Non-Commutative Instantons and the Seiberg–Witten Map

Per Kraus\textsuperscript{1} and Masaki Shigemori\textsuperscript{2}

Department of Physics and Astronomy
University of California, Los Angeles
Box 951547, Los Angeles, CA 90095-1547, USA

Abstract

We present several results concerning non-commutative instantons and the Seiberg–Witten map. Using a simple ansatz we find a large new class of instanton solutions in arbitrary even dimensional non-commutative Yang–Mills theory. These include the two dimensional “shift operator” solutions and the four dimensional Nekrasov–Schwarz instantons as special cases. We also study how the Seiberg–Witten map acts on these instanton solutions. The infinitesimal Seiberg–Witten map is shown to take a very simple form in operator language, and this result is used to give a commutative description of non-commutative instantons. The instanton is found to be singular in commutative variables.

1 Introduction

One of the most fruitful applications of non-commutative geometry has been to the construction of soliton solutions in non-commutative field theories [1]-[10]. These solutions have an elegant operator description, and can often be interpreted as D-branes in string theory. It is striking that introducing non-commutativity—which from one point of view corresponds to adding a complicated set of higher derivative interactions—can in fact greatly simplify the construction of soliton solutions.

Our focus here will be on instanton solutions in non-commutative pure Yang–Mills theory. Such solutions were originally found by Nekrasov and Schwarz [1] and by Furuuchi [2] via a non-commutative version of the ADHM construction, and have since been studied by a number of authors. One interesting fact is the existence of instantons in non-commutative $U(1)$ gauge theory, since no such nonsingular and finite action solutions exist in ordinary $U(1)$ gauge theory. According to the work of Seiberg and Witten [16], non-commutative gauge theories are related to ordinary gauge theories by a change of variables—the Seiberg–Witten (SW) map. Starting from non-commutative Yang–Mills theory, one obtains in this way an ordinary gauge theory with an infinite set of higher derivative interactions. It is then natural to ask how the SW map acts on the known soliton solutions of non-commutative field theories.
For the simplest “shift operator” solitons this question was answered in [11] using the exact form of the SW map obtained in [12, 13, 14, 15]. One typically finds a singular delta function configuration in the commutative variables. The corresponding calculation for the Nekrasov–Schwarz type instanton solutions turns out to be far more challenging due to the need to compute complicated symmetrized trace expressions. We will solve this problem by a more indirect approach.

In the course of studying this question we have obtained a number of new results concerning non-commutative instantons and the SW map. Non-commutative solitons are most simply described in the operator formalism, and so it is useful to rewrite the infinitesimal SW map in operator form. The result turns out to be quite simple. The resulting SW map states that in mapping the non-commutativity parameter from $\theta$ to $\theta + \delta \theta$ the operator configuration $X^i$ is mapped to $X^i + \delta X^i$, where

$$\delta X^i = i2\delta \theta^{kl} \theta_{km} \theta_{ln} [X^i, X^m] X^n. \quad (1)$$

To put the SW map in this simple form we have used the freedom to perform unitary transformations, as we discuss in section 2.

We use the result (1) to follow a non-commutative instanton solution from finite $\theta$ to $\theta = 0$. We will show that the solution is attracted to a fixed point of this flow, and give the form of the fixed point configuration. One can freely pass from the operator formulation to a position space formulation using the Weyl correspondence, and as with the shift operator solitons we will find that the instanton solution maps to a singular position space configuration. The explicit result for the field strength is, in complex coordinates,

$$F_{\alpha\bar{\beta}} = -i\delta^{\bar{\alpha}\bar{\beta}} \frac{\bar{z}z - z\bar{z}}{(\bar{z}z)^2} (z^\alpha \neq 0). \quad (2)$$

An additional singular contribution at the origin gives rise to the topological charge in the commutative limit. Thus away from the origin (2) gives the commutative description of the Nekrasov–Schwarz instanton.

We have also found a large new class of instanton solutions that generalize those of [1, 2] in several directions. These are found in operator form by starting from the ansatz

$$X^\alpha = U f(N) a^\alpha U^\dagger \quad (3)$$

where $U$ is a shift operator. The equations of motion reduce to a recursion relation for $f(N)$, which we solve. This procedure yields instanton solutions to non-commutative Yang–Mills theory in any even dimension. In four dimensions it also generalizes those of [1, 2] to solutions that are neither self-dual nor anti-self-dual. We give explicit examples of these solutions in dimensions 2 and 4 and evaluate their actions and topological charges in arbitrary dimensions.

The remainder of this paper is organized as follows. In section 2 we first review basic properties of non-commutative gauge theory and then show that

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the SW map can be written in a simple covariant form in operator language. In section 3, we discuss solitonic solutions in non-commutative Yang–Mills theory and find new solutions. In section 4, we will apply the operator SW map to non-commutative solitons and discuss their commutative description.

2 Non-Commutative gauge field theory

2.1 Preliminaries and conventions

Non-commutative Yang–Mills (NCYM) theory in $D = 2d$ dimensional non-commutative flat space is described by the action

$$S = -\frac{1}{4g^2} \int d^Dx \text{tr} F^{ij} * F_{ij}, \quad (i, j = 1, \ldots, 2d),$$

(4)

$$F_{ij} = \partial_i A_j - \partial_j A_i - i(A_i * A_j - A_j * A_i),$$

(5)

where the $*$-product is defined by

$$f * g(x) = e^{\frac{1}{2} \theta^{ij} \partial_i \partial_j f(x)g(x') |_{x' = x}}.$$ 

Under this rule, coordinates become non-commutative:

$$[x^i, x^j]_* = x^i * x^j - x^j * x^i = i\theta^{ij}.$$ 

When necessary, we write the $\theta$ dependence of the $*$-product explicitly as $*_{\theta}$. We consider only the case where the metric is Euclidean: $g_{ij} = \delta_{ij}$.

The action (4) is invariant under the non-commutative gauge transformation:

$$\delta_{\lambda} A_i = \partial_i \lambda + i[\lambda, A_i]_*,$$

(6)

$$\delta_{\lambda} F_{ij} = i[\lambda, F_{ij}]_*,$$

(7)

where $\lambda(x)$ is an arbitrary infinitesimal parameter. If we introduce

$$X^i \equiv x^i + \theta^{ij} A_j$$

which transforms covariantly under gauge transformation:

$$\delta_{\lambda} X^i = i[\lambda, X^i]_*,$$

we can rewrite (4) and (5) as

$$S = -\frac{1}{4g^2} \int d^Dx \text{tr} \quad \frac{1}{2} \theta^{ij} \theta^{kl} (i\theta_{im}[X^m, X^n], \theta_{nj} + \theta_{ij}) (i\theta_{kp}[X^p, X^q], \theta_{ql} + \theta_{kl}),$$

(8)

$$F_{ij} = i\theta_{ik}[X^k, X^j]_* \theta_{ij} + \theta_{ij},$$

(9)

where $\theta^{ij} \theta_{jk} = \delta^i_k$. The equation of motion derived from (8) is

$$g_{ij}[X^i, [X^j, X^k]_*]_* = 0.$$ 

(10)
It is often more convenient to work in operator language on Hilbert space rather than in c-number function language described above. Define the Weyl transformation \( \hat{f} \) of a c-number function \( f(x) \) by

\[
\hat{f} \equiv \mathcal{W}_\theta[f(x)] = \int \frac{d^Dk}{(2\pi)^D} f(x)e^{-ik\cdot\hat{x}(\theta)},
\]

where \( \hat{x}^i(\theta) \) are operators satisfying the operator commutation relation

\[
[\hat{x}^i(\theta), \hat{x}^j(\theta)] \equiv \hat{x}^i(\theta)\hat{x}^j(\theta) - \hat{x}^j(\theta)\hat{x}^i(\theta) = i\theta^{ij}.
\]

We will often drop the argument \( \theta \) when it is clear. By the isomorphism

\[
\mathcal{W}_\theta[(f *_\theta g)(x)] = \mathcal{W}_\theta[f(x)] \mathcal{W}_\theta[g(x)],
\]

we can work in c-number function language or equivalently in operator language. The inverse of the Weyl transformation is

\[
f(x) = \mathcal{W}_\theta^{-1}[\hat{f}] \equiv \int \frac{d^Dk}{(2\pi)^D} \text{Pf}(2\pi\theta) \text{Tr}[\hat{f}e^{-i\hat{x}(\theta)}]e^{ik\cdot x}.
\]

### 2.2 The Seiberg–Witten map in operator language

It is known [16] that a non-commutative gauge theory with non-commutativity parameter \( \theta \) can be equivalently described by another non-commutative gauge theory with different non-commutativity parameter \( \theta + \delta\theta \). The relation among the fields in the two descriptions is given by the so-called Seiberg–Witten (SW) map

\[
A_i(\theta + \delta\theta) = A_i(\theta) + \delta A_i(\theta) = A_i(\theta) - \frac{1}{4} \delta\theta^{kl} \{A_k(\theta), \partial_l A_i(\theta) + F_{ik}(\theta)\}_* + \mathcal{O}(\delta\theta^2),
\]

\[
\lambda(\theta + \delta\theta) = \lambda(\theta) + \delta\lambda(\theta) = \frac{1}{4} \delta\theta^{kl} \{\partial_k \lambda(\theta), A_l(\theta)\}_* + \mathcal{O}(\delta\theta^2),
\]

where \( \{f, g\}_* \equiv f* g + g* f \). The products on the right hand side are understood as *\( \theta \) products, while the fields on the left hand side are to be used with *\( \theta + \delta\theta \). We often display the non-commutativity parameter explicitly as, e.g., \( A_i(\theta) \), in order to indicate which non-commutative gauge theory is being referred to.

