdot \vec{\rho}_L) \log(\Lambda_0) + \ldots
$$

where ... denote cut-off independent pieces. Specializing to $L = i \in \Delta^0$ and using (6.14) we get that

$$
\beta_i = \sum_{K \in \Delta^+} \left( \sum_{p=1}^{k} M_{(p,K)} \right) (\vec{\rho}_K \cdot \vec{e}_i) = \sum_{j=1}^{r} C_{ij} \sum_{K \in \Delta^+} \left( \sum_{p=1}^{k} M_{(p,K)} \right) n^i_K = \sum_{j=1}^{r} C_{ij} N_j. \tag{6.15}
$$

The geometry has thus reproduced the 1-loop holomorphic beta functions (4.5).
It is simple to see that with (6.15) and (6.13) the superpotential does not have logarithmic divergences. As a by-product we have learned that the superpotential also depends on \( r \) scales \( \Lambda_i \) in the following form,

\[
W = - \sum_{L \in \Delta^+} \hat{\alpha}_L \left( \sum_{p=1}^k S_{(p,L)} \right) + \ldots
\]  

(6.16)

where

\[
\hat{\alpha}_L = \sum_{i=1}^r n^i_L \beta_i \text{Log}(\Lambda_i) \quad \text{with} \quad \bar{\rho}_L = \sum_{i=1}^r n^i_L \bar{\epsilon}_i
\]

**Leading order superpotential**

The exact effective superpotential (6.11) can be studied in the weak coupling limit. This means that the dynamically generated scales \( \Lambda_i \) with \( i = 1, \ldots, r \) are small compared to the scales set by the superpotentials \( W_i(t) \)'s. In geometrical terms this means that the compact \( S^3 \)'s are small compared to their separation in the \( t \)-plane. In order to be more precise let us introduce some notation. For zero deformation parameters we get \( kR_+ \) singular points located at the solutions of

\[
W'_K(t) \equiv \sum_{i=1}^r n^i_K W'_i(t) \equiv g_K \prod_{p=1}^k (t - a_{(p,K)}) = 0
\]

for \( K \in \Delta^+ \), the set of positive roots.

After the deformation each singular point \( t = a_{(p,K)} \) splits into two giving rise to \( S^3_{(p,K)} \). Let us denote the new two points by \( a^+_{(p,K)} \) and \( a^-_{(p,K)} \). Now the periods can be written more explicitly as follows,

\[
S_{(p,K)} = \frac{1}{2\pi i} \int_{a^-_{(p,K)}}^{a^+_{(p,K)}} \tilde{\alpha}(\bar{\rho}_K)dt \quad \text{and} \quad \Pi_{(p,K)} = \frac{1}{2\pi i} \int_{a^-_{(p,K)}}^{a^+_{(p,K)}} \tilde{\alpha}(\bar{\rho}_K)dt
\]

The weak coupling regime can therefore be defined by the following conditions

\[
| a^+_{(p,K)} - a^-_{(p,K)} | \ll | a_{(m,L)} - a_{(p,K)} | \quad \text{for all} \quad (p, K) \neq (m, L).
\]

Following [13], using monodromy arguments one can compute the \( \text{Log}(S_{(p,K)}) \) and \( \text{Log}(a_{(p,K)} - a_{(m,L)}) \) dependence of \( \Pi_{(p,K)} \) and therefore of the superpotential (6.11).
Consider first the geometry close to \( a_{(p,K)} \), this geometry can be thought of as that of a single conifold in the limit we are considering, therefore, the \( S_{(p,K)} \) period should look like \([13]\),

\[
S_{(p,K)} = \frac{1}{2\pi i} W''_K(a_{(p,K)}) \int_{a_{-(p,K)}}^{a_{(p,K)}} \sqrt{(t - a_{(p,K)})^2 - \mu_{\text{eff}}} \, dt
\]

Using Picard-Lefschetz formula for \( \mu_{\text{eff}} \to e^{2\pi i \mu_{\text{eff}}} \), we get that the corresponding dual period changes as \( \Delta \Pi_{(p,K)} = S_{(p,K)} \), therefore one can conclude that,

\[
\Pi_{p,K} = \frac{1}{2\pi i} S_{(p,K)} \log \frac{S_{(p,K)}}{W''_K(a_{(p,K)})} + \ldots
\]

Finally, let us consider how \( \Pi_{(p,K)} \) changes when we move one \( a_{(q,L)} \) around \( a_{(p,K)} \), again using P-L formula gives that,

\[
\Delta \Pi_{(p,K)} = (\rho \bar{K} \cdot \rho \bar{L}) S_{(q,L)}
\]

Notice that the coefficient in front of \( S_{q,L} \) does not depend on \( m \) or \( p \), this is because the intersection formula only sees the classes and for a given \( K \) all \( p \) have the same class.

