Einstein’s equivalence principle in classical physics is a rule stating that the effect of gravitation is locally equivalent to the acceleration of an observer. The principle determines the motion of test particles uniquely (modulo very broad general assumptions). We show that the same principle applied to a quantum particle described by a wave function on a Newtonian gravitational background determines its motion with a similar degree of uniqueness.

In this note we address one of the conceptual issues arising from the efforts to reconcile quantum theory with gravitation, the question of the status of the equivalence principle for quantum matter. In classical physics the Einstein equivalence principle is a rule making one half of the universal interdependence of geometry and matter, namely the influence of geometry on matter, more specific. It states that the effect of gravitation is locally indistinguishable from the effects arising from the acceleration of the observer. Put differently, the gravitational effects may be locally “transformed away” by an appropriate choice of the coordinate system. We adopt this version of the equivalence principle in this note as we believe that it hits the heart of the matter, other “equivalence principles” (cf. below) being more accidental or secondary.

Within Einstein’s gravitation theory one shows that the above equivalence principle implies (modulo some natural assumptions on the general nature of the equations of motion) that all test particles placed in this spacetime move along geodesics. This fact is often expressed in one of two ways: that the motion of the particle is mass-independent, or that the inertial mass of the particle is equal to its gravitational mass. These two statements are also referred to as “equivalence principles”. One should make, however, two logical qualifications. First, the two facts are logically different statements, meaning the same thing in this context but not necessarily so in another. And second, in a different context it may even happen that one of them is true while the other is not. This is, as we shall see below, what happens in the quantum case.

We turn now to this case. In the literature various opinions on the status of the equivalence principle in the quantum world are expressed [1], and various, sometimes rather far removed from the original geometrical notion of equivalence, ideas are proposed [2] (but see also the final discussion). We think, however, that the extension of the Einstein equivalence principle in the form stated above to the quantum case experiences no logical difficulty, at least in the setting in which it has often been considered. We feel, therefore, that it may be of interest to see the simplicity of its action in this setting. The setting referred to consists of a structure-less particle described by a wave function on a gravitational background of the nonrelativistic spacetime (we use this term reluctantly: it is deeply rooted in the physicists’ jargon, but highly misleading; see below). This setting has been adopted by several authors addressing the issue of covariance or equivalence [2,3], and it is also assumed in this note.

The reason for choosing the nonrelativistic rather than Einsteinean spacetime is that we want to avoid the complications arising from creation and annihilation of particles and their quantum field-theoretical description, which has to replace (“first-quantized”) quantum mechanics in this case (there existing no consistent relativistic quantum mechanics). The adopted setting is, however, nontrivial enough and, in fact, contrary to the customary name, possesses a geometrical structure (Newton-Cartan) interpretable in physical terms as a relativity theory, but with Galilean rather than Lorentzian local inertial observers [4]. We can now state the main claim of this note. Consider a quantum particle described by a wave function \( \psi \) in a geometrical background with the Newton-Cartan structure. Assume that the probability density of the particle is a scalar field. Then the Einstein equivalence principle determines the motion of the particle. This motion is not mass-independent, but the inertial and gravitational masses are necessarily equal (which is what one observes in experiment, see e.g. Ref. [5]; we shall return to the experimental aspect of the equivalence principle in the concluding discussion). The choice of the mass parameter is the only freedom of the equation. The equation itself, when written in an appropriate coordinate system, is nothing else but the usual Schrödinger equation with the Newtonian potential term. We move now on to the details.

We start by giving a brief account of the Newton-Cartan geometry. We shall not discuss the underlying axioms and the logical structure of this geometry, referring the reader to the existing literature [4],

---

Is Einstein’s equivalence principle valid for a quantum particle?