This map was originally [16] (see also [20]) derived by the condition that gauge transformations in the two descriptions are equivalent in the sense that

\[
A(A(\theta); \theta + \delta\theta) + \delta\lambda(\theta + \delta\theta) A(A(\theta); \theta + \delta\theta) = A(A(\theta) + \delta\lambda(\theta) A(\theta); \theta + \delta\theta).
\]

The gauge parameter \( \lambda(\theta + \delta\theta) \) is allowed to depend not only on \( \lambda(\theta) \) but also on the gauge field \( A(\theta) \). It is known that the condition (12) does not determine the map uniquely — there are infinitely many different solutions. However, different maps are related by gauge transformations and field redefinitions [17].

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We denote operators by hats in this section.
Now let us consider translating the SW map above into operator language. The SW map of the covariant position $X^i(\theta)$ in c-number function language is
\[
X^i(\theta) = X^i(\theta) + \delta X^i(\theta) + O(\delta^2)
\]
\[
= X^i(\theta) + \frac{i}{4} \delta \theta^k \theta_{jm} \theta_{kn} \{ X^m(\theta) - x^m, [X^n(\theta) - x^n, X^i(\theta)] \}_\ast + O(\delta^2)
\]
\[
= X^i(\theta) + \delta X^i(\theta).
\]
Equation (13)

The corresponding operator is
\[
\hat{X}^i(\theta + \delta \theta) = W_{\theta + \delta \theta}[X^i(\theta + \delta \theta)] = W_{\theta + \delta \theta}[X^i(\theta) + \delta X^i(\theta)]
\]
\[
= W_{\theta + \delta \theta}[X^i(\theta)] + \delta \hat{X}^i(\theta) + O(\delta^2),
\]
Equation (14)

where $\delta \hat{X}^i(\theta)$ is obtained by replacing $X$ functions in $\delta X^i(\theta)$ with $\hat{X}$ operators. The operators $\hat{x}^i(\theta + \delta \theta)$ operators satisfying
\[
[\hat{x}^i(\theta + \delta \theta), \hat{x}^j(\theta + \delta \theta)] = i(\theta + \delta \theta)^{ij}
\]
which are necessary for defining $W_{\theta + \delta \theta}$ are most simply constructed, in terms of $\hat{x}^i(\theta)$ operators, as
\[
\hat{x}^i(\theta + \delta \theta) = \hat{x}^i(\theta) + \frac{1}{2} \delta \theta^{ij} \theta_{jk} \hat{x}^k(\theta) \equiv \hat{x}^i(\theta) + \delta \hat{x}^i(\theta)
\]
With this choice,
\[
W_{\theta + \delta \theta}[X^i(\theta)] = \int d^D x \frac{d^D k}{(2\pi)^D} X^i(\theta) e^{-ik \cdot x} e^{i k \cdot \hat{x}(\theta + \delta \theta)}
\]
\[
= \int d^D x \frac{d^D k}{(2\pi)^D} X^i(\theta) e^{-ik \cdot x} e^{i k \cdot \hat{x}(\theta)} [\hat{\partial}_j, e^{i k \cdot \hat{x}(\theta)}]
\]
\[
= W_{\theta}[X^i(\theta)] + \delta \hat{x}^i(\theta) [\hat{\partial}_j, W_{\theta}[X^i(\theta)]
\]
\[
= \hat{X}^i(\theta) + \frac{1}{2} [\delta \hat{x}^i(\theta), [\hat{\partial}_j, X^i(\theta)]]
\]
where $\hat{\partial}_i \equiv -i \theta^{ij} \hat{x}_j$. This cancels some terms in (14) to give
\[
\delta \hat{X}^i(\theta) = \frac{i}{2} \delta \theta^{kl} \theta_{km} \theta_{ln} [\hat{X}^i(\theta), \hat{X}^m(\theta)] \hat{X}^n(\theta) + i \hat{g}, \hat{X}^i(\theta),
\]
Equation (15)

where
\[
\hat{g} \equiv \frac{1}{4} \delta \theta^{kl} \theta_{km} \theta_{ln} (-\{\hat{x}^m(\theta), \hat{X}^n(\theta)\} + X^m(\theta) X^n(\theta)).
\]
Equation (16)

As we show in appendix B, the solution to Eq. (15) and the solution to the same equation without the last term $i \hat{g}, \hat{X}^i(\theta)$ are related by unitary transformation. Namely, if we denote the two solutions as $\hat{X}_g(\theta)$ and $\hat{X}_0(\theta)$, respectively, we can always find a unitary operator $\hat{u}(\theta)$ satisfying $\hat{X}_g(\theta) = \hat{u}(\theta) \hat{X}_0(\theta) \hat{u}(\theta)^\dagger$. 

Since unitary transformation is always a symmetry of non-commutative gauge theory, we can eliminate the last term in (15) by performing suitable unitary transformation at each $\theta$. Note that the unitary transformation $\hat{u}(\theta)$ does not affect the crucial condition (12), from which the SW map (13) was derived. In general, the unitary transformation $\hat{u}(\theta)$ does not correspond to a local gauge transformation, since the operator $\hat{g}$ is not in general compact.\footnote{An operator $\hat{O}$ on a Hilbert space $\mathcal{H}$ is called compact if for any bounded sequence $\{|\psi_i\rangle\}$ ($|\psi_i\rangle \in \mathcal{H}$), the sequence $\{|\hat{O}|\psi_i\rangle\}$ contains a convergent subsequence. For example, $P_L = |0\rangle\langle 0| + |1\rangle\langle 1| + \cdots + |L-1\rangle\langle L-1|$ is compact while $\hat{S} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$ is noncompact since the image of the sequence $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$ contains no convergent subsequence.}

With the above unitary transformation understood, the SW map becomes\footnote{Applying additional unitary transformations, (17) can be also written as}

$$\delta \hat{X}^i = \frac{i}{2} \delta \theta^{kl} \theta_{km} \theta_{ln} [\hat{X}^i, \hat{X}^m] \hat{X}^n, \quad \delta \hat{\lambda} = 0. \quad (17)$$

Note that only covariant quantities appear on the right hand side, which implies that the gauge transformation operators at $\theta$ and $\theta + \delta \theta$ are the same.

In fact, we could have derived the operator SW map above directly from the condition (12) translated into operator language:

1. Gauge transformation of $\hat{X}(\theta)$ should lead to a gauge transformation of $\hat{X}(\theta + \delta \theta)$. This is obviously satisfied by taking the map to depend only on the covariant quantity $\hat{X}(\theta)$ (and consequently $\hat{\lambda}(\theta + \delta \theta) = \hat{\lambda}(\theta)$).

2. If we insert $\hat{X}^i(\theta) = \hat{x}^i(\theta)$ then the map should yield $\hat{x}^i(\theta + \delta \theta)$ satisfying $[\hat{x}^i(\theta + \delta \theta), \hat{x}^j(\theta + \delta \theta)] = i(\theta + \delta \theta)^{ij}$. This is equivalent to the statement that $\hat{A} = 0$ should be preserved under the map.

Eq. (17) gives explicitly one possible solutions to the above conditions. One can easily write down other solutions, but they all differ just by local or global gauge transformations and field redefinitions.\cite{17}

The simplest example of the map (17) is

$$\hat{X}^i(\theta) = a^i_j(\theta) \hat{x}^j(\theta),$$

from which one obtains

$$2\delta a = \theta \delta \theta^{-1} - \theta a^T \delta \theta^{-1} a.$$  

For $D = 2$ ($d = 1$), this can be solved explicitly to give

$$a^i_j(\theta) = \left[ |a_0| - (|a_0| - 1) \frac{\theta}{\theta_0} \right]^{-\frac{i}{2}} a^i_j(\theta_0),$$

These versions may be more useful in situations other than studied in this paper.

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4 An operator $\hat{O}$ on a Hilbert space $\mathcal{H}$ is called compact if for any bounded sequence $\{|\psi_i\rangle\}$ ($|\psi_i\rangle \in \mathcal{H}$), the sequence $\{|\hat{O}|\psi_i\rangle\}$ contains a convergent subsequence. For example, $P_L = |0\rangle\langle 0| + |1\rangle\langle 1| + \cdots + |L-1\rangle\langle L-1|$ is compact while $\hat{S} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$ is noncompact since the image of the sequence $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$ contains no convergent subsequence.

