Vanishing dynamical quark mass at zero virtuality? ∗

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We show that the dynamical quark mass in effective nonlocal models can vanish at zero virtuality of the quark as \( M(p^2) \propto p^2 \). Our arguments follow from the constrained-instanton model of the QCD vacuum and from QCD sum rules calculations with nonlocal condensates. The discussed models also lead to analyticity of \( M(p^2) \) in the vicinity of zero.

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The concept of dynamical quark mass generated by the spontaneous breaking of chiral symmetry has proven very useful in the description of low-energy phenomena. This mass enters the quark propagator, \( S_F^{-1}(p) = Z(p^2)[\not{p} + M(p^2)] \), and, in general, depends on the (Euclidean) momentum \( p \). Specific forms of \( M(p^2) \) are obtained in instanton models [1] or via the Schwinger-Dyson resummation techniques [2]. The quark mass in its own is not observable and does not bear physical significance. Moreover, it depends on the choice of the gluon field gauge, hence is not unique. Also, the region where, e.g., the instanton models dominate is at some intermediate values of \( p^2 \), not too high, where the model predicts a too fast dropping of the mass in the perturbative region, and not too low, since certainly the infrared physics (confinement) is not being incorporated properly. Nevertheless, the mentioned popular approaches have founded a typical image of \( M(p^2) \) in the Euclidean space: it is non-zero at

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\( p^2 = 0 \), drops monotonically with increasing \( p^2 \), and asymptotically reaches the current quark mass value, \( m_c. \)

In this paper we argue that this conventional picture need not be the case, and that we may well have

\[
M(p^2 = 0) = 0. \tag{1}
\]

As \( p^2 \) is increased, \( M(p^2) \) grows, arrives at maximum, and then decreases to reach \( m_c \) asymptotically. In fact, we will present arguments that the situation (1) can be more natural than the conventional case of \( M(p^2 = 0) > 0 \). One class of arguments comes from QCD sum rules with nonlocal condensates, another one from the so-called “constrained” instantons, where the large-distance asymptotics is improved by an additional cut-off function in the instanton size.

We will also show that although Eq. (1) looks, at first glance, unusual, it does not lead to pathologies: the chiral symmetry is broken, and all results enforced by this fact hold: the quark condensate has a non-zero value, the Gell-Mann-Oaks-Renner relation is satisfied, as are other relations coming from symmetries, anomalies are preserved, etc.

We begin by presenting a calculation in framework of the constrained instanton model, where Eq. (1) holds. We use for simplicity the “constrained” quark zero mode in the singular gauge and introduce the following ansatz for the profile function [3]:

\[
\psi_{\text{sing}}^\pm(x) = \sqrt{2} \varphi_{\text{sing}}(x) \frac{\hat{x}}{|x|} \chi^\pm, \quad \varphi_{\text{sing}}(x) = \frac{\overline{p}(x^2)}{\pi(x^2 + \overline{p}^2(x^2))^{3/2}}, \tag{2}
\]

where \( \chi \) is a color Dirac spinor given by \( \chi^\pm \chi^\mp = (\gamma_\mu \gamma_\nu / 16) (1 \pm \gamma_5) / 2 \tau^\pm_\mu \tau^\mp_\nu \) and \( \tau^\pm_\nu = (\mp i, \mp) \), with the upper (lower) signs corresponding to solutions in the instanton (anti-instanton) field. The standard instanton solution is obtained with the constant for \( \overline{p}^2(x^2) = \rho^2 \). For the constrained instantons one uses exponentially-decreasing functions \( \overline{p}^2(x) \), normalized as \( \overline{p}^2(0) = \rho^2 \). We shall use the form [4]

\[
\overline{p}^2(x^2) = \frac{2}{\Gamma(1/3) 3^{1/3}} \left( \frac{\rho}{R} \right)^2 x^2 K_{4/3} \left( \frac{2}{3} \left( \frac{x^2}{R^2} \right)^{3/4} \right), \tag{3}
\]

where \( K_\nu (z) \) is the modified Bessel function. The specific feature of the constrained instanton is that at small distances it is close to the standard instanton profile of size \( \rho \), and at large distances it has exponentially-decreasing asymptotics governed by a large-scale parameter \( R \), such as \( \rho < R \). These shapes are motivated by considering modifications of the instanton in the

\[^1\]Throughout this paper we are working in the strict chiral limit of \( m_c = 0 \).
background field of large-scale vacuum fluctuations [4]. The constrained instanton profile, as opposed to the unconstrained one, provides the correct large-distance asymptotics of the quark and gluon field correlators [4].