Now we can collect all these partial results to write,

\[
2\pi i \Pi_{(p,K)} = S_{(p,K)} \log \frac{S_{(p,K)}}{W''_K(a_{(p,K)})} + \sum_L \sum_{m=1}^k (\rho \bar{K} \cdot \rho \bar{L}) S_{(q,L)} \log (a_{(p,K)} - a_{(q,L)}) + \ldots
\]

in this formula the sum over \( L \) and \( m \) runs over all \( (q,L) \neq (p,K) \).

The leading order superpotential can then be obtained by combining (6.11), (6.17), and (6.16) to get,

\[
W = \sum_{K} \sum_{p=1}^k M_{(p,K)} \left( S_{(p,K)} \log \frac{W''_K(a_{(p,K)})}{S_{(p,K)}} + \sum_{L} \sum_{m=1}^k (\sum_{i,j=1}^r C_{ij} n_{iL} n_{jK}) S_{(q,L)} \log \frac{1}{a_{(p,K)} - a_{(q,L)}} \right) 
\]

\[
+ \sum_K \left( \sum_{p=1}^k M_{(p,K)} \sum_{i,j=1}^r C_{ij} n_{iL} \log \Lambda_i \sum_{L} n_{iL} \sum_{m=1}^k S_{(q,L)} + \ldots \right)
\]

In order to compare with the gauge theory answer from naive integrating in, let us write \( W \) collecting all terms with \( S_K \) together,

\[
W = \sum_{(p,K)} S_{(p,K)} \left( M_{(p,K)} \ln \left( \frac{W''_K(a_{(p,K)}) \prod_i \Lambda_i^{n_{iK} n_{jL} C_{ij}}}{S_{(q,K)}} \right) \right) 
\]

\[
+ \sum_{(p,K)} \sum_{(q,L) \neq (p,K)} \sum_{i,j=1}^r M_{(p,K)} S_{(q,L)} n_{iL} n_{jK} C_{ij} \ln \left( \frac{\Lambda_i}{a_{(p,K)} - a_{(q,L)}} \right)
\]
We thus find perfect agreement with the gauge theory answer (5.13). Notice that (5.13) contains linear terms in $S(p,K)$. These and possibly an infinite power expansion in $S(p,K)$’s can not be derived using monodromy arguments. A more detailed analysis of the geometry result shows that the superpotential indeed generally contains an infinite power expansion of terms which are missed by the naive integrating in analysis, as was computed for $X(k,A_1)$ in [13].

Finally, one has to check that the weak coupling approximation is self-consistent. For this it is necessary to identify the expansion parameters that enter in the infinite power series mentioned before. Let us assume that all the relevant scales set by the classical superpotentials are of the same order equal to $\Delta$. This means that $(a_{(p,K)} - a_{(q,L)}) \sim W''_j \sim \Delta$ for all $K, L, J \in \Delta_+$. Moreover, let us assume that all the scales of the individual $U(N_i)$ factors are of the same order $\Lambda_i \sim \Lambda \ll \Delta$ for $i = 1, \ldots, r$. Let us show that the natural dimensionless expansion parameter for the computation of periods is $\frac{\Lambda}{\Delta}$.