5 Applying additional unitary transformations, (17) can be also written as

$$\delta \hat{X}^i = \frac{i}{2} \delta \theta^{kl} \theta_{km} \theta_{ln} [\hat{X}^i, \hat{X}^m] \hat{X}^n, \quad \delta \hat{\lambda} = 0. \quad (17)$$

or

$$\delta \hat{X}^i = \frac{i}{4} \delta \theta^{kl} \theta_{km} \theta_{ln} \{\hat{X}^m, [\hat{X}^n, \hat{X}^i]\}.$$
where \( \theta \equiv \theta^{12} \) and \( |a_0| \equiv \det[a^i_j(\theta_0)] \). The corresponding field strength is

\[
F_{12}(\theta) = \frac{|a_0| - 1}{|a_0| \theta_0 - (|a_0| - 1)\theta} = \frac{F_{12}(\theta_0)}{1 - (\theta - \theta_0)F_{12}(\theta_0)},
\]

which is singular at \( \theta \) if \( F_{12}(\theta_0) = \frac{1}{\theta - \theta_0} \). This is consistent with the result of [16] that a commutative description \((\theta = 0)\) is impossible if \( F_{ij}(\theta_0) = (\theta_0^{-1})_{ij} \).

### 3 Instantons in pure NCYM

Pure NCYM theory has solitonic solutions [1, 7, 9, 10], some of which have counterparts in commutative theory and some of which do not. In this section, we first review shift operator solitons briefly, and present a new family of solitonic solutions.

#### 3.1 Shift operator solitons

Shift operator solitons [9, 10] are obtained from an arbitrary field \( X^i_0 \) satisfying the equations of motion by applying an “almost gauge transformation”\(^6\)

\[
X^i = U^\dagger X^i_0 U, \quad U U^\dagger = 1, \quad U^\dagger U = 1 - P, \tag{18}
\]

where \( P \) is a projection operator of a finite rank\(^7\). The \( X^i \) automatically satisfy the equations of motion because of the property \( U U^\dagger = 1 \). Note that the gauge group is unspecified; the \( X^i_0, U, \) and \( X^i \) operators do not have to be \( U(1) \), i.e., they can be matrices whose entries are operators. The simplest example of \( X^i_0 \) is the vacuum, for which

\[
X^i = U^\dagger x^i U \tag{19}
\]

and the field strength is

\[
F_{ij} = \theta_{ij} P. \tag{20}
\]

Essentially, \( U \) is a shift operator which maps one to one the subspace \((1 - P)\mathcal{H}\) to the whole space \( \mathcal{H} \), while annihilating the subspace \( P\mathcal{H} \). For example, in the \( D = 2 \) \((d = 1)\) case,

\[
U = \sum_{n^1=0}^{\infty} |n^1\rangle\langle n^1 + l| \tag{21}
\]

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\(^6\)In this section, hats on operators are omitted since we will work only in operator language.

\(^7\)Actually there is more freedom to add finite dimensional matrices to (18) corresponding to the position of the solitons. However we do not consider this generalization in this paper for simplicity. See [10].
satisfies

\[ UU^\dagger = 1, \quad U^\dagger U = 1 - \sum_{n^1=0}^{l-1} |n^1\rangle\langle n^1|. \]

In higher dimensional cases, the non-commutative ABS construction \[19, 9\] can be used to construct a \( U \) operator if the gauge group contains a \( SO(2d) \) subgroup.

### 3.2 NS-type instantons

Let us consider the case where \( \theta^{ij} \) takes the form

\[
\theta^{ij} = \begin{pmatrix}
0 & \theta & 0 & \cdots & 0 \\
-\theta & 0 & \theta & \cdots & 0 \\
0 & -\theta & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad \theta > 0, \tag{22}
\]

when skew-diagonalized. Take complex coordinates

\[
z^a = (x^{2a-1} + ix^{2a})/\sqrt{2}, \quad \bar{z}^{\bar{a}} = (x^{2a-1} - ix^{2a})/\sqrt{2}, \quad a = 1, 2, \ldots, d, \tag{23}
\]

so that

\[
[z^\alpha, \bar{z}^{\bar{\beta}}] = \theta \delta^{\alpha\bar{\beta}}, \quad \alpha = 1, \ldots, d; \quad \bar{\beta} = \bar{1}, \ldots, \bar{d}, \tag{24}
\]

where \( \delta^{1\bar{1}} = 1, \quad \delta^{1\bar{2}} = 0, \) etc. The equation of motion (10) can be written in complex coordinates as

\[
[X^\alpha, [X^{\bar{\alpha}}, X^{\beta}]] + [X^{\bar{\alpha}}, [X^\alpha, X^{\beta}]] = 0, \tag{25}
\]

where summation over identical barred and unbarred Greek letters is implied. The algebra (24) can be realized on the Hilbert space \( \mathcal{H} = \{|n^1, \ldots, n^d\}; \quad n^1, \ldots, n^d = 0, 1, 2, \ldots, \} \) by

\[
z^\alpha = \sqrt{\theta} a^\alpha, \quad \bar{z}^{\bar{\alpha}} = \sqrt{\theta} \bar{a}^{\bar{\alpha}},
\]

\[
[a^\alpha, a^{\bar{\beta}}] = \delta^{\alpha\bar{\beta}}, \quad [a^\alpha, \bar{a}^{\bar{\beta}}] = [\bar{a}^{\bar{\alpha}}, a^{\beta}] = 0.
\]

For simplicity, we take \( \theta = 1 \) henceforth in this section. Explicit \( \theta \) dependence can be recovered on dimensional grounds.

Now let us look for the solution to the equation of motion (25), taking an ansatz

\[
X^\alpha = U_i f(N) a^\alpha U_i^\dagger \equiv U_i X_0^\alpha U_i^\dagger, \quad X^{\bar{\alpha}} = (X^{\bar{\alpha}})^\dagger = U_i a^{\bar{\alpha}} f(N) U_i^\dagger \equiv U_i X_0^{\bar{\alpha}} U_i^\dagger, \tag{26}
\]

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where

\[ U_l U_l^\dagger = 1, \quad U_l^\dagger U_l = 1 - P_l = \theta(N \geq l), \quad l = 1, 2, 3, \ldots \]  \quad (27)

\[ P_l \equiv \sum_{|n| \leq l-1} |\{n\}\rangle \langle \{n\}|, \quad N = a^\alpha a^\alpha, \quad f(N)^\dagger = f(N). \]  \quad (28)

Here \{n\} is a shorthand notation for \((n^1, \ldots, n^d)\), and \(|n| \equiv n^1 + \cdots + n^d\). The function \(\theta(P)\) is 1 if the proposition \(P\) is true and 0 if \(P\) is false. For the action (8) to be finite, we require that

\[ f(N) \rightarrow 1 \quad (N \rightarrow \infty). \]  \quad (29)

From the shifting property of \(U_l\), we can set without loss of generality

\[ f(0) = f(1) = \cdots = f(l-1) = 0, \quad \text{or} \quad f(N)P_l = P_l f(N) = 0, \]  \quad (30)

because these are projected out and do not enter \(X^\alpha\).

For this ansatz, the left hand side of the equation of motion (25) is

\[
[X^\alpha, [X^\alpha, X^\beta]] + [X^\alpha, [X^\alpha, X^\beta]] \\
= U_l \left[ f(N)^3 \theta(N + 1 \geq l) \theta(N \geq l) a^\alpha a^\beta a^\alpha \\
- f(N) f(N + 1)^2 \theta(N + 1 \geq l) \theta(N + 2 \geq l) a^\alpha a^\beta a^\alpha \\
- f(N) f(N - 1)^2 \theta(N - 1 \geq l) \theta(N \geq l) a^\alpha a^\beta a^\alpha \\
+ f(N)^3 \theta(N + 1 \geq l) \theta(N \geq l) a^\beta a^\alpha a^\alpha \right] U_l^\dagger \\
= U_l f(N) \left[ (2N + d + 1) f(N)^2 \\
- (N + d + 1) f(N + 1)^2 - N f(N - 1)^2 \right] a^\beta U_l^\dagger. \]  \quad (31)

Here we used relations such as

\[ a^\alpha f(N) = f(N + 1) a^\alpha, \quad a^\alpha \theta(N \geq l) = \theta(N + 1 \geq l) a^\alpha \]

as well as (30). Because there are only states with \(N \geq l\) between \(U_l\) and \(U_l^\dagger\), we observe that the equation of motion is satisfied if

\[ (2N + d + 1) f(N)^2 - (N + d + 1) f(N + 1)^2 - N f(N - 1)^2 = 0, \]

\[ N = l, l + 1, l + 2, \ldots \]  \quad (32)

Solving this recursion equation under the initial condition

\[ f(l) = f(l + 1) = \cdots = f(L - 1) = 0, \quad f(L) \neq 0, \quad l \leq L \]

along with (29), we obtain

\[ f(N) = \sqrt{1 - \frac{L(L + 1)(L + 2) \ldots (L + d - 1)}{(N + 1)(N + 2) \ldots (N + d)}} \quad \theta(N \geq L) = f_L(N). \]  \quad (33)
Even if we take into account the $f(N)$ factor in (31), we end up with the same result (33). In addition, we could start with more general $U$ and $U^\dagger$ operators which satisfy

$$U^\dagger U = \theta(N \notin \mathcal{P}),$$

instead of (27), where $\mathcal{P}$ is a finite subset of $\mathbb{N} = \{0, 1, 2, \ldots\}$. However, this again leads to (33), with $L$ larger than any elements of $\mathcal{P}$. We will refer to the new class of solutions (26) as Nekrasov–Schwarz (NS)-type instantons, because as we will see later these include the Nekrasov–Schwarz instanton [1, 2] as a special case.

