Positivity and conservation of superenergy tensors

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Two essential properties of energy–momentum tensors $T_{μν}$ are their positivity and conservation. This is mathematically formalized by, respectively, an energy condition, as the dominant energy condition, and the vanishing of their divergence $\nabla^a T_{μν} = 0$. The classical Bel and Bel-Robinson superenergy tensors, generated from the Riemann and the Weyl tensor respectively, are rank-4 tensors. But they share these two properties with energy momentum tensors: the Dominant Property (DP) and the divergence–free property in absence of sources (vacuum). Senovilla $^{2,3}$ defined an universal algebraic construction which generates a basic superenergy tensor $T\{A\}$ from any arbitrary tensor $A$. In this construction the seed tensor $A$ is structured as an $r$-fold multivector, which can always be done. The most important feature of the basic superenergy tensors is that they satisfy automatically the DP, independently of the generating tensor $A$. In $^8$ we presented a more compact definition of $T\{A\}$ using the $r$–direct Clifford algebra $\bigotimes^r \mathcal{Cl}_{p,q}$. This form for the superenergy tensors allowed to obtain an easy proof of the DP valid for any dimension. In this paper we include this proof. We explain which new elements appear when we consider the tensor $T\{A\}$ generated by a non degree–defined $r$-fold multivector $A$ and how orthogonal Lorentz transformations and bilinear observables of spinor fields are included as particular cases of superenergy tensors. We find some sufficient conditions for the seed tensor $A$, which guarantee that the generated tensor $T\{A\}$ is divergence–free. These sufficient conditions are satisfied by some physical fields, which are presented as examples.

1 Introduction

The source of gravitation in General Relativity is the energy–momentum tensor $T_{μν}$ of the matter content of the space-time. The energy–momentum tensor of most physical fields satisfies (and is expected to satisfy) three well-known properties, which are consequence of its definition from the fields involved and of the differential equations governing these fields. It is generally symmetric, $T_{μν} = T_{(μν)}$, and it is locally conserved, i.e. divergence–free $\nabla^a T_{μν} = 0$, in absence of sources or when we consider the total energy–momentum. These two properties are automatically satisfied by the Einstein tensor $S_{μν}$ of any pseudo–Riemannian manifold. Thus, they also can be seen as a consequence or requirement of Einstein equations $S_{μν} = 8\pi G T_{μν}$. The third property of $T_{μν}$ is its ‘positivity’. For most fields, $T_{μν}$ satisfies the dominant energy condition, that is, the energy flux or momentum vector $^a j_μ = -T_{μν} u^ν$, measured by any future–pointing causal observer $u$, is also a future–pointing causal vector. When it is not satisfied, there arise problems of interpretation or rules of selection, as for the Tetrode tensor of a Dirac field. Indeed, this positivity condition (or another as the weak or the strong energy condition), is usually required for the Einstein tensor of any physically acceptable space-time.

The name of superenergy was first applied to the Bel-Robinson (BR) and Bel tensors $^1$, which are defined from the conformal Weyl tensor and from the Riemann tensor

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$^a$The sign is signature dependent. It is a minus sign in signature $(-,+,+,...)$ and a plus sign in the opposite.
respectively. The motivation for this name is that they share some properties with energy-momentum tensors. The prefix super appears because they are rank-4 tensors instead of rank-2. Senovilla defined in \(^2\) an algebraic construction which generates a basic superenergy tensors \(T\{A\}\), from an arbitrary seed tensor \(A\). A much more extensive treatment discussing properties and applications is found in \(^3\). This construction unifies in a single procedure the BR and Bel tensors and many energy-momentum tensors from different physical fields. The most important feature of the basic superenergy tensors \(T\{A\}\) is that, independently of the seed tensor \(A\), they automatically satisfy the Dominant Property (DP), which is a generalization of the dominant energy condition for arbitrary rank tensors. This property holds in any dimension, provided that the space has Lorentzian signature, when the DP can be in fact defined.

An important application of these superenergy tensors has been the study of the interchange of superenergy between different fields, including matter content and gravitation \(^4,3^\). The concept in mind is that the total superenergy may be conserved. This possibility would provide a physical relevance to the superenergy tensor of a field, in the same sense that the conservation of energy justifies even its physical ‘existence’. Success has been obtained in some cases, in which a Killing vector exists. Independently of its physical significance, there are important mathematical applications of the basic superenergy tensor \(T\{A\}\). The DP of the BR tensor is essential in the proof of the stability of Minkowski space-time \(^5\) and the fact that \(T\{A\}\) can always be constructed has been used to provide a simple geometric criterion for the causal propagation of any physical field \(A\). \(^6^\)

The original definition of \(T\{A\}\) is formulated \(^3^\) in standard index tensorial notation, but organizing tensors as \(r\)-fold exterior forms. This definition involves \(2^r\) terms, corresponding to each combination of Hodge duals which can be obtained from an \(r\)-fold form. Here, we will use the name \(r\)-fold multivector instead of \(r\)-fold form. Since vectors and forms are considered identified by the metric, the difference in the name is simply a question of tradition. Bergqvist \(^7^\) reformulated \(T\{A\}\) using 2-spinors. The spinorial expression enabled an elegant proof of the DP. However, this proof is only valid for dimension 4. A general proof for arbitrary dimension in tensorial notation was presented in \(^3^\). In \(^8^\) we presented an alternative formulation of \(T\{A\}\) using Clifford algebra. Our expression is much more compact than both, standard–tensorial and spinorial expressions, since it involves a unique term. It was used to give a simple general proof of the DP for any dimension.