Andrzej Herdegen

Physics Department, University College Cork, Ireland

and

Institute of Physics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland

Jarosław Wawrzycki

Institute of Physics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland
Moreover, vectors \( \phi \) an arbitrary time-dependent translation. Let us denote \( t, x \) a Galilean system a natural gauge of the field. This is how a gauge freedom in the choice of a metric on the whole spacetime arises. Nevertheless, both the metric properties on the hypersurfaces of constant time as well as the compatible connection are unique (gauge-independent). With respect to the thus defined connection the leaves of constant time are flat. The non-flatness of the geometry reflects only the way in which the leaves fit together to form the four-dimensional spacetime, and is encoded in one single scalar field \( \phi \). This field, however, is again non-uniquely determined by the connection, being subject to a gauge freedom. All this sounds rather more complicated than for a Lorentzian manifold of Einstein’s theory of gravity, but now a great simplification comes. In the Newton-Cartan geometry there exists a class of privileged global coordinate systems, the so-called Galilean coordinates. One of the coordinates is always the time coordinate \( t \) up to a translation by a constant. The other will be denoted by \( x^i \) \((i = 1, 2, 3) \). The space part of the coordinate basis is a Cartesian system with respect to the metric: \( (\partial/\partial x^i) \cdot (\partial/\partial x^j) = \delta_{ij} \). Moreover, vectors \( (\partial/\partial x^i) \) are parallel propagated by the connection, so the covariant derivative of \( (\partial/\partial t) \) gives the only nontrivial characteristic of the connection, and must be expressible in terms of \( \phi \). In fact, with each choice of a Galilean system a natural gauge of the field \( \phi \) is chosen by the formulas: \( \nabla_{\mu} (\partial/\partial t)^{\nu} = \phi^\nu \nabla_{\mu} t \), where in the coordinate basis the vector field \( \phi^\nu \) is given by \( \phi^0 = 0, \phi^i = (\partial \phi^0/\partial x^i) \). This fixes the choice of \( \phi \) up to an addition of an arbitrary function of time. If \((t, x^i)\) is a Galilean system, then \((t', x'^i)\) is also a Galilean system if and only if it is related to \((t, x^i)\) by a transformation of the form:

\[
 t' = t + b, \quad \vec{x}' = R \vec{x} + \vec{a}(t),
\]

where \( R \) is an orthogonal transformation and \( \vec{a}(t) \) is an arbitrary time-dependent translation. Let us denote \((t, \vec{x}) \equiv X, (t', \vec{x}') \equiv X'\) and let us write the transformation as \( X' = rX \). The two fields \( \phi \) and \( \phi' \) correlated with the two systems are then related by the transformation \( \phi'(X') = \phi(X) - \vec{a}(t) \cdot \vec{x}' + \) arbitrary function of time. We choose the function to be zero and write the transformation law in the form

\[
 \phi'(X) = \phi(r^{-1}X) - \vec{a}(t - b) \cdot \vec{x}. \tag{2}
\]

The field \( \phi \) is no longer an exact scalar, as with each of the two systems a different gauge has been fixed.

The Newton-Cartan spacetime is flat if and only if there exists a Galilean system in which \( \phi \) is a function of time only (and may be chosen equal to zero). The same holds then in all those Galilean systems which are related by a transformation from the Galilean group \((\vec{a}(t) = 0)\) to the first one (and, consequently, to each other). These special Galilean systems are called global inertial systems. If the spacetime is curved there are no global inertial systems, but, as is evident from the transformation law \((1)\) and the form of the connection, for any chosen point of spacetime there exist Galilean systems in which connection vanishes in this point. The restrictions of these systems to small neighborhood of this point are related to each other by Galilean transformations and are called local inertial systems. Both global (when they do exist) and local inertial systems have exactly the same physical interpretation in terms of special observers as in Einstein’s theory.

The Newton-Cartan spacetime is the spacetime of the Newtonian world with gravitation. The field equation of the form: “the Ricci tensor of the connection = 0” turns out to be identical in a Galilean system with the Laplace equation for \( \phi \), and the geodesic equation has in this system the form of the Newton’s second law for a particle in the gravitational potential field \( \phi \). In parallel with the Einstein theory of gravity the geodesic law of motion of test particles may be obtained by the application of the Einstein equivalence principle. And here again the two minor equivalence principles are true.

The existence of the global Galilean systems simplifies greatly the investigation of the covariance of equations. To see whether an equation has a geometrical, independent of the choice of coordinates, meaning, it is sufficient to check whether it has the same form in all Galilean systems. If it has, writing its coordinate independent form may pose some technical difficulties, but is possible. In what follows we use the Galilean systems only.

We are now prepared to place a quantum particle in this Newtonian geometry. We assume that it is described by a wave function \( \psi(X) \), such that the correlated “probability density” \( \rho(X) = \psi(X) \bar{\psi}(X) \) is a scalar field: \( \rho'(X') = \rho(X) \). We use this quantum mechanical language, but the argument is purely field theoretic in spirit, and no a priori assumptions on the integration of probability need to be made. The scalar transformation law of \( \rho \) does not determine the transformation law of \( \psi \), as it says nothing about the phase of \( \psi \). Therefore, the problem to be solved is this. Can we ascribe in each Galilean system a phase to the \( \psi \) in such a way that a consistent transformation law of the form