The field strength of the NS-type instanton is

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0, \quad F_{\alpha\bar{\beta}} = U_l(F_0)_{\alpha\bar{\beta}}U_l^\dagger,$$

(34)

where

$$(F_0)_{\alpha\bar{\beta}} = -i \left[ \delta^{\alpha\bar{\beta}} + \frac{L(L+1)\cdots(L+d-1)}{N(N+1)\cdots(N+d)} \theta(N \geq L)(N\delta^{\alpha\beta} - \bar{\alpha}a^\alpha a^\beta) \right].$$

(35)

The action and the topological charge are computed in appendix A:

$$S = (2\pi)^{d\frac{d}{2}} [dN_d(L) - N_d(l)],$$

(36)

$$Q = -\frac{1}{(-2\pi)^{d\frac{d}{2}}} \int \wedge^d F = \begin{cases} N_d(l) & (d \geq 2), \\ -(L-l) & (d = 1). \end{cases}$$

(37)

where $N_d(L) \equiv \frac{L(L+1)\cdots(L+d-1)}{d!}$ is the number of states with $N \leq L - 1$. The topological charge is equal to the number of states removed by the $U_l$ operator, except for the $D = 2 (d = 1)$ case. Note that in each topological class, the action is minimized when $L = l$ (remember that $L \geq l$).

### 3.3 Examples of NS-type instantons

**$D = 2 (d = 1)$**

From (33),

$$f_L(N) = \sqrt{\frac{N-L+1}{N+1}} \theta(N \geq L).$$

(38)

It follows from (21) and (34) that

$$X^1 = U_l f_L(N) a^1 U_l^\dagger = U_{L-l}^\dagger a^1 U_{L-l},$$

(39)

$$(F_0)_{1\bar{1}} = i\theta(l \leq N \leq L-1), \quad F_{1\bar{1}} = iF_{L-l}.$$
Therefore in this case the shift operator soliton (19) and the NS-type instanton are the same. This is consistent with the result [10] that the most general soliton solution in 2-dimensional pure NCYM is of this form up to translation.

\[ D = 4 \ (d = 2) \]

From (33),

\[
f_L(N) = \sqrt{\frac{(N - L + 1)(N + L + 2)}{(N + 1)(N + 2)}} \theta(N \geq L).
\]

Note that we can rewrite the solution in a rather suggestive way:

\[
X^\alpha = U_l \xi^{-1} a^\alpha \xi U_l^\dagger, \quad X^{\bar{\alpha}} = U_l \xi a^{\bar{\alpha}} \xi^{-1} U_l^\dagger,
\]

\[
\xi = \sqrt{\frac{(N + 2)(N + 3) \ldots (N + L + 1)}{N(N - 1) \ldots (N - L + 1)}} \theta(N \geq L)
\]

\[
= \sqrt{\frac{a^{\alpha_1} \ldots a^{\alpha_l} a^{a_L} \ldots a^{\alpha_1}}{a^{\bar{\alpha}_1} \ldots a^{\bar{\alpha}_L} a^{a_L} \ldots a^{\bar{\alpha}_1}}} \theta(N \geq L),
\]

where the inverse \( \xi^{-1} \) is defined only on \((1 - P_L)H\).

This solution gives anti-self-dual field strength if and only if \( L = l \). Although one can see this from the explicit form of the field strength (34), let us show it in a different way. In complex coordinates, the anti-self-duality condition can be written as

\[
[X^1, X^2] = 0, \quad [X^1, \bar{X}^1] + [X^2, \bar{X}^2] = 2.
\]

The first equation is trivially satisfied. The second equation is

\[
[X^1, X^2] + [X^2, \bar{X}^2] = U_l \left[ (N + 2)f(N)^2 \theta(N + 1 \geq l) - Nf(N - 1)^2 \theta(N - 1 \geq l) \right] U_l^\dagger = 2.
\]

Because \( U_l \) shifts \( N \) by \( l \), this condition is equivalent to the recursive equation

\[
(N + 2)f(N)^2 - Nf(N - 1)^2 = 2, \quad N = l, l + 1, \ldots.
\]

Solving this equation, we obtain

\[
f_L(N) = \sqrt{\frac{(N - l + 1)(N + l + 2)}{(N + 1)(N + 2)}} \theta(N \geq l),
\]

which is a special case of (41) with \( L = l \). This is the same as the instanton solution obtained by Nekrasov and Schwarz [1] (see also [2]) by the non-commutative ADHM construction, and describes \( \mathcal{N}_2(l) = \frac{l(l+1)}{2} \) instantons on top of each other at the origin.
So far, the gauge group has not been specified, but $X_0^\alpha = f(N)a^\alpha$ has always been proportional to the unit matrix. However, in this $D = 4\ (d = 2)$ case, the ansatz can be generalized to $U(2)$ gauge group:

$$X^\alpha = \begin{pmatrix} U_l f_1(N)a^\alpha U_l^\dagger & U_l \epsilon^{\alpha\beta\gamma\delta} g(N) \\ 0 & f_2(N)a^\alpha \end{pmatrix},$$

where $\epsilon^{12} = -\epsilon^{21} = 1$. For this ansatz, the anti-self-duality condition (42) reads

$$(N+2)f_1(N)^2 - Nf_1(N-1)^2 + Ng(N-1)^2 = 2, \quad N = l, l+1, \ldots,$$

$$(N+2)f_2(N)^2 - (N+2)g(N)^2 \theta(N \leq l-1) - Nf_2(N-1)^2 = 2, \quad N = 0, 1, 2, \ldots,$$

$$(N+2)f_1(N)g(N) - Ng(N-1)f_2(N-1) = 0, \quad N = l, l+1, \ldots.$$  

From these recursive equations, it follows that

$$f_1(0) = f_1(1) = \cdots = f_1(l-1) = 0,$$

$$f_2(0) = f_2(1) = \cdots = f_2(l-2) = 1,$$

$$g(0) = g(1) = \cdots = g(l-2) = 0,$$

$$f_2(l-1)^2 + g(l-1)^2 = 1.$$

Therefore we have one free parameter $f_2(l-1)$ (or equivalently, $g(l-1)$). This solution is the same as the instanton solution obtained by ADHM construction and describes $N_2(l) = \frac{l(l+1)}{2}$ instantons on top of each other at the origin, the free parameter corresponding to the size of the instantons.

Lower level solutions are

$l = 1$:

$$f_1(N)^2 = 1 - \frac{2 + \rho^2}{(N+1)(N+2 + \rho^2)},$$

$$f_2(N)^2 = 1 + \frac{\rho^2}{(N+2)(N+3 + \rho^2)},$$

$$g(N)^2 = \frac{\rho^2(2 + \rho^2)}{(N+1)(N+2)(N+3 + \rho^2)(N+3 + \rho^2)},$$

$$g(0)^2 = \frac{\rho^2}{6 + 2\rho^2}.$$

$l = 2$:

$$f_1(N)^2 = 1 - \frac{3(2 + \rho^2)(N+3)}{(N+1)(N+2)(N+3 + 3\rho^2(N+1))},$$

$$f_2(N)^2 = 1 + \frac{3\rho^2 N}{(N+2)(N+3)(N+4 + 3\rho^2(N+2))},$$

$$g(N)^2 = \frac{9\rho^2(2+\rho^2)(N)(N+3)}{(N+1)(N+2)(N+3)(N+4 + 3\rho^2(N+1))[(N+3)(N+4 + 3\rho^2(N+2))]},$$

$$g(1)^2 = \frac{\rho^2}{20 + 9\rho^2}.$$
The ansatz can be straightforwardly generalized to $U(k)$:

$$X^\alpha = \begin{pmatrix} U_1 f_1 a^\alpha U_1^\dagger & U_1 \epsilon^{\alpha\beta} \bar{a}^\beta g_1 \\ f_2 a^\alpha & \epsilon^{\alpha\beta} \bar{a}^\beta g_2 \\ \vdots & \vdots \\ f_{k-1} a^\alpha & \epsilon^{\alpha\beta} \bar{a}^\beta g_{k-1} \\ f_k a^\alpha \end{pmatrix},$$

where $f$’s and $g$’s are functions of $N$.

### 3.4 Mixing shift operator solitons and NS-type instantons

In this subsection, we go back to general $d$ and consider relaxing the assumption $UU^\dagger = 1$ in the NS-type instanton ansatz (26). We will see that this corresponds to mixing shift operator solitons and NS-type instantons.

If we only require $U^\dagger U = P$ and do not require $UU^\dagger = 1$, then $UU^\dagger$ is generally a projection operator

$$UU^\dagger = 1 - P', \quad P'^\dagger = P', \quad P'^2 = P'.$$

We require

$$\langle \{n\}|UU^\dagger|\{n\}\rangle \to 1 \quad (N \to \infty)$$

so $X^\alpha$ and $X^{\bar{\alpha}}$ approach respectively $a^\alpha$ and $a^{\bar{\alpha}}$ at large $N$. Hence we can set

$$P' = \theta(\{n\} \in P'),$$

where $P'$ is a finite subset of $\mathbb{N}^d = \{(n^1, \ldots, n^d); n^1, \ldots, n^d = 0, 1, 2, \ldots\}$. Now $U$ is essentially a shift operator which maps one to one the subspace $(1 - P)\mathcal{H}$ onto the subspace $(1 - P')\mathcal{H}$. We write this $U$ operator as $U^{(m)}_1$ henceforth, where $m$ is the number of elements of $P'$.