In this article we include the Clifford algebra formulation of superenergy tensors and the proof of the DP, presented in \(^8^\). We especially emphasize that this expression is in fact a generalization of the \(T\{A\}\) defined in \(^2,3^\), because it can be applied to non degree–defined \(r\)-fold multivectors. In section 5 we obtain the non trivial new elements and properties which appear from this generalization. In section 6 we explain how orthogonal Lorentz transformations and bilinear observables of spinors are included as particular cases of \(T\{A\}\). A complete study of null cone preserving maps has been recently presented in \(^9^\). There, it is shown that involutory Lorentz transformations are particular cases of superenergy tensors generated by simple \(n\)-vectors. Here, we show that general Lorentz transformations are also particular cases of superenergy tensors, those generated by elements of the group Pin, which are in general inhomogeneous multivectors, being
homogeneous \(n\)-vectors only for the case of involutory transformations.

In section 8 we introduce a general definition of the equation for a principal null direction \(\ell\) of an arbitrary \(r\)-fold multivector \(A\), which is the natural extension of the well-known cases for bivectors (electromagnetic field) and double bivectors (Weyl tensor). We prove the equivalence of this equation with the vanishing of the generated superenergy tensor \(T\{A\}\) when contracted \(2r\) times with the null direction \(\ell\).

Section 9 is concerned with the conservation of \(T\{A\}\). There, we introduce some sufficient conditions on the seed tensor \(A\) which guarantee that the basic superenergy tensor \(T\{A\}\) generated by it, will be divergence-free. These sufficient conditions are implied by the differential equations satisfied by some fields, which we give as examples. These results might be useful to find conserved superenergy tensors combining different fields, since conditions over the seed field \(A\) are easier to apply than over the generated tensor \(T\{A\}\). We consider that our results increase the mathematical relevance of the superenergy tensors, because any field \(A\) of \(r\)-fold multivectors satisfying the sufficient conditions, automatically generates a tensor \(T\{A\}\) which is conserved and satisfies the DP.

2 Multivectors

Let us consider the tangent vector space \(T(M)\) of some real manifold \(M\) of dimension \(d = p + q\) endowed with a metric \(g\) of signature \(p - q\). Via the exterior product, it generates the Grassmann or exterior space \(\Lambda \equiv \Lambda(T(M))\), which is a graded space \(\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_d\), where \(\Lambda_0\) contains the scalars, \(\Lambda_1\) the vectors, \(\Lambda_2\) the bivectors, and in general \(\Lambda_i\) contains the \(i\)-vectors. A general element of the Grassmann space \(A \in \Lambda\) is called a multivector, and it is said to be homogeneous if it is an \(i\)-vector for some degree \(i\), \(A \in \Lambda_i\); otherwise it is said inhomogeneous. A basic operation on multivectors is the degree projection, which projects into some subspace \(\Lambda_i\). It is denoted with angle-brackets:

\[\forall A \in \Lambda,\quad A = \sum_{i=0}^{n} \langle A \rangle_i, \quad \text{where} \quad \langle A \rangle_i \in \Lambda_i\]

To write a multivector \(A \in \Lambda\) in a complete basis of the linear space \(\Lambda\) we will use a multiindex, denoted with a latin capital letter,

\[A = A^I e_I\]

Besides the exterior product, which is metric-independent, the metric \(g\) defines on multivectors another associative \(^{10}\) product, called the Clifford geometric product\(^{11,12,13}\). We will denote the Clifford product by simple juxtaposition. As a simple case, the product of a general multivector \(A \in \Lambda\) with a vector \(b \in \Lambda_1\) can be expanded into inner and exterior product:

\[Ab = A \cdot b + A \wedge b \quad \text{and} \quad bA = b \cdot A + b \wedge A,\]

The Grassmann space \(\Lambda\) endowed with this product constitutes the Clifford geometric algebra \(\mathcal{C}_{p,q}\).
There are one natural linear involution and two linear antiinvolutions, which are independent of the metric but are fixed by the graded structure of $\Lambda$: The main involution, denoted with a star $A^*$, is defined as the involution, $(AB)^* = A^*B^*$ $\forall A, B \in \Lambda$, which changes each vector into its opposite $a^* = -a$ $\forall a \in \Lambda_1$. The reversion, denoted with a tilde $\tilde{A}$, is defined as the antiinvolution, $\tilde{AB} = \tilde{B}\tilde{A}$ $\forall A, B \in \Lambda$, which keeps vectors unchanged $\tilde{a} = a$ $\forall a \in \Lambda_1$. The Clifford conjugation, denoted with an overline $\overline{A}$, is the composition of the reversion and the main involution, $\overline{A} = A^*$. Taking, for instance, a factorizable trivector
\[
(a \wedge b \wedge c)^* = c \wedge b \wedge a = -a \wedge b \wedge c,
\]
\[
\overline{a \wedge b \wedge c} = (-c) \wedge (-b) \wedge (-a) = a \wedge b \wedge c
\]