The leading order superpotential (5.1) implies that $\langle S(p,K) \rangle_M(p,K) = \Lambda^3 M(p,K)$. Then, using (5.11) with $W''_K \sim (a_{(p,K)} - a_{(q,L)}) \sim \Delta$, and taking all $\Lambda_i \sim \Lambda$, we find for the expectation value of the gaugino fields, or in geometric language, the sizes of the $S^3$ cycles:

$$\left(\frac{\langle S_K \rangle}{\Delta^3}\right)^{M_K} = \left(\frac{\Lambda}{\Delta}\right) \sum J M_J \bar{\rho}_J \cdot \bar{\rho}_K$$

This implies that the power expansion in $\Lambda/\Delta$, and hence the superpotential (5.13), are valid approximations when $\sum J M_J \bar{\rho}_J \cdot \bar{\rho}_K > 0$. Since

$$\sum J M_J \bar{\rho}_J \cdot \bar{\rho}_K = \sum_{j=1}^r n^j_K \left(\sum_{i=1}^r N_i C_{ij}\right) = \sum_{j=1}^r n^j_K \beta_i$$

with $n^j_K \geq 0$, and $n^j_i = \delta^j_i$ for $K = j$ a simple root, the necessary condition is thus that all $U(N_i)$’s have to be asymptotically free.

This analysis shows that in cases when no weak coupling expansion is possible in terms of the parameters of a given theory two possibilities can occur. The first is that the exact superpotential (6.11) might still be computable in a power expansion in terms of the parameters of a different (dual) theory and the second is that no simple gauge theoretic interpretation exists even though the geometric description still yields exact results.
7. Duality Predictions From Geometric Construction of the A-D-E Quiver Theories

Consider the geometric engineering of the quiver theory. Consider blowing down the cycles (i.e. where the inverse couplings $1/g_i^2 = 0$). If we are just given this geometry together with some data about which classes the branes wrap (or how much flux is coming out of each vanishing $S^3$) we cannot uniquely determine the quiver theory corresponding to it. The reason for this is that in order to decipher the gauge theory we have to identify certain parameters in the geometry with a choice of simple roots of the A-D-E, and this is unique only up to the choice of a Weyl group action. This implies that with this data we cannot quite give a unique description of the quiver theory, however we can give descriptions in seemingly different looking gauge theories which have to be equivalent because they are describing the same underlying string theory. Our constructions apply equally well to A-D-E as well as the affine case. This is how geometry predicts gauge theory dualities, in one to one correspondence with elements of the Weyl group. As is well known the Weyl group is generated by Weyl reflections about simple roots, and this we identify as Seiberg-like dualities in the corresponding quiver theory.

In the original geometric engineering we have blown up $S^2$'s and which $S^2$'s we blowup picks a particular ‘preferred’ description for which the gauge couplings $1/g_i^2 > 0$. Of course they can be viewed as analytic continuation of the other dual descriptions where some of the gauge couplings squared are negative. This phenomenon, taking into account the dimensional transmutation, becomes part of the data of matching of scales between the dual theories.

Let us consider a given theory with branes $N_i$ wrapping the corresponding dual cycles, undergoing a transition to Higgs branch with branch number degeneracies $M(p,K)$ where $K$ labels the positive roots and $p$ an integer between $1,\ldots,k$. Now consider a different choice of positive roots given by Weyl reflection about $\vec{e}_{i_0}$. This affects the roots by

$$\vec{e}''_j = \vec{e}_j - (\vec{e}_j \cdot \vec{e}_{i_0})\vec{e}_{i_0}. \quad (7.1)$$

The conservation of brane charge determines the rank of the gauge groups after transitions, as in [3,4] and we find

$$\sum N_i \vec{e}_i = \sum N'_i \vec{e}'_i. \quad (7.2)$$

It follows from this that $N'_j = N_j$ for $j \neq i_0$, and

$$N'_{i_0} = N_f - N_{i_0}$$
where \( N_f = \sum_{i \neq i_0} (-\vec{e}_i \cdot \vec{e}_{i_0})N_i \) denotes the number of flavors of the \( U(N_{i_0}) \) theory. The Weyl group also acts on the couplings, which correspond to Kahler volumes of the \( \vec{e}_i' \), as

\[
\frac{1}{g'^2_i} = \frac{1}{g^2_i} - \frac{\vec{e}_i \cdot \vec{e}_{i_0}}{g^2_{i_0}}. \tag{7.3}
\]

Similarly it acts on the superpotentials by the integral of the holomorphic 3-form over the relevant cycle which is

\[
W_i \rightarrow W_i - (\vec{e}_i \cdot \vec{e}_{i_0})W_{i_0} \tag{7.4}
\]

In the IR, i.e. at scales below the scale of the superpotential we also have to choose which branches we are in. This makes sense assuming that the coupling of the gauge theory is weak at the scale of the superpotential, so that the classical analysis is reliable. In this case we have branches labeled by the positive roots \( \vec{\rho}_K \). Under the Weyl reflection the positive roots get permuted except for \( \vec{\rho}_K = \vec{e}_{i_0} \) which goes to minus itself (it is also easy to see, using (7.4) that the choices within a given branch get mapped in a canonical way).