With this change, the ansatz (26) can be generalized to

$$X^\alpha = U_1^{(m)} f(N) a^\alpha U^{(m)}_1^\dagger = U_1^{(m)} X_0 a^\alpha U^{(m)}_1^\dagger,$$

$$X^{\bar{\alpha}} = U_1^{(m)} \bar{a}^{\bar{\alpha}} f(N) U^{(m)}_1 = U_1^{(m)} X_0 \bar{a}^{\bar{\alpha}} U^{(m)}_1.$$

(43)

In order for this $X$ to satisfy the equation of motion, $f(N)$ should still be given by (33). The field strength is modified from (34) to

$$F_{\alpha\bar{\beta}} = U_1^{(m)} (F_0)_{\alpha\bar{\beta}} U^{(m)}_1^\dagger - i \delta^{\bar{\alpha}\beta} P',$$

(44)

\footnote{Just as stated in footnote 7, there exists the additional freedom of adding finite matrices in the subspace $P'\mathcal{H}$ to (43). However we do not consider this generalization for simplicity.}
where \((F_0)_{\alpha\beta}\) is still given by (35). The values of the action (36) and the topological charge (37) are modified as

\[
S = (2\pi)^{d/2} \left[ d(\mathcal{N}_d(L) + m) - \mathcal{N}_d(l) \right],
\]

\[
Q = \begin{cases} 
\mathcal{N}_d(l) - m & (d \geq 2), \\
-(L - l) - m & (d = 1). 
\end{cases}
\] (45)

Therefore, introducing \(P'\) changes the action and the topological charge by \(m\).

As mentioned earlier, this generalization (43) amounts to adding \(m\) shift operator solitons (19) to the NS-type instanton. This is clear because the new solution (43) can be obtained from the old solution (26) by an “almost gauge” transformation:

\[
U_l^{(m)} X_0 U_l^{(m)\dagger} = V^\dagger (U_l X_0 U_l^\dagger) V,
\] (46)

\[
VV^\dagger = 1, \quad V^\dagger V = 1 - P'.
\] (47)

Note that the shift operator soliton gives an opposite sign contribution to the topological charge (45). This is consistent with the fact that in the \(D = 4\) case a shift operator soliton can be obtained by taking the radius \(\rho \to 0\) limit of a \(U(2)\) anti-self-dual instanton obtained via the ADHM construction [2] in the case of anti-self-dual non-commutativity. On the other hand, because we are taking self-dual non-commutativity, our solution, which becomes anti-self-dual when \(L = l\), should have opposite topological charge as compared to shift operator solitons.

In the \(D = 2 (d = 1)\) case, we can take for example

\[
U = \sum_{n^1 = 0}^{\infty} |n^1 + m\rangle \langle n^1 + l|, \quad m, l \geq 0,
\] (48)

which satisfies

\[
U^\dagger U = 1 - P_l, \quad U U^\dagger = 1 - P_m.
\] (49)

Using (38), one can show

\[
X^1 = U f_L(N) a^1 U^\dagger = U^\dagger_{-l+m} a^1 U_{-l+m}.
\]

This is again consistent with the result [10] concerning the most general soliton solution in 2-dimensional pure NCYM.

In the \(D = 4 (d = 2)\) case, it is clear that the field strength (44) can never be anti-self-dual for non-vanishing \(P'\). This can be also seen in terms of relations following from (45):

\[
\frac{S}{(2\pi)^2} + Q = 2\mathcal{N}_d(L) + m,
\]

\[
\frac{S}{(2\pi)^2} - Q = 2[\mathcal{N}_d(L) - \mathcal{N}_d(l)] + 3m.
\]

The right hand sides never become zero unless \(m = 0\).
4 Seiberg–Witten map of NCYM solitons

In this section, we consider the SW map (17) of solitonic solutions in NCYM. We derive a differential equation describing the evolution of fields under the SW map. The shift operator soliton is shown to be invariant, while the NS-type instantons are shown to approach a fixed point in the $\theta \to 0$ limit. We discuss the commutative description of the NS-type instantons using the properties of the fixed point.

4.1 Shift operator soliton

Let us consider the SW map of the shift operator soliton (19):
\[
\hat{X}^i(\theta) = \hat{U}^\dagger \hat{x}^i(\theta) \hat{U}, \quad \hat{U}^\dagger = \hat{U}^\dagger, \quad \hat{U}^\dagger \hat{U} = \hat{P}.
\]
(50)

By inserting this into (17), we find
\[
\hat{X}^i(\theta + \delta \theta) = \hat{U}^\dagger \left( \hat{x}^i(\theta) + \frac{1}{2} \delta \theta^j \hat{\theta}^j \hat{x}^k(\theta) \right) \hat{U} = \hat{U}^\dagger \hat{x}^i(\theta + \delta \theta) \hat{U}.
\]

Therefore, the shift operator soliton is invariant under the SW map.

For non-commutativity of the form (22) and for $\hat{P} = |\vec{0}\rangle \langle \vec{0}|$, the inverse Weyl transformation of the field strength (20) is
\[
F_{\alpha \bar{\beta}}(x) = -\frac{i \delta \alpha \bar{\beta}}{\theta} \int \frac{d^d k}{(2\pi)^d} (2\pi \theta)^d \text{Tr}[\hat{P} e^{-ik \cdot \hat{x}(\theta)}] e^{ik \cdot x} = -\frac{2^d i \delta \alpha \bar{\beta}}{\theta} e^{-x^2 / \theta}.
\]

Therefore, in the $\theta \to 0$ limit, we obtain the commutative description of the shift operator solitons:
\[
F = F^2 = \cdots = F^{d-1} = 0, \quad F^d \propto \delta^{(2d)}(x),
\]
which is consistent with the result obtained by direct calculation using the exact form of the SW map [11].

4.2 NS-type instanton

With non-commutativity parameter (22), if the solution is of the form
\[
\hat{X}^\alpha(\theta) = \hat{U} f(\hat{N}; \theta) \hat{x}^\alpha(\theta) \hat{U}^\dagger, \quad \hat{U}^\dagger \hat{U} = \hat{P}, \quad \hat{U}^\dagger \hat{U} = \hat{P}', \quad f(\hat{N}; \theta) \hat{P} = \hat{P} f(\hat{N}; \theta) = 0,
\]
(51)
(52)

then the SW map (17) is
\[
\delta \hat{X}^\alpha(\theta) = \frac{\delta \theta}{2\theta} \hat{U} f(\hat{N}; \theta) [(\hat{N} + 1) f(\hat{N}; \theta)^2 - \hat{N} f(\hat{N} - 1; \theta)^2] \hat{x}^\alpha(\theta) \hat{U}^\dagger.
\]
(53)
This is again of the form of (52), and thus we obtain an equation which describes the evolution of the field under the SW map:

$$\frac{\partial f(\hat{N}; \theta)}{\partial \theta} = \frac{f(\hat{N}; \theta)}{2\theta} (\hat{N} + 1) f(\hat{N}; \theta)^2 - \hat{N} f(\hat{N} - 1; \theta)^2 - 1,$$

where the last term comes from rewriting $\hat{x}(\theta + \delta \theta)$ in terms of $\hat{x}(\theta)$. Defining $t \equiv \ln(\theta/\theta_0)$ and $h(\hat{N}; t) \equiv (\hat{N} + 1) f(\hat{N}; \theta)^2 \geq 0$, this equation can be rewritten as

$$\frac{\partial h(\hat{N}; t)}{\partial t} = h(\hat{N}; t) [h(\hat{N}; t) - h(\hat{N} - 1; t) - 1]. \quad (54)$$

If $h(\hat{N}; t)$ is convergent in the $\theta \to 0$ (or $t \to -\infty$) limit, the limiting value $h(\hat{N}; -\infty)$ should be a fixed point for which the right hand side of (54) vanishes:

$$h(\hat{N}; -\infty)[h(\hat{N}; -\infty) - h(\hat{N} - 1; -\infty) - 1] = 0.$$

This gives

$$h(\hat{N}; -\infty) = (\hat{N} - L + 1) \theta(\hat{N} \geq L), \quad L = 1, 2, 3, \ldots, \quad (55)$$

or,

$$f(\hat{N}; \theta = 0) = \sqrt{\frac{\hat{N} - L + 1}{N + 1}} \theta(\hat{N} \geq L). \quad (56)$$

Indeed, provided $h(L - 1; t = 0) = 0$, as is the case with the NS-type instanton, Eq. (54) gives $h(N; t) = (1 - Ce^t)^{-1} \to 1$ ($t \to -\infty$) with $C$ a constant, which coincides with (55). On the other hand, if $h(N - 1; t) \sim N - L$ for $t \to -\infty$, one can show that $h(N; t) \sim N - L + 1$. Therefore the NS-type instanton approaches the fixed point (56) as $t \to -\infty$. Note that in the $d = 1$ ($D = 2$) case, the fixed point (56) gives the NS-type instanton itself, which is the same as the shift operator soliton and invariant under the SW map.

