We show, under general assumptions which are well satisfied in relativistic heavy-ion collisions, that the geometric relation of centrality $c$ to the impact parameter $b$, namely $c \approx \pi b^2/\sigma_{\text{inel}}$, holds to a very high accuracy for all but most peripheral collisions. More precisely, if $c(N)$ is the centrality of events with the multiplicity higher than $N$, then $b$ is the value of the impact parameter for which the average multiplicity $\bar{n}(b)$ is equal to $N$. The corrections to this geometric formula are of the order $(\Delta n(b)/\bar{n}(b))^2$, where $\Delta n(b)$ is the width of the multiplicity distribution at a given value of $b$, hence are very small. In other words, the centrality effectively measures the impact parameter.

$$c(N) \approx \frac{\pi b^2}{\sigma_{\text{inel}}}, \quad \text{for } b < \bar{R} \
$$

(1)

where $\sigma_{\text{inel}}$ is the total inelastic nucleus-nucleus cross section, and $\bar{R}$ is of the order of the sum of the radii of the colliding nuclei. The centrality $c(N)$ is the centrality of events with the multiplicity higher than $N$, while $b(N)$ is the value of the impact parameter for which the average multiplicity $\bar{n}(b)$ is equal to $N$. As will be shown, Eq. (1) holds to a high accuracy for all but most peripheral collisions. Note that it is geometric in nature, and does not involve explicitly the variable $n$ needed to categorize the data (multiplicities, number of participants, number of binary collisions, etc.). At first glance, this fact may seem a bit surprising.

One can explain the geometric nature of (1), and the fact that it does not explicitly depend on $n$, with the following pedagogical example. Consider a competition where archers are shooting at a target of radius $R$, each of them once. The archers are very poor, such that they shoot randomly. They are paid accordingly to their aim: more central, higher reward. We are not allowed to watch the competition, hence do not know which spot on the target has been hit, but later we review the reward records. Suppose a large number of archers scored (here we take only 10 in order to write down the results explicitly), and are ranked according to their prizes, which are: 100$, 100$, 50$, 50$, 50$, 10$, 10$, 10$, 10$, 10$. The two archers that got the highest prize (100$ in this case) had to hit the bull’s eye. Since these are the 20% of all archers, and they were shooting randomly, we can immediately determine (neglecting the statistical error) the radius $b$ of the bull’s eye, since 20% is the ratio of the area of the bull’s eye to the total area of the target: $20\% = \pi b^2/\pi R^2$. Therefore $b = R \sqrt{20\%}$. Now, imagine another competition is held, with all rules the same but the prizes differently assigned to the rings of the target. Suppose the ten archers got 500$, 500$, 100$, 100$, 100$, 50$, 50$, 50$, 50$, 50$. Again, we can determine that the 20% of the highest rewards correspond to hitting the central spot, and can determine its radius exactly $b$ as before. Note that in the determination of $b$ we are not using the actual values of the rewards at all – the function used can be any monotonic function of the centrality. The rewards are only used to categorize the data. Once this is done, we can identify the $c$ “most central” archers and determine $b$ according to Eq. (1), irrespectively of the function used for categorizing. Our example can be translated into heavy-ion collisions in the following way: archery competition – heavy-ion experiment, archer that scored – event, rewards in competition I – number of participants, rewards in competition II – multiplicity of produced particles, percentile of highest-scoring archers – centrality, radii of rings on the target – impact parameters.
The above example shows the essence of our argument, valid for the classical physics of relativistic heavy-ion collisions. There are, however, two additional features which need to be considered. First, a collision at a particular impact parameter $b$ produces values of $n$ which are statistically distributed around some mean value $\bar{n}(b)$ with a distribution width $\Delta n(b)$. As we will show, Eq. (1), formally valid at $\Delta n(b) \ll \bar{n}(b)$, is accurate even for realistically large $\Delta n(b)$, such as obtained from statistical models of particle production. Second, there are boundary effects near $b \sim R$ — at lower values of $b$ the inelastic cross section is the cross section for colliding black disks, whereas at the boundary the target gradually becomes transparent.