Thus \( M_{p,K} = M'_{p,w_{e_{i_0}}(K)} \), for \( K \neq e_{i_0} \) where \( w_{e_{i_0}} \) denotes the Weyl reflection by \( e_{i_0} \), and \( M'_{p,K} = -M_{p,K} \) for \( K = e_{i_0} \). Note that this latter action on the branches would yield negative multiplicities unless \( M_{p,e_{i_0}} = 0 \). So only for this case we can formally use the dual. We will elaborate on the geometric meaning of this later. However, we emphasize that even if \( M_{p,e_{i_0}} \neq 0 \) in a formal sense the dual theory makes sense. What we mean by this is that when we set up the dual geometry and write the corresponding superpotential, replacing the flux coming from the branch corresponding to \( \vec{e}_{i_0} \) with a negative number does make sense, and would yield an identical description of the geometry. Thus at the level of setting up the dual geometry description we simply have an ambiguity of reading off the gauge theory. Thus the geometry predicts gauge theoretic dualities which we will verify in the next section.

8. Dualizing a gauge group factor

Consider a particular \( U(N_{i_0}) \) gauge group factor in our general \( \mathcal{N} = 1 \) quiver labelled by \( k \) and \( G \) or \( \hat{G} \). We write the superpotential for the fields charged under \( U(N_{i_0}) \) as

\[
W = \frac{s}{k+1} \text{Tr}\phi^{k+1} + \text{Tr}\phi\overline{QQ} + \text{Tr}m\overline{Q}, \tag{8.1}
\]

where \( Q \) is a \( N_f \times N_c \) matrix, with \( \overline{QQ} \) in the adjoint of \( U(N_c) \) singlet under \( U(N_f) \) and \( M = \overline{QQ} \) a \( U(N_c) \) singlet and in the adjoint of \( U(N_f) \). The \( N_f = \sum_{i \neq i_0} (-\vec{e}_i \cdot \vec{e}_{i_0})N_i \)
fundamentals arise from the bi-fundamentals connecting to the neighboring nodes of the quiver diagram, and the mass $m$ in (8.1) is a matrix in the flavor space, which is actually given by the expectation values of the adjoints of the neighboring nodes’ gauge groups. We treat the neighboring nodes as weakly gauged flavor symmetries.

As we briefly review, the above theory can be dualized to a $U(N_f - N_{i_0})$ gauge theory for all $k$. This is naturally related to the $U(N_f - N_{i_0})$ which arises in the $\mathcal{N} = 2$ theory (setting $s = 0$ in (8.1)) at the base where the “baryon branch” intersects the Coulomb branch [16]. Before discussing the details of the duality, we note a few of the most important features.

As seen in the geometry, the duality corresponds to a $G$ Weyl reflection, or $\hat{G}$ Weyl reflection in the affine case. The duality does not act on the $N_i$ of the other nodes, which correspond to flavor symmetries, unchanged, and replaced $N_{i_0} \equiv N_c$ with $N_f - N_c$, i.e.

$$N'_i = N_i \quad \text{for} \quad i \neq i_0, \quad N'_{i_0} = N_{i_0} - \sum_j \bar{e}_{i_0} \cdot \bar{e}_j N_j.$$

As discussed in the previous section, we can write this as

$$\vec{N} \equiv \sum_{i=1}^r N_i \vec{e}_i = \sum_{i=1}^r N'_i \vec{e}'_i,$$  \hspace{1cm} (8.3)

with

$$\vec{e}'_i = \vec{e}_i - (\bar{e}_{i_0} \cdot \vec{e}_i) \bar{e}_{i_0},$$  \hspace{1cm} (8.4)

which is precisely the action of a Weyl reflection about the simple root $\vec{e}_{i_0}$. Such transformations for all the nodes generate the entire Weyl group (or affine Weyl group for the case of the affine quiver diagrams).