Let us define the 4-dimensional Fourier transform of the quark zero mode profile \( \tilde{\varphi}_{\text{sing}}(p) \). The explicit form of the Fourier transform of the standard (unconstrained) zero mode in the singular gauge is well known,

\[
\tilde{\varphi}_{\text{sing}}^I(p^2) = \pi \rho^2 \frac{d}{dz} [I_1(z)K_1(z) - I_0(z)K_0(z)]_{z=\rho p/2},
\]  

and has the asymptotics

\[
\tilde{\varphi}^I(p^2) = \begin{cases} 
\frac{2\pi \rho}{p} & \text{as } p^2 \to 0, \\
\frac{12\pi}{p^3 \rho^3} & \text{as } p^2 \to \infty.
\end{cases}
\]  

The constrained zero mode has the same large-\( p^2 \) asymptotics, but it goes to a constant at \( p^2 \to 0 \)

\[
\tilde{\varphi}^{CI}(p^2) = \begin{cases} 
\pi^2 I_{CI} & \text{as } p^2 \to 0, \\
\frac{12\pi}{p^3 \rho^3} & \text{as } p^2 \to \infty,
\end{cases}
\]  

where \( I_{CI} = \int_0^\infty du u \varphi_{CI}(u) \). The quark mass generated by the one-instanton background is equal to [5]

\[
M(p^2) = C p^2 \tilde{\varphi}^2(p^2),
\]  

where the constant \( C > 0 \) is determined from the gap equation [5,6]

\[
\int \frac{d^4k}{(2\pi)^4} \frac{M^2(k^2)}{k^2 + M^2(k^2)} = \frac{n}{4N_c},
\]

with \( n = 0.0016 \text{ GeV}^4 \) [1] denoting the instanton density. From the asymptotic behavior (5) and (6) we see immediately, that for the standard zero mode \( M_I(0) > 0 \), whereas constrained zero modes give \( M_{CI}(0) = 0 \). In Fig. 1 we show the quark mass function for the constrained and unconstrained zero modes. As advocated above, the constrained zero modes lead to (1). The momentum at which \( M(p^2) \) starts dropping as \( p^2 \) is being decreased is controlled by the large-scale parameter \( R \). When \( R \) is increased, the constrained profile tends to the standard zero mode solution everywhere except the point \( p^2 = 0 \).

Let us pass to another approach hinting to Eq. (1), namely QCD sum rules with nonlocal condensates \( \tilde{Q}(x^2) \equiv \langle \bar{q}(x)P \exp \{ig \int_0^x dz A(z)\}q(0) \rangle \), etc [7]. The local quark condensates, \( \langle \bar{q}q \rangle, \langle \bar{q}(D^2)^n q \rangle, n = 1,2,... \) appear as expansion coefficients of the correlator \( \tilde{Q}(x^2) \) in the variable \( x^2/4 \). Now, we
Fig. 1. Dynamical quark mass as a function of the square of the Euclidean momentum. The dashed line corresponds to the standard zero mode solution of Eq. (4), while the solid line corresponds to the constrained zero mode profile, Eq. (3). The parameters are $\rho = 1.7 \text{GeV}^{-1}$ and $R = 3\rho$.

make a crucial assumption needed for our argumentation: the quark condensate and its moments can be identified with the expressions obtained at the one-quark-loop level in the effective quark model. This is valid in the large-$N_c$ limit. We have therefore (for the single-flavor)\(^2\)

$$\langle \bar{q}q \rangle = -4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M(p^2)}{p^2 + M^2(p^2)}, \quad (9)$$

and

$$\chi^2_{q^n} \equiv \frac{\langle \bar{q}D^{2n}q \rangle}{\langle \bar{q}q \rangle} = -4N_c \int \frac{d^4p}{(2\pi)^4} p^{2n} \frac{M(p^2)}{p^2 + M^2(p^2)}, \quad (10)$$

can be found\(^3\)

$$Q(p^2) = 2 \frac{M(p^2)}{p^2 + M^2(p^2)}, \quad (11)$$

where $Q(p^2)$ is proportional to the Fourier transformation of the scalar quark

\(^2\)For simplicity, we are neglecting the effects of the quark wave-function renormalization by setting $Z(q^2) = 1$ in Eq. (8). This amounts to neglecting possible vector interactions, as is the case of the instanton calculations.