We now proceed with a formal derivation. Let $P(n)$ denote the probability of obtaining value $n$ for the categorizing function (multiplicity of produced particles, number of participants, number of binary collisions, etc.). For simplicity of the language we call it the multiplicity, bearing in mind it could be any of these quantities. The centrality $c$ is defined as the cumulant of $P(n)$, namely

$$c(N) = \sum_{n=N}^{\infty} P(n).$$

Thus $c(N)$ is the probability of obtaining an event with multiplicity larger or equal to $N$. A particular value of multiplicity $n$ may be collected from collisions with various impact parameters $b'$, thus we can write

$$c(N) = \sum_{n=N}^{\infty} \int_{a}^{b} 2\pi b'db' \rho(b') P(n|b'),$$

where $2\pi b'db'$ is the area of the ring between impact parameters $b'$ and $b'+db'$, the quantity $\rho(b')$ is the probability of an event (inelastic collision) at impact parameter $b'$, and $P(n|b')$ is the conditional probability of producing multiplicity $n$ provided the impact parameter is $b'$. The function $\rho(b')$ is unity for $b'$ below $R$, and drops smoothly to zero at $b'$ around $R$, reflecting the washed-out shape of the nuclear density functions at the edges. The interpretation of Eq. (3) is clear: the probabilities for hitting the ring between $b'$ and $b'+db'$, the probability for an event to occur at $b'$, and the probability to produce multiplicity $n$ (provided the event occurred at $b'$) are multiplied, as requested by the classical nature of the problem. Since we have $\sum_{n=0}^{\infty} P(n|b') = 1$, and, by definition, $\int_{a}^{b} 2\pi b'db' \rho(b') = \sigma_\text{inel}$, we verify the proper normalization in Eq. (3), namely $c(1) = 1$. Furthermore, for heavy nuclei we may use the continuity limit, $\sum_{n=N}^{\infty} \int_{a}^{b} 2\pi b'db' \rho(b') = \int_{a}^{b} \rho(b') |db'|$.

The function $P(n|b')$ is not known, yet, by the statistical nature of the particle production, and by experience of various models, we expect that for large values of $n$ it is narrowly peaked around an average value $\bar{n}(b')$. Thus we begin our study by taking the limit of an infinitely-narrow distribution, $P(n|b') = \delta(n - \bar{n}(b'))$. In this case

$$c(N) = \int_{\bar{n}(b')}^{\infty} dn \frac{\rho(b')}{\sigma_\text{inel}} \delta(n - \bar{n}(b')) = \int_{0}^{\infty} 2\pi b'db' \rho(b') \theta(\bar{n}(b') - N).$$

Since $\bar{n}(b')$ is a monotonically decreasing function of $b'$, we have $\theta(\bar{n}(b') - N) = \theta(b'(N - b'))$, where $b(N)$ is the solution of the equation $\bar{n}(b) = N$. Therefore

$$c(N) = \int_{0}^{\infty} 2\pi b'db' \rho(b') \theta(b(N) - b') = \int_{0}^{\infty} 2\pi b'db' \rho(b') = \sigma_\text{inel}(b(N)),$$

where $\sigma_\text{inel}(b(N))$ is the inelastic cross section accumulated from $b' \leq b(N)$. Equation (5) is a generalization of formula (1). In Ref. [11] it has been quoted in the context of the Glauber model. We notice that although $c$ and $b$ depend implicitly on $N$, their relation does not explicitly involve $N$.

We now turn to a quantitative analysis of dispersion effects. Assume

$$P(n|b') = \frac{1}{\Delta n(b') \sqrt{2\pi}} \exp \left( -\frac{(n - \bar{n}(b'))^2}{2\Delta n(b')^2} \right),$$

which is a good approximation for $\Delta n(b') < \bar{n}(b')$. Then

$$c(N) = \int_{0}^{\infty} 2\pi b'db' \rho(b') \left\{ \frac{1}{2} \left[ \text{erf} \left( \frac{\bar{n}(b') - N}{\sqrt{2\Delta n(b')}} \right) + 1 \right] \right\},$$

