QCD$_2$ IN THE AXIAL GAUGE REVISITED

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Abstract. The ’t Hooft model for the two-dimensional QCD in the limit of infinite number of colours is studied in the axial gauge. The mass-gap and the bound-state equations are derived using the two consequent Bogoliubov-like transformations. Chiral properties of the model are studied in the Hamiltonian and matrix approaches to the latter. Special attention is payed to the explicit pionic solution of the bound-state equation.

The model for the two-dimensional QCD in the limit of infinite number of colour was suggested in 1974 [1] and it is a marvelous playground for testing methods and approaches used in QCD$_4$. It is described by the Lagrangian

\[ L(x) = -\frac{1}{4} F^a_{\mu\nu}(x) F^{a\mu\nu}(x) + \bar{q}(x)\gamma_\mu \partial_\mu - g A^a_\mu t^a \gamma_\mu - m q(x), \]  

whereas the large $N_C$ limit implies that $\gamma \equiv \frac{g^2 N_C}{4\pi}$ remains finite, so that a nontrivial set of diagrams (planar diagrams) appear to be of the same order in $N_C$ and should be summed.

Following [2, 3] we consider the model (1) in the axial gauge $A^a_1(x_0, x) = 0$ and use the principal-value prescription to regularize the infrared divergences. We also define the dressed quark field

\[ q_{\alpha i}(t, x) = \int \frac{dk}{2\pi} e^{ikx} [b_{\alpha}(k, t) u_i(k) + d^+_{\alpha}(-k, t) v_i(-k)], \]  

\[ \{b_{\alpha}(t, p)b^+_{\beta}(t, q)\} = \{d_{\alpha}(t, p)d^+_{\beta}(t, -q)\} = 2\pi \delta(p - q) \delta_{\alpha\beta}, \]  

\[ u(k) = T(k) \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad v(-k) = T(k) \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad T(k) = \exp \left[ -\frac{1}{2} \theta(k) \gamma_1 \right], \]  

where the Bogoliubov angle $\theta(p)$ is the solution to the mass-gap equation. To derive the latter, we organize the normal ordering of the Hamiltonian in the new basis,

\[ H = LN_C E_v + : H_2 : + : H_4 :, \]  

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and demand that the quadratic part be diagonal in terms of the dressed-quark creation and annihilation operators. This implies that $\theta$ is a solution to the following equation:

$$p \cos \theta(p) - m \sin \theta(p) = \frac{\gamma}{2} \int \frac{dk}{(p-k)^2} \sin[\theta(p) - \theta(k)],$$

(6)

and the dressed-quark dispersive law is

$$E(p) = m \cos \theta(p) + p \sin \theta(p) + \frac{\gamma}{2} \int \frac{dk}{(p-k)^2} \cos[\theta(p) - \theta(k)].$$

(7)

An alternative way to derive the mass-gap equation (6) is to minimize the vacuum energy (the first term on the r.h.s. in (5)),

$$E_v = \int \frac{dp}{2\pi} Tr \left\{ \gamma_5 p \Lambda_-(p) + \frac{\gamma}{4\pi} \int \frac{dk}{(p-k)^2} \Lambda_+(k) \Lambda_-(p) \right\},$$

(8)

with the projectors being $\Lambda_\pm(p) = T(p) \frac{1 \pm \gamma_0}{2} T(p)$.

In the meantime, one should be extremely careful with the divergences which show up in the second term of (8). Indeed, defining the difference between the vacuum energy of a nontrivial vacuum and the free-theory one,

$$\Delta E_v[\theta] = E_v[\theta] - E_v[\theta_{\text{free}}], \quad \theta_{\text{free}}(p) = \frac{\pi}{2} \text{sign}(p),$$

(9)

one can use the following trick [4]. If $\theta(p)$ is the function which provides the minimum of the vacuum energy (9), then the function with the same profile, but stretched with an arbitrary parameter $A$, $\theta(p) \to \theta(p/A)$, should enlarge the energy. A simple analysis demonstrates, that this is, indeed, the case, since $\Delta E_v$ has the following behaviour as a function of $A$:

$$\Delta E_v = \frac{1}{2} C_1 A^2 - \gamma C_2 \ln A + \gamma C_3$$

(10)

with $C_{1,2,3}$ being positive constants. The logarithm in the second term on the r.h.s. of (10) is due to the infrared divergence of the integral in (8), and the constant $C_3$ contains the logarithm of the cut-off. The function (10) has a minimum for a nonzero $A$ and, what is more, there is no minimum for $A = 0$ provided $\gamma \neq 0$. The latter fact means that there is no phase of the theory with the angle $\theta = \frac{\pi}{2} \text{sign}(p)$ found in [2], which leads to zero chiral condensate and unbroken chiral symmetry. Thus the 't Hooft model has the only non-trivial vacuum which provides spontaneous breaking of the chiral symmetry and which is defined by the numerical solution of equation (6) found in [5]. Formula (10) can be written in a more physically transparent form if one notices that the chiral condensate transforms linearly under the above-mentioned
transformation (\(\Sigma \equiv \langle \bar{q}q \rangle \rightarrow A\Sigma\)), so that one can use \(\Sigma\) instead of \(A\):

\[
\Delta \mathcal{E}_v = C_1 \left[ \frac{1}{2} (\Sigma/\Sigma_0)^2 - \ln |\Sigma/\Sigma_0| \right] + \gamma C_3
\]

(11)

with \(\Sigma_0\) being the real condensate of the model which will be discussed below.