To see how (7.3) occurs in the field theory duality, consider the holomorphic beta functions of the $\mathcal{N} = 1$ quiver diagram theories (above the scale $\Delta$ where the adjoints get masses); these coincide with (4.1), and can be written as in (4.2). The beta functions of the theory after the duality transformation are

$$\beta'_i = \vec{e}'_i \cdot \sum_j \vec{e}_j N'_j = \vec{e}'_i \cdot \sum_j \vec{e}'_j N'_j = \vec{e}'_i \sum_j \vec{e}_j N_j = \beta_i - (\bar{e}_{i_0} \cdot \vec{e}_i) \beta_{i_0},$$  \hspace{1cm} (8.5)

where we used (8.4), $\vec{e}'_i \cdot \vec{e}'_j = \vec{e}_i \cdot \vec{e}_j = C_{ij}$, and (8.3). So the holomorphic functions transform precisely as under the Weyl transformation (8.4). We can formally integrate these beta functions to get the similar transformation of the couplings $g_i^{(-2)}$, as in (7.3).
A similar transformation as (8.5) would hold for the exact physical beta functions (4.7) if all \( \gamma(\phi_i) \) are equal and \( \beta(\lambda_{ij}) = 0 \); likewise for (4.9) if \( \tilde{\gamma}(Q_{ij}) = 0 \).

Consider matching the running gauge couplings, given by (4.5), before and after the duality transformation on some particular \( U(N_{i_0}) \); the matching occurs at the scale \( \mu = \Lambda_{i_0} \) where \( U(N_{i_0}) \) gets strong:

\[
e^{2\pi i \tau_j(\mu)} = \left( \frac{\Lambda_j}{\mu} \right)^{\beta_j} = \left( \frac{\Lambda_j'}{\mu} \right)^{\beta_j'} \quad \text{at} \quad \mu = \Lambda_{i_0}.
\] (8.6)

Using (8.5) the matching relation obtained from (8.6) is

\[
\Lambda_i \beta_i = \Lambda_i \beta_i \Lambda_{i_0}^{-\tilde{e}_i \cdot \tilde{e}_{i_0}},
\] (8.7)
i.e. (aside from the case \( \tilde{A}_1 \) where \( C_{01} = -2 \))

\[
\Lambda_i \beta_i \Lambda_{i_0} \beta_{i_0} = 1, \quad \Lambda_i \beta_j = \Lambda_i \beta_i |s_{i_0}| \Lambda_j \beta_j \quad j \neq i_0.
\] (8.8)

The first relation (8.8), which gives \( \Lambda_{i_0} = \Lambda_{i_0}' \), is similar to the duality relation [41] for \( \mathcal{N} = 1 \) SQCD without the adjoint \( \phi \)

\[
\Lambda_{SQCD}^{3N_c - N_f} \tilde{\Lambda}_{SQCD}^{3N_c - N_f} \sim \mu^{N_f},
\] (8.9)

where \( \mu \) is the scale appearing in the dual superpotential as \( W_{mag} = \mu^{-1} M q \bar{q} \). Indeed, for \( k = 1 \) the adjoint \( \phi_{i_0} \) has mass \( m = s \) from (8.1) and can be integrated out from both the electric and magnetic theories, giving \( \Lambda_{SQCD}^{3N_c - N_f} = m^{N_c} \Lambda_{i_0}^{\beta_{i_0}} \) and \( \tilde{\Lambda}_{SQCD}^{3N_c - N_f} = m^{N_c} \Lambda_{i_0}^{\beta_{i_0}'} \), and then (8.8) agrees with (8.9) for \( \mu = m \).

Integrating the beta function equations, in order to have all \( g_i^{-2} \geq 0 \), we should have

\[
\Lambda_{NAF} > \mu > \Lambda_{AF},
\] (8.10)

where \( \mu \) is the energy scale and \( \Lambda_{NAF} \) is the dynamical scale \( \Lambda_i \) of those \( i \) which are not asymptotically free, \( \beta_i < 0 \), and \( \Lambda_{AF} \) is that of those \( i \) which are. In particular, the \( \Lambda_{NAF} > \Lambda_{AF} \). As we lower the scale \( \mu \), eventually we get to \( \Lambda_{i_0} \) of the asymptotically free \( U(N_{i_0}) \), which we dualize as above. According to (8.5), \( U(N_{i_0}') \) is not asymptotically free and \( U(N_i) \) is more asymptotically free than it was before if nodes \( i \) and \( i_0 \) are linked. The relation (8.7) ensures that the new scales satisfy (8.10), e.g. if \( U(N_j) \) is NAF we have \( \Lambda_j > \Lambda_{i_0} \) and then we get \( \Lambda_j' > \Lambda_{i_0} \) if \( U(N_j') \) is NAF or \( \Lambda_j' < \Lambda_{i_0} \) if \( U(N_j') \) is AF.