For small $\Delta n(b')$ the function in curly brackets resembles the function $\theta(\bar{n}(b') - N)$, washed out over the range $\Delta n(b')$. Thus, we introduce the function

$$d(x) = \frac{1}{2} \left[ \text{erf} \left( \frac{x}{\sqrt{2\Delta n}} \right) + 1 \right] - \theta(x).$$

The integral of $d(x)$ with a regular function $f(x)$ can be expanded in even powers of $\Delta n$ as follows (this is analogous in spirit to the Sommerfeld expansion of the Fermi-Dirac distribution function at low temperatures):

$$\int dx f(x)d(x) = - \sum_{j=1,3,5,...} a_j (\Delta n)^{j+1} \left. \frac{d^j f(x)}{dx^j} \right|_{x=0},$$

with the coefficients

$$a_{j+1} = \frac{1}{j!} \int_{-\infty}^{\infty} dx x^j d(x) = \frac{2^{j+3}/\Gamma(\frac{5}{2} + 1)}{\sqrt{\pi}(j + 1)!}, a_1 = 1, a_3 = \frac{1}{4}, a_5 = \frac{1}{48}, a_7 = \frac{1}{576}, ...$$

We rewrite the integral in Eq. (7) as $\int 2\bar{n}'db' = \int d\bar{n} db'^2/d\bar{n}$, and use expansion (9) to obtain
Again, the model curves for \( c(N) \) proportional to \( d^2(\tilde{b}(\tilde{n})) / d\tilde{n}^2 \). In the models considered below this quantity is proportional to \( 1 / \tilde{n}^2 \), and as a result \( c(N) = \sigma_{inel}(b(N)) / \sigma_{inel} + O(\Delta n^2 / \tilde{n}^2) \), quantitatively showing that the geometric identification (1), or (5), is good for narrow distributions.

In order to illustrate the above results and to obtain more detailed numerical estimates for the corrections we consider two models: a model inspired by the wounded-nucleus model [12], and the optical limit of the Glauber model [13] for the binary collisions. A combination of these models has been used to explain the observed hadron multiplicities produced in RHIC [14]. We look at the \( Au + Au \) reaction, with the nucleus density profile \( \rho_A(r) \) described by the standard Woods-Saxon function with the radius \( r_0 = (1.12A^{1/3} - 0.86A^{-1/3}) \text{fm} \), with \( A = 197 \), and the width parameter \( a = 0.54 \text{fm} \). The nucleus-nucleus thickness function is given by \( T_A(s) = \int_{-\infty}^{\infty} dz \rho_A(\sqrt{s^2 + z^2}) \), and the average number of wounded nucleons is

\[
\bar{n}(b) = 2A \int_0^{\infty} dsd \varphi T_A(\sqrt{s^2 + b^2 + 2sb \cos \varphi}) \times (1 - (1 - \sigma T_A(s))^4),
\]

where, following Ref. [14], we take \( \sigma = 40 \text{mb} \) as the nucleon-nucleon inelastic cross section. The total nucleus-nucleus cross section obtained in this model is \( \sigma_{inel} = 7.05 \text{mb} \). The expressions for the dispersion of wounded nucleons produced at a given \( b \) is very complicated. Instead of computing multidimensional integrals, we explore, for our illustrative purpose, two cases: \( \Delta n \sim \bar{n} \), and \( \Delta n \sim \sqrt{\bar{n}} \). Led by the sample numerical results for the distributions given in Fig. 1 of Ref. [15], we take i) \( \Delta n = \bar{n}/10 \), or ii) \( \Delta n = \sqrt{\bar{n}} \). In Fig. 1 we show the results of computing \( c(N) \) according to Eqs. (7,12) with \( \rho(b') = \theta(\sqrt{\sigma_{inel}/\pi - b'}) \), and for the choices i) and ii) (dot-dashed and dashed lines, respectively). These are compared to \( \pi b(N)/\sigma_{inel} \) (solid line), where \( b(N) \) is defined as the solution of the equation \( \bar{n}(b) = N \). The curves overlap within the width of the line, except for tiny regions at very low \( N \), \( (N < 2) \), corresponding to very peripheral collisions, and at large \( N \), corresponding to \( b \) around 0. The discrepancy at large \( N \) follows from the fact that \( c(N) \) evaluates exactly continues to be non-zero till the maximum value of wounded nuclei, \( N = 2A \), whereas \( b(N) \) by construction goes to zero at \( N = \bar{n}(b = 0) \approx 377 \). This effect is visible in Fig. 1 only for the choice i) for the widths.