From the original paper [1] it is known that the "physical" degrees of freedom of the model are the quark-antiquark mesons. The operators creating and annihilating mesonic states can be defined via the dressed quark states as [6]

\[
m_n = \int \frac{dq}{2\pi\sqrt{N_C}} \left\{ d_i(P-q)b_i(q)\varphi_n^+(q, P) + b_i^+(q)d_i^+(P-q)\varphi_n^+(q, P) \right\},
\]

(12)

where the two wave function appeared for each meson. The "+" function describes the motion of the \(q\bar{q}\) pair in meson forward in time, whereas the "-" one describes the backward motion. The orthogonality and completeness conditions for \(\varphi\)’s contain the negative sign between the "+" and the "-" parts, e.g.,

\[
\int \frac{dp}{2\pi} \left( \varphi_+^n(p, Q)\varphi_-^m(p, Q) - \varphi_-^n(p, Q)\varphi_+^m(p, Q) \right) = \delta_{nm},
\]

\[
\int \frac{dp}{2\pi} \left( \varphi_+^n(p, Q)\varphi_-^m(p, Q) - \varphi_-^n(p, Q)\varphi_+^m(p, Q) \right) = 0.
\]

(13)

This sign is a consequence of the second Bogoliubov-like transformation which is to be performed in the theory to find the form (12) of the mesonic operators, so that the two components of the mesonic wave function play the role of the standard Bogoliubov amplitudes \(u\) and \(v\) [6].

The Hamiltonian of the model takes the diagonal form in the new basis,

\[
H = LN_C\mathcal{E}_v + \sum_{n=0}^{+\infty} \int \frac{dp}{2\pi} P_0^n(P)m_n^+(P)m_n(P) + O\left(\frac{1}{\sqrt{N_C}}\right),
\]

(14)

if the wave functions obey the bound-state equation in the form of a system of two coupled equations [2,3]:

\[
\begin{aligned}
[K(p, P) - P_0] \varphi_+(p, P) &= \gamma \int \frac{dk}{(p-k)^2} [C\varphi_+(k, P) - S\varphi_-(k, P)] \\
[K(p, P) + P_0] \varphi_-(p, P) &= \gamma \int \frac{dk}{(p-k)^2} [C\varphi_-(k, P) - S\varphi_+(k, P)],
\end{aligned}
\]

(15)

with \(K(p, P) = E(p) + E(P - p), C = \cos \frac{\theta(p) - \theta(k)}{2} \cos \frac{\theta(P-p) - \theta(P-k)}{2}\) and \(S = \sin \frac{\theta(p) - \theta(k)}{2} \sin \frac{\theta(P-p) - \theta(P-k)}{2}\). Note that the system (15) can be written in the form of a Dirac-type equation in the Hamiltonian form,

\[
\hat{\mathcal{H}}\psi = Q_0\psi,
\]

\[
\hat{\mathcal{H}} = \left( \begin{array}{cc} K - \hat{C} & \hat{S} \\ -\hat{S} & -K + \hat{C} \end{array} \right) = \gamma_0(K - \hat{C}) + \gamma_1\hat{S},
\]

(16)
where for an arbitrary function $F(p, P)$ operators $\hat{S}$ and $\hat{C}$ act as

$$\hat{C}(\hat{S})F(p, P) \equiv \gamma \int \frac{dk}{(p-k)^2} C(S)(p, k, P)F(k, P).$$  \hspace{1cm} (17)

The operator $\hat{H}$ is not Hermitian, and the distorted form of the norm (13) is a remnant of this fact. Nevertheless the positiveness of its eigenvalues can be proved explicitly [7].

Using the Hamiltonian approach to the model developed above, one can study its chiral properties, among which we mention

- The chiral pion — the exact massless (in the chiral limit) solution of the bound-state equation (15) [7]:
  $$\varphi_{\pm}^\pi(p, Q) = \sqrt{\frac{\pi}{2Q}} \left( \cos \left( \frac{\theta(Q-p) - \theta(p)}{2} \right) \pm \sin \left( \frac{\theta(Q-p) + \theta(p)}{2} \right) \right).$$  \hspace{1cm} (18)

- The pion decay constant, which can be defined for the above-mentioned state ($|\Omega\rangle$ is the vacuum annihilated by mesonic operators (12)):
  $$\langle \Omega | \bar{q}(x)\gamma_\mu \gamma_5 q(x) | \pi(Q) \rangle = f_\pi Q_\mu \frac{e^{-iQx}}{\sqrt{2Q_0}}, \quad f_\pi = \sqrt{\frac{N_C}{\pi}}.$$  \hspace{1cm} (19)