3 r-fold multivectors

An r-fold multivector $A$ is an element of the direct product of $r$ copies of the Clifford algebra $\mathcal{C}_{p,q}$.

\[
A \in \mathcal{C}_{p,q}^r \equiv \bigotimes^r \mathcal{C}_{p,q} = \bigotimes^r \Lambda \equiv \Lambda^r
\]

Each multivector space $\Lambda$ will be called a block. Its expression in a basis, using multi-indices, is
\[
A = A^{I_1J_1} \cdots I_rJ_r e_{I_1} \otimes e_{I_2} \otimes \cdots \otimes e_{I_r} .
\]

We observe that terms of the form $e_{I_1} \otimes 1 \otimes e_{I_3}$ are possible and are different from $e_{I_1} \otimes e_{I_3}$.

The natural associative product defined between two r-fold multivectors is the r-direct Clifford product, which involves an independent Clifford product in each block.

\[
AB = \left( A^{I_1 \cdots I_r} e_{I_1} \otimes \cdots \otimes e_{I_r} \right) \left( B^{J_1 \cdots J_r} e_{J_1} \otimes \cdots \otimes e_{J_r} \right)
\]

\[
= A^{I_1 \cdots I_r} B^{J_1 \cdots J_r} (e_{I_1} e_{J_1}) \otimes \cdots \otimes (e_{I_r} e_{J_r})
\]

We will also consider the product between an r-fold and an s-fold multivector with $r \neq s$. It is defined by an independent Clifford product in each block, starting by the left-hand side. If $A \in \Lambda^r$ and $B \in \Lambda^s$ where, say, $s < r$ then

\[
AB = A^{I_1 \cdots I_r} B^{J_1 \cdots J_s} (e_{I_1} e_{J_1}) \otimes \cdots \otimes (e_{I_s} e_{J_s}) \otimes e_{I_{s+1}} \otimes \cdots \otimes e_{I_r}
\]

Notice that this product is equivalent to the product with $r = s$, if we complete the right-hand side of the shortest factor with the necessary number of ‘ones’

\[
B^{I_1 \cdots J_s} e_{J_1} \otimes \cdots \otimes e_{J_s} \implies B^{I_1 \cdots J_s} e_{J_1} \otimes \cdots \otimes e_{J_s} \otimes 1 \otimes \cdots \otimes 1
\]

In order to shorten expressions we introduce multi-fold multiindices. The collect a list of multiindices and are denoted with an underline, $\underline{L} \equiv \{I_1, I_2, \cdots, I_s\}$. Using them, expression (1) simplifies to $A = A^L e_L$. They will also be used as a shorthand to make a single block explicit.

\[
A = A^{L} e_{L} \otimes e_{I_1} \otimes e_{D} = e_{C} \otimes A^{L} e_{D}
\]

where $C = \{I_1, \cdots, I_{r-1}\}$ and $D = \{I_{r+1}, \cdots, I_r\}$. Note that $A^{L} \equiv A^{L} e_{I_1} e_{I_r} \in \Lambda$ is not just a scalar component but a multivector.
The basic operations of degree projection, reversion and Clifford conjugation, acting on multivectors, can be extended to $r$-fold multivectors. Thus, we define the $r$-fold degree projection, the $r$-fold reversion and the $r$-fold Clifford conjugation as the result of applying the corresponding operation independently on every block, and will use the same notation as with simple multivectors.

\[ (A)_{s_1,s_2,\ldots,s_r} = A_{I_1,I_2\ldots,I_r} \epsilon_{I_1} \otimes \epsilon_{I_2} \otimes \cdots \otimes \epsilon_{I_r}, \quad \langle A \rangle_s \equiv \langle A \rangle_{s,s,\ldots,s} \]

\[ \bar{A} = A_{I_1,I_2\ldots,I_r} \epsilon_{I_1} \epsilon_{I_2} \epsilon_{I_3} \cdots \epsilon_{I_r} \quad \text{and} \quad \bar{A} = A_{I_1,I_2\ldots,I_r} \epsilon_{I_1} \epsilon_{I_2} \epsilon_{I_3} \cdots \epsilon_{I_r} \]

As remarked by Senovilla, any tensor $\hat{A} = \tilde{A}^{\mu_1 \cdots \mu_s} \epsilon_{\mu_1} \otimes \cdots \otimes \epsilon_{\mu_s}$ can be considered as an $r$-fold multivector, by reordering and grouping antisymmetric indices in separated blocks. Thus, the reordered tensor $A$ will be an $r$-fold $(n_1,n_2,\ldots,n_r)$-vector.

\[ A \in \Lambda^{n_1} \otimes \Lambda^{n_2} \otimes \cdots \otimes \Lambda^{n_r} \subset \tilde{\Lambda} = \tilde{\mathcal{O}}_{p,q} \]

where $n_1 + n_2 + \cdots + n_r = s$. An $r$-fold multivector $A$ which is homogeneous at each block will be said to be degree-defined.