The above analysis implies that the commutative description of the NS-type instanton can be extracted from the $\theta \to 0$ behavior of

$$X^\alpha_{\text{fp}}(\theta) \equiv \mathcal{W}^{-1}_\theta [\hat{X}^\alpha_{\text{fp}}(\theta)],$$

where

$$\hat{X}^\alpha_{\text{fp}}(\theta) \equiv \hat{U} f(\hat{N}; -\infty) \hat{x}^\alpha(\theta) \hat{U}^\dagger.$$

Note that $\hat{X}_{\text{fp}}(\theta)$ is not the SW map of $\hat{X}(\theta)$. The corresponding field strength is

$$[\hat{F}_{\text{fp}}(\theta)]_{\alpha\beta} = -\frac{i}{\theta} \hat{U} \left[ \delta^{\alpha\beta} \theta(l \leq \hat{N} \leq L - 1) \right. \left. + \frac{L}{N(N + 1)} \theta(\hat{N} \geq L)(\hat{N}\delta^{\alpha\beta} - \hat{a}^\alpha \hat{a}^\beta) \right] \hat{U}^\dagger - \frac{i}{\theta} \hat{P}^\dagger \delta^{\alpha\beta}. \quad (57)$$
The topological charge associated with $\hat{F}_{\text{ip}}(\theta)$ can be evaluated in the same way as in appendix A:

$$Q_{\text{fp}} = N_d(l) - m - \frac{L^d}{d!},$$

which is different from the topological charge of the NS-type instanton (45) because of the last term, except for the $d = 1$ case. Since the topological charge cannot change under the continuous SW map, this implies that $f(\tilde{N}, \theta)$ converges nonuniformly in the $\theta \to 0$ limit. One can show that the last term comes from an $O(1/N)$ term in $f(\tilde{N}, \theta = 0)$ which is absent in $f(\tilde{N}, \theta)$ for any $\theta \neq 0$. If we want to make sense of the $\theta \to 0$ limit of $X$, $F$, $F^2$ etc., we may have to somehow correct them so that they give the expected topological charge.

Specifically, let us first consider the case where $d = 2$ ($D = 4$), $l = L = 1$, and $\tilde{U} = U^t = 1 - |0, 0)(0, 0|$ hence $\tilde{P} = \tilde{P}' = |0, 0)(0, 0|$. According to subsection 3.4, this case corresponds to a shift operator soliton on top of a Nekrasov–Schwarz instanton and has no net topological charge. Denoting this solution by $\text{NS}'$, Eq. (57) is

$$[\hat{F}_{\text{ip}}^{\text{NS}'}(\theta)]_{\alpha \beta} = -\frac{i}{\theta}(1 - \tilde{P}) \left[ \frac{1}{N(N + 1)}(\tilde{N}\delta^{\alpha \beta} - \hat{a}^{\alpha} \hat{a}^{\beta}) \right] (1 - \tilde{P}) - i \tilde{\delta}^{\alpha \beta} \tilde{P}' \phantom{0} (58)$$

$$= -\frac{i}{\theta} \left[ \frac{\delta^{\alpha \beta}}{N + 1} - \frac{1}{N(N + 1)} \hat{a}^{\alpha} \hat{a}^{\beta} \right]. \phantom{0} (59)$$

In c-number function language (using the relations in appendix C),

$$[F_{\text{ip}}^{\text{NS}'}(\theta)]_{\alpha \beta} = -\frac{i}{\theta} \left[ \delta^{\alpha \beta} \left( \left( \frac{1}{\rho} + \frac{1}{2\rho^2} \right) - \left( \frac{2}{\rho} + \frac{1}{2\rho^2} \right) e^{-2\rho} \right) \right.$$

$$\left. - \tilde{\xi}^{\alpha} \tilde{\xi}^{\beta} \left( \left( \frac{1}{\rho} + \frac{1}{\rho^2} \right) - \left( \frac{4}{\rho} + \frac{3}{2\rho^2} + \frac{1}{\rho^3} \right) e^{-2\rho} \right) \right],$$

where $\xi^{\alpha} \equiv z^{\alpha}/\sqrt{\theta}$ and $\rho \equiv \tilde{\xi}^{\alpha} z^{\alpha} = 1/\theta \sum_{i=1}^{D} (x^{i})^2$. Taking the $\theta \to 0$ limit, we find

$$[F_{\text{ip}}^{\text{NS}'}(\theta = 0)]_{\alpha \beta} = -i \delta^{\alpha \beta} \tilde{z} z - \tilde{z}^{\alpha} \tilde{z}^{\beta} \left( \frac{\tilde{z}}{(\tilde{z})^2} \right) (x \neq 0), \phantom{0} (60)$$

where $\tilde{z} z \equiv \tilde{z}^{\alpha} z^{\alpha}$. This result can be understood as follows. Roughly, $\tilde{N}$ corresponds to $\rho = \tilde{z} z/\theta$ and $\theta \to 0$ corresponds to $\rho \sim \tilde{N} \to \infty$, hence only the leading term in the $1/\tilde{N}$ expansion contributes in the $\theta \to 0$ limit. In addition, if the leading term is $O(C N^{-\alpha})$ and if we are considering a quantity $O_{\text{ip}}(\theta)$ with mass dimension $\delta$, then $O_{\text{ip}}(\delta) \sim \frac{1}{\theta^{\alpha} \rho^{\delta}} = \frac{\rho^{\alpha - \delta/2}}{(\tilde{z} z)^{\delta/2}}$ by dimensional analysis. Therefore i) if $\alpha > \delta/2$, then $O_{\text{ip}}(0) = 0$, ii) if $\alpha = \delta/2$, then $O_{\text{ip}}(0) \sim (\tilde{z} z)^{\delta/2}$, and iii)
\( \alpha < \delta/2 \), then \( O_{fp}(0) \) is not well-defined. Note that this argument cannot be used if \( x = 0 \).

Now let us turn to the topological charge density of the NS' solution. Defining \( \hat{\sigma}(\theta) \) and \( \sigma(x; \theta) \) by

\[
Q = (2\pi \theta)^d \text{Tr}[\hat{\sigma}(\theta)] = \int d^D x \sigma(x; \theta),
\]

we obtain

\[
\hat{\sigma}^{\text{NS}'}(\theta) = \frac{1}{2(2\pi \theta)^2} \left[ \frac{1}{N} (1 - |0, 0\rangle\langle 0, 0|) - \frac{1}{(N + 1)^2} - \frac{1}{N + 1} \right].
\]

The corresponding c-number function is, from direct calculation or from the argument above,

\[
\sigma^{\text{NS}'}(x; \theta) \sim \frac{1}{2(2\pi \theta)^2} \frac{1}{\rho^3} \frac{\theta}{(zz)^{1/2}} (x \neq 0),
\]

which vanishes in the \( \theta \to 0 \) limit. We can extend this result to \( x = 0 \) so that it gives the desired topological charge \( Q^{\text{NS}'} = \int d^D x \sigma^{\text{NS}'} = 0 \):

\[
\sigma^{\text{NS}'}(x; \theta = 0) = 0 \quad \text{(for all } x \text{)}.
\]

Second, let us consider the case of the Nekrasov–Schwarz (NS) instanton, namely \( d = 2 \) (\( D = 4 \)), \( l = L = 1 \), and \( \hat{U} \hat{U}^\dagger = 1, \hat{U}^\dagger \hat{U} = 1 - |0, 0\rangle\langle 0, 0| \). We choose [2]

\[
\hat{U} = (1 - \hat{P}_2) + \hat{S}_1 \hat{P}_2,
\]

where subscripts of bras and kets refer to the two subspaces of the whole Hilbert space. We find

\[
[\hat{F}^{\text{NS}'}_{fp}(\theta)]_{1\bar{1}} = [\hat{F}^{\text{NS}'}_{fp}(\theta)]_{1\bar{1} \theta} + \frac{i}{\theta} |0, 0\rangle\langle 0, 0|,
\]

\[
[\hat{F}^{\text{NS}'}_{fp}(\theta)]_{2\bar{2}} = [\hat{F}^{\text{NS}'}_{fp}(\theta)]_{2\bar{2} \theta} + \frac{1}{\theta (\tilde{N}_1 + 1)(\tilde{N}_1 + 2)} |0\rangle\langle 0|_{22},
\]

\[
[\hat{F}^{\text{NS}'}_{fp}(\theta)]_{1\bar{2}} = [\hat{F}^{\text{NS}'}_{fp}(\theta)]_{1\bar{2} \theta} - \frac{i}{\theta} \left( \frac{1}{(\tilde{N}_1 + 1)(\tilde{N}_1 + 2)} a^\dagger - \frac{1}{\sqrt{\tilde{N}_1 + 1}(\tilde{N}_1 + 2)} \right) |0\rangle_{22},
\]

\[
[\hat{F}^{\text{NS}'}_{fp}(\theta)]_{2\bar{1}} = [\hat{F}^{\text{NS}'}_{fp}(\theta)]_{2\bar{1} \theta},
\]

where \( \tilde{N}_1 \equiv \hat{a}^\dagger \hat{a}^\dagger \). If we translate this result into c-number function language, the difference between \( F^{\text{NS}'} \) and \( F^{\text{NS}} \) vanishes in the \( \theta \to 0 \) limit. For example,

\[
[\hat{F}^{\text{NS}'}_{fp}(x; \theta)]_{2\bar{2}} - [\hat{F}^{\text{NS}'}_{fp}(x; \theta)]_{2\bar{2} \theta} \sim \frac{1}{\theta} \frac{i}{\rho^2} e^{-2\rho^2} = \frac{i \theta}{(z^2 z^1)^2} e^{-2z^2 z^2/\theta} (x \neq 0),
\]