33
A final relation, which will occupy the rest of this section is the transformation (7.4) of the superpotential:

$$ W_i(\phi_i) \rightarrow W_i(\phi_i) - (\vec{e}_i \cdot \vec{e}_{i_0}) W_{i_0}(\phi_i). \quad (8.11) $$

To show that this is indeed the case, we need to show that the dual of our theory (8.1) for the $U(N_{i_0})$ charged fields is $U(N_f - N_{i_0})$ with the superpotential

$$ \widetilde{W} = -s \frac{k+1}{k+1} \text{Tr} \tilde{\phi}^{k+1} + s \frac{k+1}{k+1} \text{Tr} \tilde{m}^{k+1} + \text{Tr} \tilde{\phi} q \overline{q} + \text{Tr} \tilde{m} q \overline{q}. \quad (8.12) $$

Here $\tilde{\phi}$ is the $U(N_f - N_{i_0})$ adjoint of the dual theory and $q$ and $\overline{q}$ are the $N_f$ dual matter fields. The opposite sign of the first term in (8.12), as compared with (8.1), corresponds to the result of (8.11) for $i = i_0$: $W_{i_0} \rightarrow -W_{i_0}$. The transformation in (8.11) for the nodes $i \neq i_0$ corresponds to the second term in (8.12). This is because the mass $m$ in (8.1) are actually the adjoints $\phi_i$ of the nodes linked to $i_0$, so the second term in (8.12) will properly lead to (8.11) for the nodes $i \neq i_0$ with $\vec{e}_i \cdot \vec{e}_{i_0} = -1$.

We will first outline how the predicted superpotential (8.12) indeed arises for the case of $k = 1$; after that we’ll discuss $k > 1$.

8.1. $k = 1$ case

Consider first the case $k = 1$, where $\phi_i$ is massive, with mass $s$, and can be integrated out for scales $\mu < s$. The relevant duality for the low-energy theory is then that of [24]. When $s$ is large, the low energy theory is $\mathcal{N} = 1$ SQCD with $N_f$ flavors and the additional tree-level superpotential

$$ W_{\text{elec}} = -\frac{1}{2s} \text{Tr}(\overline{Q}Q)^2 + \text{Tr} M \overline{Q}, \quad (8.13) $$

obtained by integrating out $\phi$ from (8.1) via its equation of motion. For $s$ large it’s a good description to simply add this extra superpotential to the usual SQCD dynamics.

For $N_f > N_c$, we can dualize the SQCD theory [24] to $U(N_f - N_c)$, with superpotential

$$ W_{\text{mag}} = \frac{1}{\mu} M \overline{q} q - \frac{1}{2s} \text{Tr} M^2 + \text{Tr} M. \quad (8.14) $$

$M$ is massive and can be integrated out by its equation of motion, $M = s(\mu^{-1} \overline{q} q + m)$, leading to

$$ W_{\text{mag}} = \frac{1}{2s} \text{Tr}(m + \frac{1}{\mu} \overline{q} q)^2 = \frac{s}{2\mu^2} \text{Tr}(\overline{q} q)^2 + \frac{s}{\mu} \text{Tr}(m \overline{q} q) + \frac{s}{2} \text{Tr} M^2. \quad (8.15) $$
Taking $\mu = s$, this superpotential is precisely what we would obtain from (8.12) upon integrating out the massive adjoint $\tilde{\phi}$. In particular, corresponding to the Weyl reflection, the sign of the quartic term in (8.15) is opposite to that of (8.13), and we have the additional term $W_i(m)$ in (8.15).

As an aside, we briefly review the vacuum structure of the $U(N_c)$ theory with superpotential (8.1), for $k = 1$, thinking of the linked nodes as a $U(N_f)$ flavor symmetry. The relevant detailed analysis has been presented in [42]. A semi-classical analysis of the vacua, for general quark masses $m$ leads to $\binom{N_f}{r}$ vacua where the gauge group is Higgsed as $U(N_c) \to U(N_c - r)$ for $r = 0 \ldots \min(N_c, N_f)$; each unbroken $SU(N_c - r)$ has no massless flavors and thus has $N_c - r$ susy vacua via gaugino condensation.