\[
 \psi'(X) = e^{-i\theta(rX)} \psi(r^{-1}X) \tag{3}
\]

would hold and \( \psi \) would satisfy a form invariant equation in all those systems? Wishing to make use of the equivalence principle we first have to answer this ques-
Finally, the condition for $g = 0$. This case is trivial. We can assume then that the condition for translations and rotations one finds that also $\vec{f} \cdot \vec{\partial}_n$ is an invariant vector field, thus Re $\vec{\partial} \chi$ is a vector field with respect to rotations. But $\phi$, the only characteristic of the geometry, is a scalar field with respect to rotations, so there is no local way in which a $\chi$ giving rise to a nonzero vector field $\vec{\partial} \times \chi$ can be formed with it. Hence $\chi(X) = \vec{\partial} \chi$. We observe now, that this longitudinal field may be absorbed into the phase of $\psi$ (with the appropriate modification of $\Lambda$), so one can assume $\chi = 0$. The transformation condition then simplifies to $\theta(r, X) = -(m/\hbar)\vec{\partial}(t-b) \cdot \vec{x} + \vec{\theta}(r, t)$. The covariance condition for the $\Lambda$ term now takes the form

$$\Lambda(X) = \Lambda'(X') - m\vec{\alpha}(t) \cdot \vec{x} - (m/2)\vec{\alpha}^2(t) + \hbar \vec{\partial}_t \vec{\theta}(r, t').$$

At this point let us look back once more to the flat space case and assume that $X$ is an inertial system. Then $\Lambda(X) = 0$ and we find that the additional terms in the operator acting on $\psi'$ arising from the non-inertiality of the system $X'$ are $m\vec{\alpha}(t'-b) \cdot \vec{x} + (m/2)\vec{\alpha}^2(t'-b) - \hbar \vec{\partial}_t \vec{\theta}(r, t')$. We learn two things. First, the terms are real, so by the equivalence principle $\Lambda(X)$ is real in general. Second, a change of coordinates produces definite terms up to linear order in $\vec{x}$. The equivalence principle then implies that in the general case it should be possible by a change of coordinates to eliminate in the neighborhood of a given point $X_0$ terms independent of, and linear in $\vec{x} - \vec{x}_0$. Put differently, it should be possible to transform away the value and the first derivative $\vec{\partial}$ of $\Lambda(X)$ at this point. Let

$$\left[i\hbar \vec{\partial}_t + \frac{\hbar^2}{2m} \vec{\partial}^2\right]\psi = 0, \quad \theta(r, X) = -\frac{m}{\hbar} \vec{x} \cdot \vec{x} + \frac{m}{2\hbar} \vec{\partial}^2 t,$$

which, of course, is the standard free particle theory.

Einstein’s equivalence principle implies now that if in the flat space we transform the Schrödinger equation to all arbitrary Galilean systems (noninertial), then we can identify all local modifications to the equation which can appear in an arbitrary Galilean system in curved spacetime. We assume the transformation law (3) and find that the transformed equation differs from the Schrödinger equation at most by additional terms on the l.h.s. of the form $i\vec{\chi} \cdot \vec{\partial} \psi + \Lambda \psi$, where $\chi$ is real. In curved spacetime the Einstein equivalence principle gives then the equation

$$\left[i\hbar \vec{\partial}_t + \frac{\hbar^2}{2m} \vec{\partial}^2 + i\chi(X) \cdot \vec{\partial} + \Lambda(X)\right] \psi(X) = 0,$$

where $\chi(X)$ and $\Lambda(X)$ are now fields characterising geometry. We assume that these fields are determined locally by the geometry. Assuming the transformation law of the form (3) and demanding the covariance of the equation we find the condition

$$\chi(X) = R^{-1} \left[\chi'(X') - \hbar\vec{\alpha}(t) - \frac{\hbar^2}{m} \vec{\partial} \theta(r, X')\right].$$

Applying $\vec{\partial} \times$ to this equation we find that $\vec{\partial} \times \chi$ is a vector field, in particular it is a vector field with respect to rotations. But $\phi$, the only characteristic of the geometry, is a scalar field with respect to rotations, so there is no local way in which a $\chi$ giving rise to a nonzero vector field $\vec{\partial} \times \chi$ can be formed with it. Hence $\chi(X) = \vec{\partial} \chi$. We observe now, that this longitudinal field may be absorbed into the phase of $\psi$ (with the appropriate modification of $\Lambda$), so one can assume $\chi = 0$. The transformation condition then simplifies to $\theta(r, X) = -(m/\hbar)\vec{\partial}(t-b) \cdot \vec{x} + \vec{\theta}(r, t)$.
us introduce \( \Lambda(X) \) by \( \Lambda(X) = -m \phi(X) + \hat{\Lambda}(X) \). Using (2) we find that the covariance condition now takes the form