- Partial conservation of the axial-vector current, which holds true in the operator form,
  $$J_5^\mu(x) = i f_\pi \partial^\mu \Psi_\pi(x), \quad f_n = f_\pi \delta_{n\pi}.$$  \hspace{1cm} (20)

- The chiral condensate,
  $$\langle \bar{q} q \rangle (m = 0) = -N_C \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \cos \theta(p) = -0.29 N_C \sqrt{2\gamma}.$$  \hspace{1cm} (21)

- The Gell-Mann-Oakes-Renner relation,
  $$f_\pi^2 M_\pi^2 = -2m \langle \bar{q} q \rangle,$$  \hspace{1cm} (22)

which can be checked explicitly.

Now we turn to the properties of currents in the 't Hooft model. The conservation laws can be derived for both, vector and axial-vector, currents in the chiral limit, which read:

$$V_\mu^M(P) = \langle \Omega|\bar{q}_\gamma \mu q|M, P\rangle, \quad P_0^M V_0^M - PV^M = 0.$$  \hspace{1cm} (23)
\[ A^M_\mu(P) = \langle \Omega|\bar{q} \gamma_\mu \gamma_5 q|M,P\rangle, \quad P_0^M A^M_\mu - PA^M = 0. \] (24)

For the current-quark-antiquark vertices one has the vector and the axial-vector Ward identities [3,7,8]

\[ -iP_\mu v_\mu(p,P) = S^{-1}(p) - S^{-1}(p - P), \] (25)
\[ -iP_\mu a_\mu(p,P) = S^{-1}(p)\gamma_5 + \gamma_5 S^{-1}(p - P), \] (26)

and, as a result, the following conservation laws:

\[ (P_\mu - P'_\mu)(M,P|v_\mu|M',P') = 0, \] (27)
\[ (P_\mu - P'_\mu)(M,P|a_\mu|M',P') = 0. \] (28)

In derivation of the latter formulae the matrix approach to the model was used [7,8], in which an effective diagrammatic technique is defined containing the matrix form of the wave function obeying the matrix bound-state equation, the dressed quark propagator \( S(p) \), the dressed meson-quark-antiquark vertex \( \Gamma(\bar{\Gamma} \text{ for outgoing meson}) \), the dressed quark-quark scattering amplitude, and, finally, the effective coupling constant \( -i\gamma/\sqrt{N_C} \), which is to be prescribed to each meson-quark-antiquark vertex. For example, for the pion one can find the following vertex function:

\[ \Gamma_\pi(p,P) = S^{-1}(p)(1 + \gamma_5) - (1 - \gamma_5)S^{-1}(p - P), \] (29)

which can be easily identified with the divergences of the vector and axial-vector currents (25) and (26):

\[ \Gamma_\pi(p,P) = -iP_\mu v_\mu(p,P) - iP_\mu a_\mu(p,P). \] (30)

It is not surprise that the pion couples to both, the vector and the axial-vector currents, since in two dimensions the two currents are dual to one another. The interested reader can find details of the approach in the review paper [3].

As a next application of the Hamiltonian and the matrix approaches discussed above, let us study hadronic decays in this theory. If meson \( A \) decays into mesons \( B \) and \( C \), then the amplitude of such a process can be found in two ways: in the Hamiltonian technique,

\[ M(A \rightarrow B + C) = \langle B(P_B)C(P_C)|\Delta H|A(P_A)\rangle, \] (31)
or using the matrix approach, directly from the corresponding decay diagrams:

\[ M(A \rightarrow B + C) = -\frac{i\gamma^3}{\sqrt{N_C}} \int \frac{d^2k}{(2\pi)^2} Sp[\Gamma_A(k + P_B, P_A)S(k - P_C) \times \bar{\Gamma}_C(k, P_C)S(k) \bar{\Gamma}_B(k + P_B, P_B)S(k + P_B)] + (B \leftrightarrow C). \] (32)
Both equations, (31) and (32), lead to one and the same 6-term form of the amplitude (see [3, 7] for details), which possesses contributions of both components of the wave function for each meson. If one of the final states is the pion and the chiral limit is used, then the explicit form of the vertex (29) can be substituted into (32) that immediately leads to the result that the amplitude vanishes,

\[ M(A \to \pi + C) \equiv 0, \quad (33) \]

which also follows from the identification (30) and the vector and axial-vector conservation laws.

The result (33) could be anticipated, since the pion decay constant is dimensionless in the ’t Hooft model. As a result, the Adler selfconsistency condition for the amplitudes with pions involved [9] has to hold true for any pion momentum.

In conclusion let us say, that the two-dimensional ’t Hooft model possesses many features which are known to be inherent to QCD and it can be used as a test laboratory to see how all these beautiful properties may appear.

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