Consider the quantum theory in the limit of large $s$, where we simply add (8.13) to the usual $U(N_{i0})$ dynamics. For example, for $N_f < N_c$ the theory is described by the mesons $M$ with superpotential

$$W = -\frac{1}{2s} \text{Tr} M^2 + \text{Tr} m M + (N_c - N_f) \left( \frac{s^{N_c} \Lambda^{2N_c-N_f}}{\det M} \right)^{1/(N_c-N_f)}.$$  

This superpotential has $\frac{1}{2}(2N_c - N_f) \binom{N_f}{r}$ vacua where $\langle M \rangle$ expectation values break $U(N_f) \to U(N_f - r) \times U(r)$, even in the $m \to 0$ limit, for every $r = 0 \ldots N_f$. These give all the vacua for $N_f < N_c$ [42]. For $N_f > N_c$ we can analyze the vacua using the $U(N_f - N_c)$ dual. The result (see [42]) are vacua of two types. One type is visible semi-classically in the dual theory, with $U(N_f - N_c)$ Higgsed to $U(N_f - N_c - r)$ and $U(N_f)$ is unbroken in the $m \to 0$ limit. The other comes from strong coupling dynamics in the dual theory: when rank($M$) = $N_f$, the dual quarks are all massive and a dynamical superpotential is generated in the dual, e.g. via gaugino condensation; as usual, this superpotential is the continuation of (8.16) to $N_f > N_c$. These vacua again have $U(N_f) \to U(N_f - r) \times U(r)$ for $r = 0 \ldots N_f$.

One can also analyze the problem in the limit where the adjoint mass $s \ll \Lambda$, with $\Lambda$ the scale of the theory with $\phi$ included. The theory can then be usefully analyzed in terms of the curve of the $\mathcal{N} = 2$ SQCD theory, breaking to $\mathcal{N} = 1$ by the small adjoint mass $s$. This analysis leads to two sorts of vacua [16,42]. One set, existing for all $N_f$, are vacua with the entire gauge group confined, and the flavor symmetry broken as $U(N_f) \to U(N_f - r) \times U(r)$ for all $r \leq \lfloor N_f/2 \rfloor$ via monopole condensation. The other set exists for $N_f > N_c$ and have unbroken $U(N_f)$; they are visible semi-classically in the dual $U(N_f - N_c)$ theory of [16].

35
Now consider (8.1) with $k > 1$. As discussed in [43] for $k = 2$ and more generally in [44], these theories, without the term $m\overline{Q}Q$ in (8.1), are dual to a $U(N_f - N_c)$ theory with superpotential

$$\overline{W} = -\frac{s}{k + 1} \text{Tr}\overline{\phi}^k + \text{Tr}\overline{\phi}\overline{q}, \quad (8.17)$$

(Comparing with [44], we have normalize $\overline{q}$ and $q$ so that the coefficient of the Yukawa term in (8.17) is the same as in the electric theory.) As discussed in [44], this duality can be obtained from that of [45,46,33] by deforming by the $Q\overline{\phi}Q$ term in (8.1). The dual theory is of the same form as the original theory, and does not contain the gauge singlet mesons found in the original $\mathcal{N} = 1$ dualities of [24,45,46,33]; all of the mesons usually required in the dual are massive for $0 \neq s_i < \infty$ [44].

Following the $\text{Tr}m\overline{Q}Q$ in (8.1) through the duality is a little more involved. As was discussed in [43] for the case $k = 2$, one finds various vacua. Our interest is in showing that one of these vacua has $U(N_f - N_c)$ gauge group, with the terms involving $m$ in the superpotential, as in (8.12).