We can treat the dependence on \( N \) as parametric, and plot \( c(b(N)) vs. b(N) \). The results is shown in Fig. 2(a). Again, the model curves for \( c(b) \) for choices i) and ii) overlap with the curve \( \pi b^2 / \sigma_{inel} \) except for very peripheral \( (b > 14 \text{fm}) \) and very central \( (b < 2 \text{fm}) \) collisions. This behavior directly reflects the behavior of Fig. 1. The size of the correction of Eq. (11) is, at intermediate \( b \), of the order of \( 10^{-3} \).

As another illustrative example we consider the Glauber model of nucleus-nucleus collisions and analyze binary collisions, \( n = n_{coll} \). We use the optical limit of the model, which results in simple expressions. In this model

\[
c(N) = \sum_{n=N}^{A^2} \frac{2\pi b'db'}{\sigma_{inel}} P_G(n, b') = \sum_{n=N}^{A^2} \int_0^{\infty} \frac{2\pi b'db'}{\sigma_{inel}} \left( \frac{A^2}{n} \right) [T(b') \sigma]^n [1 - T(b') \sigma]^{A^2-n},
\]

where for \( P_G(n, b') \) we have used the formula for the probability of the occurrence of \( n \) inelastic baryon-baryon collisions at an impact parameter \( b' \) [13] (note that \( P_G \) plays the role of the product \( \rho(b')P(n|b') \) from the previous discussion). Here \( T(b) \) is the nucleus-nucleus thickness function,

\[
T(b) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dz_A \int_{-\infty}^{\infty} dz_B \int_0^{2\pi} d\varphi \times \rho(\sqrt{s^2 + z_A^2}) \rho(\sqrt{s^2 + b^2 + 2sb \cos \varphi + z_B^2}).
\]

The sum in Eq. (13) can be carried out exactly, yielding, with the notation \( x = T(b') \sigma \), the expression

\[
c(N) = \int_0^{\infty} \frac{2\pi b'db'}{\sigma_{inel}} \left( \frac{A^2}{n} \right) \times \frac{(1 - x)^{A^2 - N} x^N 2F_1(1, N - A^2; N + 1; x)}{x - 1}).
\]

We perform the integration in Eq. (15) numerically. On the other hand, the average number of collisions at a fixed value of the impact parameter \( b \) is \( \bar{n}(b^2) = A^2 T(b) \sigma \).
Replacing the sum by the integral in (16) we find

\[ I_{\text{incomplete beta function}} = \beta \text{ and } n_{\text{coll}} \text{ with the distribution of particles produced in an elementary event (by the wounded nucleon or in a single binary collision), may result in a broadening effect in the observed distribution of the multiplicity of the produced particles, } n. \text{ However, we expect this broadening to be negligible in the ratio } \Delta n/\bar{n}, \text{ which is the quantity controlling the accuracy of Eq. (1). In particular, for the wounded nucleon model [15] one has } (\Delta n/\bar{n})^2 = 2 (\Delta H_H)^2/(\bar{n}_H \bar{n}_H^2) + (\Delta n_w/\bar{n}_w)^2, \text{ where the subscript } H_H \text{ refers to the nucleon-nucleon collision. Assuming } (\Delta H_H)^2 \sim \bar{n}_H, \text{ we find that the contribution from the first term is smaller than from the second term already for moderately large } \bar{n}_w, \text{ and } \Delta n/\bar{n} \approx \Delta n_w/\bar{n}_w. \text{ This indicates that Eq. (1) remains very accurate when multiplicities of produced particles are used as the centrality criterion.}

We wish to thank Andrzej Białaś, Andrzej Budzanowski, Wiesław Czyż, Roman Holýński, Pasi Huovinen, and Kacper Zalewski for useful discussions.

* Supported by the Polish State Committee for Scientific Research, grant 2 P03B 90419.