\(^3\)Here and below we silently assume that nonlocal condensates are defined in gauge-invariant way by putting in appropriate definitions the path-ordered Schwinger phase factor. In QCD sum rules this factor becomes unity throughout the use of the Fock-Schwinger gauge. A similar situation occurs in the instanton model, where we have to transform the fields from standard (singular or regular) gauges to the axial one [13]. This results in modification of Eq. (7). We omit these complications in the present work, since the influence of the phase factor is not important for the qualitative analysis of the dynamical quark mass.
condensate $\tilde{Q}(x^2)$. In QCD sum rules one typically makes a hypothesis for $Q(p^2)$. For instance, one of the ansatze successfully applied in QCD sum rules [7] is

$$Q(p^2) = -\frac{\langle \bar{q}q \rangle}{2N_c} \frac{16\pi^2}{\Lambda^4} \exp\left(-\frac{p^2}{\Lambda^2}\right),$$

with average quark virtuality in QCD vacuum being

$$\lambda_q^2 = 2\Lambda^2.$$  

Then, the moments of the quark condensate become parameterized in a simple way. Thus, in QCD sum rules its the nonlocal quark condensate $Q(p^2)$ which is the basic quantity, which is in a sense opposite to quark models or the instanton approach, where we start with $M(p^2)$. The ansatz (12) is compatible with recent lattice measurements of scalar quark correlator, $\tilde{Q}(x^2)$, on the lattice [8,9]. The dynamical quark mass $M(p^2)$ is related to the nonlocal quark condensate $Q(p^2)$ via Eq. (11).

We shall now explore the consequences of Eq. (11) and try to invert it in order to obtain the quark mass as a function of the condensate (assumed to be known from the QCD sum rules [7] or lattice simulations [8]). A few implications of (11) are immediate. We can see that due to the rapid decrease of the profiles $Q(p^2)$ and $M(p^2)$ at large $p^2$ the relation becomes linear asymptotically:

$$M(p^2) = \frac{1}{2}p^2 \frac{Q(p^2)}{p^2} \text{ at large } p^2.$$  

(14)

This is the dilute instanton medium regime, where the single-instanton effects dominate. Next, from (11) it follows that for real valued $M(p^2)$

$$\sqrt{p^2} Q(p^2) \leq 1$$

(15)

for any value of $p^2 > 0$. Thus, in general, one can consider three different situations. First, if the model profile function for the nonlocal condensate $Q(p^2)$ violates the bound (15), then one cannot find a quark-model representation for the results in terms of real valued functions. The other two possibilities, discussed by us below, correspond to the case when the bound (15) is saturated at some points $\sqrt{p^2} \equiv m_i$, or, finally, to the case of the strict inequality for all momenta, $\sqrt{p^2} Q(p^2) < 1$. Let us first consider the last possibility, where we find

$$M(p^2) = \frac{1 - \sqrt{1 - \frac{p^2 Q^2(p^2)}{Q(p^2)}}}{Q(p^2)}.$$  

(16)

The sign of the root is chosen in such way that the large-$p^2$ asymptotics (14) is satisfied. The feature of this solution is that $M(p^2 = 0) = 0$. Indeed, at
small $p^2$ we get
\[
M(p^2) = \frac{1}{2} Q(0)p^2 + O(p^4).
\] (17)

In the case where the bound Eq. (15) is saturated at real valued points $\sqrt{p^2} \equiv m$, the solution can smoothly jump from one branch to another. We illustrate this for the case of a single point, $m_0$. At this point we have the identity
\[
M(m_0^2) = m_0.
\] (18)

In this case the inverse of (11) is
\[
M(p^2) = \frac{1 - \text{sgn}(p^2 - m_0^2) \sqrt{1 - p^2 Q^2(p^2)}}{Q(p^2)}.
\] (19)

The sign function flips the solution from one branch to another at the point $p^2 = m_0^2$. This prescription ensures the continuity of $M(p^2)$ with all its derivatives. Note that now, due to branch-switching, we have $M(0) = \frac{2}{Q(0)} > 0$.

We illustrate the above considerations in the model (12). The condition (18) occurs for
\[
\Lambda_\text{crit}^3 = -\frac{8\pi^2 \langle \bar{q}q \rangle}{\sqrt{2e N_c}},
\] (20)

which numerically gives $\Lambda_\text{crit} = 505\text{MeV}$ if one fixes $\langle \bar{q}q \rangle = -(225\text{MeV})^3$. At $\Lambda > \Lambda_\text{crit}$ we have only one branch. At $\Lambda = \Lambda_\text{crit}$ we have the branch-switching, according to (19). The situation is depicted in Fig. 2.