4 Superenergy tensors

The procedure for the construction of the basic superenergy tensor\(^3\) $T\{A\}$ starts by the arrangement of the seed tensor $A$ as an $r$-fold multivector, in the way that has been commented above. Strictly speaking, we will consider that $A$ is a seed $r$-fold multivector. The procedure was originally defined for the signature $p-1 (d = p+1)$, while its expression in 2-spinors language, obtained by Bergqvist, is naturally written for signature $1-p$. Here, we will introduce the Clifford algebra formalism expression for the signature $p-1$, that is, for the algebra $\mathcal{O}_{p,1}$, but the corresponding expression for the opposite signature is also given.

Senovilla’s definition of the basic superenergy tensor has the following form\(^3\)

\[ T\{A\} = \frac{1}{2} \sum_P A_P \times A_P \quad (2) \]

where $A_P$ denotes the $r$-fold multivector $A$ transformed by a combination of duals: that is, $P$ codifies the action of taking the Hodge dual on some blocks and keeping the rest of blocks unchanged. Thus, the summation runs through the $2^r$ possible combinations.

The cross product $A \times A$ used in (2) is defined as the contraction in every block of all the indices except one in each factor. Therefore, $T\{A\}$ has $r$ pairs of indices:

\[ (A \times A)_{\mu_1 \nu_1 \cdots \mu_r \nu_r} = \left( \prod_{i=1}^{r} \frac{1}{(n_i - 1)!} \right) A_{\mu_1 \lambda_{12} \cdots \lambda_{1n_1} \cdots \mu_r \lambda_{2r} \cdots \lambda_{rn_r}} A_{\nu_1 \lambda_{12} \cdots \lambda_{1n_1} \cdots \nu_r \lambda_{2r} \cdots \lambda_{rn_r}} \]

The basic superenergy tensor $T\{A\}$ is a generalization of some well-known superenergy and energy–momentum tensors. A number of examples are given in\(^3\). Among them: the Weyl conformal tensor $C$ generates the Bel-Robinson superenergy tensor $T\{C\}$, the
Riemann tensor $R$ generates the Bel tensor $T\{R\}$, the gradient of a Klein-Gordon massless scalar field $\nabla \phi$ generates its standard energy–momentum tensor $T\{\nabla \phi\}$ and, the Faraday bivector $F$ generates the standard electromagnetic energy–momentum tensor $T\{F\}$. We refer the reader to section 9 and to the bibliography for more details.

The use of the $r$-fold Clifford algebra $\mathcal{Cl}_{p,1}$ enables us to introduce an alternative and much more compact definition of the basic superenergy tensor. Our expression is inspired by the Clifford geometric algebra formulation of the standard electromagnetic energy–momentum tensor \cite{14} which defines this tensor as a vector endomorphism:

$$T(u) = -\frac{1}{2} FuF \in \Lambda_1 \quad \forall u \in \Lambda_1$$

Equivalently one can write it as applied to a pair of vectors, or obtain its components in a basis by simply applying the tensor to the elements of this basis (first found in this form in \cite{15}):

$$T(v, u) = -\frac{1}{2} \langle v FuF \rangle_0 \quad \text{and} \quad T_{\mu \nu} = -\frac{1}{2} \langle e_\mu Fe_\nu F \rangle_0$$

This kind of sandwich–like formula is really a standard procedure in Clifford algebra formulations which, as we will see below, also appears in the implementation of isometries and for obtaining the bilinear observables of a spinor field.

Generalizing this expression for $r$-fold multivectors $A$, we define the associate superenergy tensor as an endomorphism on the direct product of $r$ copies of the vector space,

$$T\{A\} : \otimes^r \Lambda_1 \rightarrow \otimes^r \Lambda_1 \quad u \mapsto T\{A\}(u) := (-1)^r \frac{1}{2} \langle A u \bar{A} \rangle_1$$

Thus, applied to $r$ vectors:

$$T\{A\}(u_1 \otimes u_2 \otimes \cdots \otimes u_r) = (-1)^r \frac{1}{2} \langle A (u_1 \otimes \cdots \otimes u_r) \bar{A} \rangle_1$$

We can also write it as applied to $r$ pairs of vectors

$$T\{A\}(v_1 \otimes \cdots \otimes v_r, u_1 \otimes \cdots \otimes u_r) = (-1)^r \frac{1}{2} \langle (v_1 \otimes \cdots \otimes v_r) A (u_1 \otimes \cdots \otimes u_r) \bar{A} \rangle_0$$

or obtaining its components in a basis $\{e_\mu\}$

$$T\{A\}_{\mu_1 \nu_1 \cdots \mu_r \nu_r} = (-1)^r \frac{1}{2} \langle (e_{\mu_1} \otimes \cdots \otimes e_{\mu_r}) A (e_{\nu_1} \otimes \cdots \otimes e_{\nu_r}) \bar{A} \rangle_0$$

The expression of the superenergy tensor for the signature $1 - p$ is slightly more simple: $T\{A\}(u) = \frac{1}{2} \langle Au\bar{A} \rangle_1$. Indeed, the reason for the sign $(-1)^r$ is that, for the signature $p - 1$, a time-like vector must be past–pointing to give a positive quantity when contracted with a future–pointing vector.