As in [44], we obtain the duality by flowing from that of [45,46,33], which relates the theory (8.13) to a magnetic $U(kN_f - N_c)$ theory with superpotential

$$\overline{W} = -\frac{s}{k + 1} \text{Tr}Y^k + \frac{s}{\mu^2} \sum_{j=1}^{k} M_j \overline{q}Y^{k-j} + \lambda M_2 + mM_1. \quad (8.18)$$

The $\text{Tr}\phi\overline{Q}Q$ perturbation in (8.18), with coefficient $\lambda$ which we’ll take to equal $1_{N_f}$, leads to a Higgsing of the magnetic theory to $U(N_f - N_c)$ [44]. We now consider the effect of the added $m$ perturbation in (8.18). The $F$-term conditions required for a vacuum of the theory (8.18) are

$$\frac{s_i}{\mu^2} \overline{q}Y^p q = -m\delta_{p,k-1} - \lambda\delta_{p,k-2},$$

$$Y^k = \mu^{-2} \sum_{j=1}^{k-1} (k-j)M_j \overline{q}Y^{k-j-1}, \quad (8.19)$$

$$\sum_{j=1}^{k} M_j \overline{q}Y^{k-j} = 0.$$

The vacuum solution of [44] for $m = 0$ Higgses $U(kN_f - N_c)$ to $U(N_f - N_c)$. This solution can now be modified to account for $m \neq 0$. For simplicity, we just discuss the case $k = 2$. Considering first the first flavor, the vacuum of [44] has $q_1^\alpha = b\delta^{\alpha-1}$ and $\overline{q}_1^\alpha = b\delta^{\alpha-1}$, with
\[ b^2 = -\lambda_1 \mu^2 / s, \] satisfying (8.19) for \( p = 0 \). We can satisfy (8.19) for \( p = 1 \) by taking \( Y_1 = -m_1 / \lambda_1 \). In order to satisfy the other equations in (8.19) we also need \((M_1)_1^1\) and \((M_2)_1^1\) to be non-zero; these non-zero values will not contribute to the low-energy superpotential, since the linearity of (8.18) in the \( M_j \) ensures that the coefficients of the \( M_j \) have zero expectation value.

We now expand (8.18) around this vacuum, where \( U(2N_f - N_c) \) is Higgsed to \( U(2N_f - N_c - 1) \). Though the \( q_1 \) and \( \tilde{q}_1 \) flavor is eaten, we get back a flavor from \( F^\alpha \sim Y_1^\alpha \) and \( \tilde{F}_\alpha \sim Y_1^\alpha \). Expanding out the \(-s/3 \text{Tr} Y^3\) term of the \( U(2N_f - N_c) \) theory gives

\[
-\frac{s}{3} \text{Tr} Y^3 \rightarrow -\frac{s}{3} \left( \left(-\frac{m_1}{\lambda_1}\right)^3 + \text{Tr} \hat{Y}^3 \right) + \lambda_1 \text{Tr} \hat{F} \hat{Y} F + m_1 \hat{F} F, \tag{8.20}
\]

where \( \hat{Y} \) is the part of \( Y \) in the unHiggsed \( U(2N_f - N_c - 1) \) adjoint, and \( F \) has been normalized so that the Yukawa coupling in (8.20) coefficient is \( \lambda_1 \). Continuing this process for all flavors, and taking the \( \lambda = 1_{N_f} \), we eventually get a \( U(N_f - N_c) \) theory with superpotential precisely as in (8.12), just as we wanted to verify.

We can also see the above \( U(N_f - N_c) \) dual in the limit where we treat the coefficient \( s \) of the \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 1 \) superpotential term in (8.1) as being small, via an analysis similar to that of [16]. In the undeformed \( \mathcal{N} = 2 \) theory, at the root of the baryon branch, there is a free-magnetic \( U(N_f - N_c) \times U(1)^{2N_c - N_f} \) theory. Deforming by the term \( W_i = \frac{s}{k+1} \text{Tr} \phi^{k+1} \) leads to various vacua, the one of interest for us being that where the \( U(N_f - N_c) \) remains unbroken and the \( U(1)^{2N_c - N_f} \) is Higgsed entirely by monopoles, which condense due to the \( W_i = \frac{s}{k+1} \text{Tr} \phi^{k+1} \) deformation. Carrying out this analysis along the lines of [16], it can be seen how all the terms in the expected superpotential (8.12) can indeed arise.

9. \textit{A}_2 \textit{ example}

In this section we study the \( A_2 \) quiver theory with \( k = 1 \) as an example of how the dualities enter the description of the theory both in the field theory analysis and in the geometric analysis. We first present the field theory analysis and then discuss how it is realized geometrically. The other A-D-E cases work in a similar fashion.