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which vanishes in the $\theta \to 0$ limit. Here $\rho_1 \equiv \tilde{\xi}^1 \xi^1$ and $\rho_2 \equiv \tilde{\xi}^2 \xi^2$. Therefore we obtain

$$[F_{ip}^{NS}(\theta = 0)]_{\alpha\bar{\beta}} = [F_{ip}^{NS'}(\theta = 0)]_{\alpha\bar{\beta}} = -i \frac{\delta^{\bar{\alpha}\bar{\beta}} \bar{z} z - \bar{z} \bar{z}^{\bar{\alpha}\bar{\beta}}}{(\bar{z} z)^2} \quad (x \neq 0).$$

Similarly,

$$\hat{\sigma}_{NS}(\theta) = \hat{\sigma}_{NS'}(\theta) + \frac{1}{2(2\pi\theta)^2} \left[ -\frac{1}{N_1} (1 - |0\rangle_1 \langle 0|) + \frac{1}{(N_1 + 1)^2} + \frac{1}{N_1 + 1} \right.$$

$$\left. + \frac{1}{(N_1 + 1)(N_1 + 2)^2} |0\rangle_2 \langle 0| \right]$$

leads to

$$\sigma_{NS}(x; \theta = 0) = \sigma_{NS'}(x; \theta = 0) = 0 \quad (x \neq 0).$$

Since the desired topological charge is $Q_{NS} = \int d^D x \sigma_{NS} = 1$, we extend this result as

$$\sigma_{NS}(x; \theta = 0) = \delta^{(4)}(x) \quad \text{(for all } x).$$

Note that the commutative description does not depend on the specific choice (61), which just corresponds to non-commutative gauge freedom. Actually one can say much more: the commutative description for $x \neq 0$ should be the same for any choice of $\hat{U}$ operator, i.e., any values of $l$, $m$ and $L$. This is because these finite shifts in the Hilbert space are negligible in the $\theta \to 0$ limit, since what determine the commutative description at $\rho \neq 0$ are the states with $\hat{N} \sim \rho/\theta \gg 1$. In other words, the commutative description away from the origin is determined only by the fixed point $f(N, \theta = 0)$ and the finite shift introduced by $\hat{U}$ and $\hat{U}^\dagger$ can only affect the singularity at the origin.

To summarize, we have found the commutative description of non-commutative instantons by observing that the solutions are attracted to a fixed point of the flow governed by the SW map. In operator language the fixed point is given by (56). We then applied an inverse Weyl transformation to compute the form of the solution in commutative variables at $\theta = 0$, yielding (60). Due to non-uniform convergence of the configuration, it was necessary to augment the solution with a singular contribution located at the origin so that the topological charge was held fixed as $\theta \to 0$.

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A Actions and topological charges of NS-type instantons

In this appendix, we compute the action and the topological charge (d-th Chern character) of NS-type instantons.

The action can be evaluated as

$$\frac{S}{(2\pi)^d} = -\frac{1}{2} \text{Tr} F_{\alpha\beta} F_{\beta\alpha} = -\frac{1}{2} \text{Tr} (F_0)_{\alpha\beta} (F_0)_{\beta\alpha}$$

$$= \frac{1}{2} \text{Tr} \left[ \delta^{\alpha\beta} \delta^{\beta\alpha} \theta(l \leq N \leq L) + F(N)^2 \theta(N \geq L)(N\delta^{\alpha\beta} - d a^\alpha a^\beta)(N\delta^{\beta\alpha} - d a^\beta a^\alpha) \right]$$

$$= \frac{d}{2} \sum_{N=l}^{L-1} D_d(N) + \frac{1}{2} \sum_{N=L}^{\infty} F(N)^2 d(d-1)N(N+d) D_d(N)$$

$$= \frac{d}{2} \left[ dN_d(L) - N_d(l) \right] + \frac{d(d-1)}{2} N_d(L)$$

$$= \frac{d}{2} \left[ dN_d(L) - N_d(l) \right]. \quad (62)$$

Here

$$F(N) \equiv \frac{L(L+1)\ldots(L+d-1)}{N(N+1)\ldots(N+d)},$$

$$D_d(N) \equiv \frac{(N+1)(N+2)\ldots(N+d-1)}{(d-1)!},$$

$$N_d(L) \equiv \sum_{N=0}^{L-1} D_d(N) = \frac{L(L+1)\ldots(L+d-1)}{d!}.$$
non-vanishing component of the field strength is \( F_{\alpha \beta} \), we find,

\[
Q \equiv \frac{-1}{(2\pi)^d} \int d^d F \wedge d^d F \\
= \frac{-1}{(2\pi)^d} \int (F_{\alpha_1 \beta_1} dz^{\alpha_1} \wedge dz^{\beta_1}) \wedge \cdots \wedge (F_{\alpha_d \beta_d} dz^{\alpha_d} \wedge dz^{\beta_d}) \\
= \frac{-1}{(2\pi)^d} \int \epsilon^{\alpha_1 \cdots \alpha_d \epsilon^{\beta_1 \cdots \beta_d} F_{\alpha_1 \beta_1} \cdots F_{\alpha_d \beta_d} dz^{\alpha_1} \wedge dz^{\beta_1} \wedge \cdots \wedge dz^{\alpha_d} \wedge dz^{\beta_d} \\
= \frac{-1}{(2\pi)^d} \int \epsilon^{\alpha_1 \cdots \alpha_d \epsilon^{\beta_1 \cdots \beta_d}} \frac{d^D x}{i^d} \\
= \frac{1}{(2\pi)^d} \epsilon^{\alpha_1 \cdots \alpha_d \epsilon^{\beta_1 \cdots \beta_d}} \text{Tr}(F_{\alpha_1 \beta_1} \cdots (F_{\alpha_d \beta_d} \\
= \frac{1}{(2\pi)^d} \epsilon^{\alpha_1 \cdots \alpha_d \epsilon^{\beta_1 \cdots \beta_d}} \text{Tr}(F_0)_{\alpha_1 \beta_1} \cdots (F_0)_{\alpha_d \beta_d},
\]

where summation over identical upper and lower indices is implied. The totally antisymmetric \( \epsilon \) symbol is defined as \( \epsilon^{12\cdots d} = \epsilon^{12\cdots d} = 1 \). Plugging in the explicit form of \( (F_0)_{\alpha \beta} \),

\[
\epsilon^{\alpha_1 \cdots \alpha_d \epsilon^{\beta_1 \cdots \beta_d}} (F_0)_{\alpha_1 \beta_1} \cdots (F_0)_{\alpha_d \beta_d} \\
= (-i)^d \left[ \epsilon^{\alpha_1 \cdots \alpha_d \epsilon^{\beta_1 \cdots \beta_d}} (F_0)_{\alpha_1 \beta_1} \cdots (F_0)_{\alpha_d \beta_d} \\
= (-i)^d \left[ dt \theta(l \leq N \leq L - 1) \\
+ F(N)^d \theta(N \geq L) \sum_{k=0}^d (dk) \epsilon^{\alpha_1 \cdots \alpha_k \alpha_{k+1} \cdots \alpha_d \epsilon^{\beta_1 \cdots \beta_k \beta_{k+1} \cdots \beta_d} \\
\times (-d)^k (a^{\alpha_1 \beta_1}) \cdots (a^{\alpha_k \beta_k}) (N^{\delta^{\alpha_{k+1} \beta_{k+1}}}) \cdots (N^{\delta^{\beta_{k+1} \beta_{k+1}}}) \\
= (-i)^d \left[ dt \theta(l \leq N \leq L - 1) + F(N)^d \theta(N \geq L) \\
\times \sum_{k=0}^d (dk) (d-k)! (-d)^k N^{-d-k} \delta^{\alpha_1 \cdots \alpha_k; \beta_1 \cdots \beta_k} (a^{\alpha_1 \beta_1}) \cdots (a^{\alpha_k \beta_k}),
\right]
\]

where

\[
F(N) \equiv \frac{L(L+1) \cdots (L+d-1)}{N(N+1) \cdots (N+d)}
\]

and we have used the relation

\[
\epsilon^{\alpha_1 \cdots \alpha_k \epsilon^{\beta_1 \cdots \beta_k} \epsilon^{\alpha_{k+1} \cdots \alpha_d}} = (d-k)! \delta^{\alpha_1 \cdots \alpha_k; \beta_1 \cdots \beta_k},
\]

\[
\delta^{\alpha_1 \cdots \alpha_k; \beta_1 \cdots \beta_k} \equiv k! \left( \text{antisymmetrization of } \delta^{\alpha_1 \beta_1} \cdots \delta^{\alpha_k \beta_k} \right),
\]