In the remaining part of the section we will show that both expressions, the Senovilla’s standard–tensorial definition (2) and the Clifford algebra formulation (5), correspond to the same object. The method to prove this identity will be to expand (5) in order to obtain (2). We must remark here that Senovilla’s definition applies to degree–defined
r-fold multivectors. In contrast, the expression (5) can be applied to general r-fold multivectors. We will comment on this generalization later. But evidently, for this proof, we must consider only degree-defined r-fold multivectors.

Let us concentrate on a single arbitrary block. Using multi-fold multiindices, the components (6) can be written as

\[ T\{A\}_{\mu_1 \nu_1 \ldots \mu_r \nu_r} = \]

\[ (-1)^{r} \frac{1}{2} \left( e_{\mu_1 \ldots \mu_r} e_{\nu_1 \ldots \nu_r} e_\Phi \right)_0 \left( e_{\mu_1} A^{\mu_1} e_\nu A^{\nu} \right)_0 \left( e_{\mu_1+1 \ldots \mu_r} e_\nu A^{\nu} e_{\nu_1 \ldots \nu_r} e_\Phi \right)_0 \]

To re-express the result in this \( r \)th block, we take into account two essential facts. The first is that the Clifford product of any multivector with a vector can be split into inner and exterior products.

\[ e_{\mu_1} A^{\mu_1} = e_{\mu_1} \cdot A^{\mu_1} + e_{\mu_1} \wedge A^{\mu_1} \]

With this expansion the block \( \langle (e_{\mu_1} A^{\mu_1}) (e_\nu A^{\nu}) \rangle_0 \) should split into 4 terms. But since \( A \) is degree-defined, the cross terms cannot contract completely and vanish. Thus, it splits into 2 terms.

\[ \left\langle e_{\mu_1} A^{\mu_1} e_\nu A^{\nu} \right\rangle_0 = \left\langle (e_{\mu_1} \cdot A^{\mu_1}) (e_\nu \cdot A^{\nu}) \right\rangle_0 + \left\langle (e_{\mu_1} \wedge A^{\mu_1}) (e_\nu \wedge A^{\nu}) \right\rangle_0 \quad (7) \]

The second fact is that an exterior product can be written, with the help of the Hodge duality, as an inner product. Applying it, the second term has the same structure as the first term, but where in the first we have the original \( n_i \)-vector, \( A^{\mu_1} \), in the second we have the dual \( (n-i) \)-vector, \( *A^{\mu_1} \).

\[ \left\langle e_{\mu_1} A^{\mu_1} e_\nu A^{\nu} \right\rangle_0 = \left\langle (e_{\mu_1} \cdot A^{\mu_1}) (e_\nu \cdot A^{\nu}) \right\rangle_0 + \left\langle (e_{\mu_1} \cdot [*A^{\mu_1}]) (e_\nu \cdot [*A^{\nu}]) \right\rangle_0 \]

Repeating this expansion for every block we obtain \( 2^r \) terms, corresponding to all possible combinations that take the dual in some blocks and leave unchanged the rest.

\[ T\{A\}_{\mu_1 \nu_1 \ldots \mu_r \nu_r} = (-1)^r \frac{1}{2} \sum_{\mathcal{P}} \left\langle (e_{\mu_1} \cdot \ldots \cdot e_{\mu_r} \cdot A_P) (e_{\nu_1} \cdot \ldots \cdot e_{\nu_r} \cdot \overline{A_P}) \right\rangle_0 \quad (8) \]

where the dot denotes the inner product in every block, and we have used \( \mathcal{P} \) again to indicate each combination of Hodge duals. Finally, comparing this last expression with (2), we only have to check that the terms of this summation (8) coincide with the Senovilla’s cross product (3).

\[ (-1)^r \left\langle (e_{\mu_1 \ldots \mu_r} \cdot A) (e_{\nu_1 \ldots \nu_r} \cdot \overline{A}) \right\rangle_0 = \left\langle (e_{\mu_1 \ldots \mu_r} \cdot A) (e_{\nu_1 \ldots \nu_r} \cdot \overline{A}) \right\rangle_0 = (A \times A)_{\mu_1 \nu_1 \ldots \mu_r \nu_r} \]

This can be seen by realizing that the dot products in (8) fix one index in each factor for each block. The scalar projection of the product selects the terms that correspond to the contraction of the rest of the indices. It is easy to check that the signs coincide. Then, the proof is complete.
5 Symmetries

The basic superenergy tensor defined in \( ^3 \) is symmetric by construction in each pair of indices:

\[
T\{A\}_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_r\nu_r} = T\{A\}_{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}
\]

However, this property only holds for a degree–defined \( r \)-fold multivector \( A \). When \( A \) is non degree–defined, the formula (7) is no longer valid because there can be contributions from terms mixing different degrees. Let us see which non trivial elements appear when we consider general \( r \)-fold multivectors.