\[
\hat{\Lambda}(X) = \hat{\Lambda}'(X') - \frac{m}{2} \bar{a}^2(t) + \hbar \partial_v \hat{\theta}(r, t') ,
\]

which implies \( \partial \hat{\Lambda}'(X') = R \partial \hat{\Lambda}(X) \). It is now clear that if \( \partial \hat{\Lambda}(X) \neq 0 \) at some point then it cannot be transformed away. Therefore \( \hat{\Lambda}(X) = \hat{\Lambda}(t) \), and may be removed by a change of phase of \( \psi \). Thus the unique solution for \( \Lambda \) is \( \Lambda = -m \phi \). We now see the geometrical meaning of the condition that the first derivative \( \partial \hat{\Lambda} \) may be removed at a point by a change of coordinates: this derivative is equivalent to the connection, so the meaning is exactly the same as in the classical case. The covariance condition now simplifies to \( -(m/2) \ddot{a}(t) + \hbar \partial_v \dot{\theta}(r, t') = 0 \). In this way we finally obtain the equation

\[
\left[ i \hbar \partial_t + \frac{\hbar^2}{2m} \bar{\partial}^2 - m \phi(t, \vec{x}) \right] \psi(t, \vec{x}) = 0 , \quad (4)
\]

and the transformation exponents

\[
\theta(r, X) = -\frac{m}{\hbar} \dot{a}(t-b) \cdot \vec{x} + \frac{m}{2 \hbar} \int_0^t \dot{a}^2(\tau-b) \, d\tau . \quad (5)
\]

Until now we have considered the relation between two coordinate systems only. Is the resulting structure, the equation (4) and the transformation laws (2) and (5), consistent with the composition of transformations? That is, do we get the same result if we choose to break the transformation \( X \mapsto X' \) into two steps with an intermediate system on the way: \( X \mapsto X'' \mapsto X' \)? The answer is that the two final gravitational potentials differ in general by a time-dependent (\( \vec{x} \)-independent) additive term, while the two final wave functions differ by a time-dependent phase factor. This, however, poses no difficulty. The difference in the potentials is consistent with the freedom in their definition, while a time dependent phase factor in the wave function does not change with the freedom in their definition, while a time dependent phase factor. This, however, poses no difficulty. Thus the unique solution for \( \Lambda \) is \( \Lambda = -m \phi \).

We have thus shown that Eq. (4) is uniquely determined by Einstein’s equivalence principle. In particular, we have shown that the principle implies equality of inertial and gravitational masses. The equation, of course, is standard, and has been discussed many times, but its geometrical uniqueness is, we believe, a new result. For example, Kuchař [3] has derived the equation by canonical quantization of the geodesic motion. (Where in the process is the mass independence lost? It is, of course, when after going over to the Hamiltonian formalism, in which mass appears, the momentum looses any memory of the mass upon replacement by \( -i\hbar \partial \).) However, canonical quantization is a heuristic procedure (it is rather classical mechanics, which is believed, in principle, to be derivable from the quantum mechanics) and it is unable to decide the uniqueness question or to clarify the intrinsic structure at play. On the other hand Lämmerzahl [2] gives arguments to the effect that Eq. (4) is favored by a principle which he introduces and calls quantum equivalence principle. This principle formulates a condition for a possibility of the extraction of mass-independent characteristics from experimental results. However, there is no obvious connection of this principle with Einstein’s geometrical idea and its compelling persuasiveness (in fact, Lämmerzahl avoids the covariance question completely). We do believe Lämmerzahl’s results are important and interesting, but see their role on the experimental side rather than as a theoretical paradigm. What we mean, more precisely, is this. Einstein’s principle is a local principle. For a classical test particle, which is a local object, its content translates itself rather directly into experimental predictions. The quantum mechanical wave function, on the other hand, is a nonlocal object, and there is no simple analogous translation - in general gravity cannot be eliminated on any hypersurface of constant time. Lämmerzahl’s papers show how to extract experimental consequences of Einstein’s equivalence principle from experimental data. Having said this, however, we also want to express disagreement with the opinion that nonlocality of the wave function precludes operational meaning of Einstein’s principle. It may be not obvious how to reveal such meaning, but we can see no fundamental obstacle on this way. Measurements are done locally, which is the operational foundation of the Einstein’s principle.