One determines the values of model parameters, such as $\Lambda$, by fitting observables. In quark models one typically fits the pion decay constant, $F_\pi$, to its experimental value, which in the chiral limit equals $86\text{ MeV}$ [10]. The expression for $F_\pi$ is [6,11]
\[
F_\pi^2 = \frac{N_c}{4\pi^2} \int_0^\infty du \frac{M(u)^2 - uM(u)M'(u) + u^2M'(u)^2}{(u + M(u)^2)^2},
\] (21)
or it may be rewritten in terms of $Q$ as:
\[
F_\pi^2 = \frac{N_c}{64\pi^2} \int_0^\infty du \frac{4Q(u)^2 + 4uQ(u)Q'(u) + 4u^2Q'(u)^2 - 3uQ(u)^4}{1 - 4uQ(u)^2}.
\] (22)

Above, $M'(u) = \frac{d}{du}M(u)$ and $Q'(u) = \frac{d}{du}Q(u)$. Figure 3 shows $F_\pi$ evaluated with (12), plotted as a function of $\Lambda$. We note that the experimental value of $F_\pi$ favors $\Lambda \approx 0.55\text{ GeV}$, comfortably above the critical value, such that indeed we have a one-branch situation and, consequently, (1). The average quark
Fig. 2. Dynamical quark mass for the nonlocal condensate given by Eq. (12), plotted as a function of the square of the Euclidean momentum expressed in units of the cut-off parameter $\Lambda$. The curve labeled $\Lambda > \Lambda_{\text{crit}}$ satisfying condition (1) corresponds to $\Lambda = 554$ MeV, which fits the value of the pion decay constant in the chiral limit, $F_\pi = 86$ MeV. The curve labeled $\Lambda = \Lambda_{\text{crit}}$ displays the branch-switching. At high momenta the solution is given by the solid line, and at low momenta by the dashed line.

Fig. 3. $F_\pi$ as a function of $\Lambda$ (solid line) and the experimental value of 86 MeV (dashed line).

virtuality (13) is estimated as $\lambda_q^2 = 0.6$ GeV$^2$ and fits the value obtained in the QCD sum rules $\lambda_q^2 = 0.5 \pm 0.1$ GeV$^2$ [12]. Another interesting feature of ansatz (12) is that the quark propagator does exhibit analytical confinement of quarks. This is because the equation $p^2 = M^2(-p^2)$ has no real solution at positive (Minkowskian) $p^2$.

The important feature of effective models is the compliance to symmetries, in particular to the chiral symmetry. The consistency of quark models with nonlocal regulators with symmetries has been shown in Ref. [11] and we are not going to repeat it here. Since the proofs never use the explicit form of $M(p^2)$,
the results are true for any mass function, and thus also hold for functions with the property (1).

Finally, we examine the analytic properties of the pion propagator. The purpose here is to check whether the vanishing quark mass does not induce an infrared cut, which would be unphysical. In nonlocal quark models the inverse pion propagators is given by [11,14–16]

\[
K_{\pi}^{-1}(q) = \text{const} \int \frac{d^4p}{(2\pi)^4} \times \left[ \frac{M(p + \frac{q}{2})M(p - \frac{q}{2})}{(p^2 - \frac{q^2}{4} + M(p + \frac{q}{2})M(p - \frac{q}{2}))} - \frac{M(p)^2}{p^2 + M(p)^2} \right].
\]

(23)

It is easy to verify that the mass factors \(M(p + \frac{q}{2})M(p - \frac{q}{2})\) in the numerator cancel the infrared divergence from the denominator, and as a result the pion propagators has no cuts at low \(q\). Its analytic structure in the low-energy region consists of the pole at \(q^2 = 0\), as it should be. If it were not for the presence of \(M(p + \frac{q}{2})M(p - \frac{q}{2})\) in the numerator, expression (23) would have developed a cut in the \(q^2\) variable reaching from 0 to \(-\infty\). Analogous result holds for the \(\sigma\)-field propagator.

We conclude that the scenario of a vanishing dynamical quark mass at zero quark virtuality is possible, as well as free of pathologies. In forthcoming work we are going to study the processes that are sensitive to infrared behaviour of nonlocal nonperturbative propagator. In particular, in [17] it was shown that the pion light-cone wave function in the middle region is related to derivative of the vacuum nonlocality in the infrared region. Finally, the models discussed in this paper lead to, in addition to Eq. (1), analyticity of \(M(p^2)\) in the vicinity of zero. This is not the case of, e.g., standard zero modes, where the function develops a cut at \(p^2 = 0\), extending along the whole Minkowski region. This nonanalyticity becomes an obstacle in soliton calculations in such models [18], where other regulators must be used, such that they can be continued to the Minkowski momenta near the origin. Models discussed in this paper are free of that problem.

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References