First, consider as seed a simple multivector, split into its different degrees \( A = \sum_{k=0}^d A[k] \), where \( A[k] \in \Lambda_k \). Then, the contribution of each degree to the tensor \( T\{A\} \) is given by:

\[
-2T\{A\}(u,v) = \sum_{k=0}^d \left\langle u A[k] v A[k] \right\rangle_0
\]

\[
= \sum_k \left\langle (u \cdot A_{[k+1]} + u \wedge A_{[k-1]}) (v \cdot A_{[k+1]} + v \wedge A_{[k-1]}) \right\rangle_0
\]

\[
= \sum_k \left[ \left\langle (u \cdot A_{[k+1]})(v \cdot A_{[k+1]}) \right\rangle_0 + \left\langle (u \wedge A_{[k-1]})(v \wedge A_{[k-1]}) \right\rangle_0 
\]

\[
+ \left\langle (u \cdot A_{[k+1]})(v \wedge A_{[k-1]}) \right\rangle_0 + \left\langle (u \wedge A_{[k-1]})(v \cdot A_{[k+1]}) \right\rangle_0
\]

The first pair of terms was already present in formula (7), while the last pair collects the non vanishing terms mixing different degrees. Observe that these extra terms only contain factors differing in 2 degrees. This means, for instance, that a bivector combines only with scalars and 4-vectors. Besides, the contribution of these terms is always antisymmetric in the pair of indices. Then, the total superenergy tensor \( T\{A\} \) can be written as the sum of the symmetric superenergy tensors generated by each degree and the antisymmetric mixing terms:

\[
T\{A\}(u,v) = \sum_k T\{A[k]\}(u,v) + \sum_k \left\langle A_{[k+2]} A_{[k]} \right\rangle_2 (v \wedge u)_0
\]

or written with tensorial indices

\[
T\{A\}_{\mu\nu} = \sum_k T\{A[k]\}_{\mu\nu} + \sum_k \frac{1}{k!} (A_{[k+2]} A_{[k]})_{\mu\nu\rho_1\cdots\rho_k} (A[k])^{\rho_1\cdots\rho_k}
\]

For the more general case of \( r \)-fold multivectors, this splitting will occur in each block. Thus, it can appear any combination of terms symmetric in some blocks and antisymmetric in the rest, so that no symmetry is satisfied in general.

Another possible symmetry is the one involving the interchange of different pairs of indices. It is easy to see that, if \( A \in \Lambda \) is symmetric or antisymmetric in any two blocks, then, \( T\{A\} \) will be symmetric in the interchange of the respective pair of indices.

\[
A^{I_1\cdots I_j\cdots I_r} = \pm A^{I_1\cdots I_j\cdots I_r} \quad \Rightarrow \quad T\{A\}_{\mu_1\nu_1\cdots\mu_j\nu_j\cdots\mu_r\nu_r} = T\{A\}_{\mu_1\nu_1\cdots\mu_j\nu_j\cdots\mu_r\nu_r}
\]

This property is independent of \( A \) being degree–defined or not.
6 Lorentz Transformations and Bilinear Observables of Spinors

A general Lorentz transformation of a vector \( u \in \Lambda_1 \) is performed by a versor \( R \in \text{Pin}_{p,1} \subset \mathcal{O}_{p,1} \), by means of the formula\(^{16}\):

\[
R(u) = RuR^* - 1
\]  

(9)

The group \( \text{Pin}_{p,1} \) is the double covering group of \( O(p, 1) \), and is defined by

\[
\text{Pin}_{p,1} \equiv \{ R \in \mathcal{O}_{p,1} \mid R\bar{R} = \pm 1 \, \text{and} \, RuR^* - 1 \in \Lambda_1 \ \forall u \in \Lambda_1 \}
\]

By means of the Cartan-Dieudonné theorem it is shown\(^{16}\) that \( \text{Pin}_{p,1} \) can be equivalently defined by

\[
\text{Pin}_{p,1} \equiv \{ R = v_1v_2 \cdots v_n \in \mathcal{O}_{p,1} \mid v_i \in \Lambda_1, \ v_i^2 = \pm 1 \ \text{and} \ n = 0, 1, \cdots, 2d \} \quad (10)
\]

The sign \( R\bar{R} = \pm 1 \) splits the group into two unconnected parts, \( \text{Pin}_{p,1}^\pm \), which implement respectively orthochronous and time–reversing Lorentz transformations. Taking into account this sign to compute the inverse in expression (9), we obtain

\[
R(u) = \pm RuR \quad \text{when} \quad R \in \text{Pin}_{p,1}^\pm
\]

This can be seen as a particular case of the basic superenergy tensor \( T\{ R \} \) when \( R \in \text{Pin}_{p,1} \). The opposite of a time–reversing transformation, \( -R(u) \), is an orthochronous transformation. This implies that \( RuR = \mp R(u), \ R \in \text{Pin}_{p,1}^\pm \) is always orthochronous. This proves the following

**Theorem 1** A mapping \( L : \Lambda_1 \to \Lambda_1 \) is an orthochronous Lorentz transformation iff there exists some \( R \in \text{Pin}_{p,1}^\pm \) for which \( L(u) = -2T\{ R \}(u) = RuR \). \( \square \)

This theorem together with the expression (10) for the Pin group can be considered a generalization of corollary 4.2 in reference\(^{9}\). This corollary states that a superenergy tensor generated by a non-null simple \( n \)-vector \( A = v_1 \wedge v_2 \wedge \cdots \wedge v_n \) defines a mapping \( b -T\{ A \}(u) \) proportional to an involutory orthochronous Lorentz transformations.