\[
\text{with respect to } \beta_1 \cdots \beta_k.
\]
Furthermore, using the relation
\[ \delta_{\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_k} (a^{\alpha_1} a^{\beta_1}) \ldots (a^{\alpha_k} a^{\beta_k}) \]
\[ = (-1)^{k-1}(d-1)(d-2) \ldots (d-k+1)N, \quad k \geq 1, \]
(64) becomes
\[ \epsilon^{\alpha_1 \ldots \alpha_d \bar{\beta}_1 \ldots \bar{\beta}_d} (F_0)_{\alpha_1 \bar{\beta}_1} \ldots (F_0)_{\alpha_d \bar{\beta}_d} \]
\[ = (-i)^d \left[ \frac{d!}{d} \sum_{N=l}^{N=L-1} F(N)^d \theta(N \geq L)(d-1)! (dN^d + N^{d+1} - N(N + d)^d) \right]. \]
Putting this back into (63), we obtain the final result
\[ Q = \frac{-1}{d!} \left[ \frac{d!}{d} \sum_{N=l}^{N=L-1} D_d(N) \right. \]
\[ + (d-1)! \sum_{N=l}^{\infty} F(N)^d (dN^d + N^{d+1} - N(N + d)^d) D_d(N) \]
\[ = \begin{cases} \frac{d!}{d} \left[ \sum_{N=l}^{N=L} (N_d(L) - N_d(l)) - (d-1)! \cdot dN_d(L) \right] \quad (d \geq 2), \\ \frac{d!}{d} \left[ \sum_{N=l}^{N=L} (N_d(L) - N_d(l)) + 0 \right] \quad (d = 1) \end{cases} \]
\[ = \begin{cases} N_d(l) \quad (d \geq 2), \\ -(L - l) \quad (d = 1). \end{cases} \]

**B Two Seiberg–Witten maps are related by unitary transformation**

In this appendix, we will show that solutions to the two SW maps (15) and (17), namely,
\[ \frac{dX^i_g}{d\tau} = \frac{i}{2} \frac{d\theta^{kl}}{d\tau} \theta_{km} \theta_{ln} [X^i_g, X^m_g] X^n_g + i[g(X_g, x), X^i_g] \quad (65) \]
and
\[ \frac{dX^i_0}{d\tau} = \frac{i}{2} \frac{d\theta^{kl}}{d\tau} \theta_{km} \theta_{ln} [X^i_0, X^m_0] X^n_0, \quad (66) \]
are related by a unitary transformation. Here \( \tau \) parametrizes the trajectory \( \theta(\tau) \), and \( g(X, x) \) is assumed to be Hermitian for any Hermitian \( X \).

For \( X_0 \) and \( X_g \) to be connected by a unitary transformation
\[ X_g = uX_0u^{-1}, \]
\( u(\tau) \) must satisfy

\[
\frac{du}{d\tau} = iu g(X_0, u^{-1}xu).
\]  

(67)

Therefore the question is reduced to whether we can solve Eq. (67) for a unitary operator \( u(\tau) \).

Let us expand as

\[
u(\tau) = e^{i \sum_{n=1}^{\infty} \tau^n H^{(n)}}, \quad H^{(n)\dagger} = H^{(n)}
\]

and try to determine \( H^{(n)} \) order by order. We also define

\[
u^{(m)}(\tau) = e^{i \sum_{n=1}^{m} \tau^n H^{(n)}},
\]

Suppose that we have solved (67) to order \( \tau^{m-2} \), namely, we have found a unitary \( u^{(m-1)} \) satisfying

\[
\frac{du^{(m-1)}}{d\tau} = iu^{(m-1)} g(X_0, u^{(m-1)-1}xu^{(m-1)}) + O(\tau^{m-1}),
\]

or

\[
\left[ \frac{du^{(m-1)}}{d\tau} \right]_{m-2} = \left[ iu^{(m-1)} g(X_0, u^{(m-1)-1}xu^{(m-1)}) \right]_{m-2}.
\]

Here we have defined

\[
\left[ \sum_{n=0}^{\infty} a_n \tau^n \right]_m = \sum_{n=0}^{m} a_n \tau^n.
\]

Now try to solve (67) at order \( \tau^{m-1} \):

\[
\left[ \frac{du^{(m)}}{d\tau} \right]_{m-1} = \left[ iu^{(m)} g(X_0, u^{(m)-1}xu^{(m)}) \right]_{m-1}.
\]

Since \( [u^{(m)}]_m = [u^{(m-1)}]_m + i\tau^m H^{(m)} \), the left hand side can be rewritten as

\[
\left[ \frac{du^{(m)}}{d\tau} \right]_{m-1} = \frac{d}{d\tau} [u^{(m)}]_m = \frac{d}{d\tau} [u^{(m-1)}]_m + im\tau^{m-1} H^{(m)}
\]

\[
= \left[ \frac{du^{(m-1)}}{d\tau} \right]_{m-1} + im\tau^{m-1} H^{(m)}.
\]

Therefore

\[
im\tau^{m-1} H^{(m)} = - \left[ \frac{du^{(m-1)}}{d\tau} \right]_{m-1} - \left[ iu^{(m-1)} g(X_0, u^{(m-1)-1}xu^{(m-1)}) \right]_{m-1},
\]
where we replaced \( u^{(m)} \) with \( u^{(m-1)} \) in \([ \cdot ]_{m-1} \). Now that \( H^{(m)} \) only appears on the left hand side, the question is whether the right hand side is anti-Hermitian. By assumption, the right hand side is zero to order \( \tau^{m-2} \). Therefore, up to order \( \tau^{m-1} \) we can multiply the right hand side by \([u^{(m-1)}]_{m-1} \) to obtain

\[
\begin{align*}
im \tau^{m-1} H^{(m)} &= - \left[ \frac{d u^{(m-1)}}{d \tau} u^{(m-1)-1} \right]_{m-1} \\
&\quad - \left[ i u^{(m-1)} g(X_0, u^{(m-1)-1} x u^{(m-1)-1}) \right]_{m-1} + \mathcal{O}(\tau^{m}).
\end{align*}
\]

The two terms on the right hand side are easily shown to be anti-Hermitian using the unitarity of \( u^{(m-1)} \) and the assumption that \( g(X, x) \) is Hermitian for any Hermitian \( X \). Therefore \( H^{(m)} \) is Hermitian.

Since \( H^{(1)} = g \) is Hermitian, \( H^{(n)} \) is Hermitian for all \( n \) and the proof is complete.

### C Useful formulae

Using relations

\[
\begin{align*}
\text{Tr}[e^{-\tau \hat{N}} e^{-i k \cdot \hat{x}}] &= e^{-\frac{\hbar k}{2 \sinh \frac{\tau}{2}}}, \\
\frac{\partial}{\partial k^\alpha} e^{-i k \cdot \hat{x}} &= (-i \hat{a}^\alpha - \frac{1}{2} k^\alpha) e^{-i k \cdot \hat{x}} , \\
\frac{\partial}{\partial k^\alpha} e^{-i k \cdot \hat{x}} &= \left( -i \hat{a}^\alpha + \frac{1}{2} k^\alpha \right) e^{-i k \cdot \hat{x}},
\end{align*}
\]

and \( \frac{1}{N+a} = \int_0^\infty d\tau e^{-\frac{(N+a)\tau}{2}} \), one can derive

\[
\begin{align*}
\text{Tr} \left[ \frac{1}{N+a} e^{-i k \cdot \hat{x}} \right] &= 2^{1-d} \int_1^\infty dy (y+1)^{d-a-1}(y-1)^{a-1} e^{-\frac{\hbar k}{2} y}, \\
\int d^D k e^{i k \cdot x} \text{Tr} \left[ \frac{1}{N+a} e^{-i k \cdot \hat{x}} \right] &= 2(2\pi)^d \int_0^1 d\eta (1+\eta)^{d-a-1}(1-\eta)^{a-1} e^{-2\bar{z} \eta}, \\
\text{Tr} \left[ \frac{1}{N+a} \hat{a}^\alpha \hat{a}^\beta e^{-i k \cdot \hat{x}} \right] &= 2^{-d} \left[ \delta^{\alpha\beta} \int_1^\infty dy (y+1)^{d-a-1}(y-1)^{a} e^{-\frac{\hbar k}{2} y} - \frac{k^\alpha k^\beta}{2} \int_1^\infty dy (y+1)^{d-a-1}(y-1)^{a} e^{-\frac{\hbar k}{2} y} \right],
\end{align*}
\]

\[
\begin{align*}
\int d^D k e^{i k \cdot x} \text{Tr} \left[ \frac{1}{N+a} \hat{a}^\alpha \hat{a}^\beta e^{-i k \cdot \hat{x}} \right] &= (2\pi)^d \left[ -\delta^{\alpha\beta} \int_0^1 d\eta (1+\eta)^{d-a-1}(1-\eta)^{a} e^{-2\bar{z} \eta} + 2\bar{z} \eta^{\beta} \right],
\end{align*}
\]

where \( k \cdot \hat{x} \equiv k^\alpha \hat{a}^\alpha + k^\alpha \hat{a}^\alpha, \ k \cdot x \equiv k^\alpha z^\alpha + k^\alpha z^\alpha, \ \bar{z} k \equiv k^\alpha k^\alpha, \ \text{and} \ \bar{z} \bar{z} \equiv \bar{z}^\alpha \bar{z}^\alpha. \)
References