Observe that in general the versor \( R \in \text{Pin}_{p,1} \) in theorem 1 is not an homogeneous multivector. This is the reason why general orthochronous Lorentz transformations are not included as superenergy tensors in the work\(^{9}\). \( R \in \text{Pin}_{p,1} \) is homogeneous only when it can be decomposed into orthogonal vectors:

\[
R = v_1v_2 \cdots v_n = v_1 \wedge v_2 \wedge \cdots \wedge v_n
\]

In this case, \( R \) is a simple \( n \)-vector and implements an involutory transformation, which is implied by the property \( R^2 = \pm 1 \).

The special case of orthochronous *proper* Lorentz transformation is performed by a rotor \( R \in \text{Spin}_{p,1}^\pm \equiv \text{Pin}_{p,1}^\pm \cap \mathcal{O}_{p,1}^\pm \). The group \( \text{Spin}_{p,1}^\pm \) is the connected double covering group of \( SO^+(p, 1) \), and \( \mathcal{O}_{p,1}^\pm \) is the even subalgebra of \( \mathcal{O}_{p,1} \).

\(^{b}\)The minus sign is due to the signature \( p-1 \). It is not present in\(^{9}\), which uses signature \( 1-p \).
Let us now consider an \( r \)-fold 1-vector \( u \in \bigotimes^r \Lambda_1 \). Then, we can implement simultaneously an independent orthochronous Lorentz transformation at each block, by means of an \( r \)-fold versor

\[
R = R_1 \otimes R_2 \otimes \cdots \otimes R_r , \quad R_i \in \text{Pin}^+_p,q , \quad \mathcal{R}(u) = R u \overline{R} = (-1)^r T\{R\}(u) \tag{11}
\]

We can extend the group of Lorentz transformation to include dilations. The simplest way of implementing this transformation is to extend the Pin\(^+_p,1\) group by adding a scalar factor:

\[
\Gamma^+_p,1 \equiv \{ \lambda R \mid \lambda \in \mathbb{R}^+, R \in \text{Pin}^+_p,1 \} , \quad u \mapsto (\lambda R)u(\overline{\lambda R}) = \lambda^2 R(u)
\]

\( \Gamma^+_p,1 \) is the positive subgroup of the Clifford-Lipschitz group\(^{16}\)

\[
\Gamma^+_p,1 = \{ R \in \mathcal{A}^+_p,1 \mid RuR^{-1} \in \Lambda_1 \ \forall u \in \Lambda_1 \}
= \{ R \in \mathcal{A}^+_p,1 \mid RuR \in \Lambda_1 \ \forall u \in \Lambda_1 \ \text{and} \ R\overline{R} \in \mathbb{R}^+ \}
\]

The extension for the transformation of \( r \)-fold 1-vectors is completely analogous to (11).

The singular limiting set

\[
\Gamma^0_p,1 \equiv \{ R \in \mathcal{A}^+_p,1 \mid RuR \in \Lambda_1 \ \forall u \in \Lambda_1 \ \text{and} \ R\overline{R} = 0 \}
\]

is a semigroup which contains non invertible even multivectors. Each element \( R \in \Gamma^0_p,1 \) maps any vector \( u \) into a unique null vector \( \ell = RuR \), up to a scalar factor. Thus, it defines a null direction.

Pauli, Dirac, Weyl and Majorana spinor fields can be treated as operator spinors\(^{17,18}\). The concept of operator spinor is based on identifying spinors as elements of the even subalgebra \( \mathcal{A}^+_p,q \). The name is motivated because a spinor is then interpreted as a generalization of a rotor, which links the observer tetrad field \( \{e_\mu\} \) to the bilinear observables associated to the spinor. Typically, for a Dirac spinor \( \Psi \in \mathcal{A}^+_3,1 \), the current and the spin vector, and the magnetization bivector are given by

\[
j = \Psi e_0 \overline{\Psi} , \quad s = \Psi e_3 \overline{\Psi} \quad \text{and} \quad M = \Psi e_{12} \overline{\Psi}
\]

For Weyl and Majorana spinors, these expressions are also valid. However, these spinors are not general even multivectors but elements of two specific subsets of \( \Gamma^0_p,1 \cap \mathcal{A}^+_p,1 \). Thus, their current \( j \) is light-like.

Operator spinors are represented by multivectors. But they are not tensors because they relate the observables, which are objective quantities (tensors), with the observer. They can be considered ‘pseudo’-tensors in the same sense as affine connections. Thus, a spinor is inseparably related to the observer tetrad. For this reason, its associated superenergy tensor \( T\{\Psi\}_{\mu\nu} \), is only a tensor for the second index, since the first is related to the observer: \( j_\nu = -2T\{\Psi\}_{\mu\nu}(e_0)^\mu \equiv -2T\{\Psi\}(0)_{\mu} \). Thus, while a tensorial multivector \( A \) generates a rank-2 tensor \( T\{A\} \), a spinor \( \Psi \) generates a rank-1 tensor \( T\{\Psi\}(e_0) \). Compare this treatment with the 2-spinor formulation in\(^6\).
7 Dominant Property (DP)

In this section we present a simple proof of the DP for the superenergy tensor \( T\{A\} \), using its expression in the \( r \)-fold Clifford algebra \( {}^r\mathcal{O}_{1,p} \). We must underline that this proof is valid for a general \( r \)-fold multivector \( A \). Even if the seed tensor \( A \) is degree–defined, the essential point of the present proof is the definition of a second \( r \)-fold multivector \( A' \), which in general is no longer degree–defined. We see, then, that this generalization is fundamental for the proof.

**Definition 1** A superenergy tensor satisfies the Dominant Property (DP) if for all collection \( \{u_i, v_i\} \) of causal and future-pointing (f-p) vectors

\[
T\{A\}(u_1 \otimes \cdots \otimes u_r, v_1 \otimes \cdots \otimes v_r) = \frac{1}{2} \left\langle (u_1 \otimes \cdots \otimes u_r) A (v_1 \otimes \cdots \otimes v_r) \overline{A} \right\rangle_0 \geq 0 .
\]

**Theorem 2** Let \( A \in {}^r\Lambda \) be any \( r \)-fold multivector, then \( T\{A\} \) satisfies the DP. □

**Proof:** Let us recall, first, that a time-like f-p vector can always be expressed as the result of applying a local orthochronous Lorentz transformation and a dilation to a chosen unitary time-like f-p vector \( e_0 \). As seen in the previous section, this transformation is performed by means of a multivector:

\[
u = R_u e_0 R_u^\dagger , \quad R_u \in \Gamma^+_{p,1} \subset \mathcal{O}_{p,1}
\]

The same expression applies for a null vector \( u \) with \( R_u \in \Gamma^0_{p,1} \). For the tensor product of \( r \) f-p vectors \( \v u \equiv u_1 \otimes u_2 \otimes \cdots \otimes u_r \in \bigotimes^r \Lambda_1 \), we can construct

\[
R_{\v u} \equiv R_{u_1} R_{u_2} \cdots R_{u_r} \quad \text{and} \quad e_0 \equiv e_0 \otimes \cdots \otimes e_0 , \quad \text{so that} \quad \v u = R_{\v u} e_0 R_{\v u}^\dagger
\]

Our proof of the DP proceeds in two steps. First, using the operators \( R_{\v u} \in {}^r\mathcal{O}_{p,1} \), we express the result of applying \( T\{A\} \) to any set of \( 2r \) f-p vectors \( \v u, \v v \) as the \( \{0, \ldots, 0\} \) component of another superenergy tensor \( T\{A'\} \):

\[
(-1)^r 2T\{A\}(\v u, \v v) = \left\langle \v u A \v v \overline{A} \right\rangle_0 = \left\langle \left( R_{\v u} e_0 R_{\v u}^\dagger \right) A \left( R_{\v u} e_0 R_{\v u}^\dagger \right) \overline{A} \right\rangle_0 \geq 0 , \quad (12)
\]

where \( A' \equiv R_{\v u} A R_{\v u} \in {}^r\mathcal{O}_{p,1} \) is also an \( r \)-fold multivector.

The second step proves that, for all \( A' \in {}^r\mathcal{O}_{p,1} \), this component is non negative:

\[
T\{A'\}(e_0, e_0) = (-1)^r \frac{1}{2} \left\langle e_0 A' e_0 \overline{A'} \right\rangle_0 \geq 0 \quad \forall A' \in {}^r\mathcal{O}_{p,1}
\]

To see this, we split \( A' \) into parts orthogonal and parallel to the direction \( e_0 \). This splitting corresponds to the isomorphism of linear spaces, though not as algebras,

\[
{}^r\mathcal{O}_{p,1} \simeq {}^r\mathcal{O}_{p,0} \otimes {}^r\mathcal{O}_{0,1}
\]
where $\mathcal{V}_{0,1}$ is the space generated by the vector $e_0$ and $\mathcal{V}_{p,0}$ is the space generated by the Euclidean space orthogonal to $e_0$. A basis for $\mathcal{V}_{0,1}$ has the $2^r$ elements:

$$\{e_P\} = \{1 \otimes \cdots \otimes e_0, 1 \otimes \cdots \otimes e_0 \otimes 1, \ldots, e_0 \otimes \cdots \otimes e_0 \otimes e_0\}$$

Now, we expand $A'$ in the basis $\{e_P\}$ of $\mathcal{V}_{0,1}$ with components in $\mathcal{V}_{p,0}$

$$A' = A' P e_P, \quad A' P \in \mathcal{V}_{p,0} \subset \mathcal{V}_{p,1}, \quad e_P \in \mathcal{V}_{0,1} \subset \mathcal{V}_{p,1}$$

Observe that both, 'components' and 'basis', are elements of the algebra $\mathcal{V}_{p,1}$, thus the product in $A' P e_P$ is the $r$-direct Clifford product. This finally completes the proof

$$(-1)^r \left< e_0, A' e_0, \mathcal{V}_{0,1} \right>_0 = \left< (A' P e_P) e_0, (\bar{e}_Q A\bar{Q}) e_0^{-1} \right>_0 = \left< (A' P e_P) (\bar{e}_Q A\bar{Q}) \right>_0$$

This last summation is always positive since, being $\mathcal{V}_{p,0}$ an algebra generated by an Euclidean metric, $\left< B\bar{B} \right>_0$ is a positive definite norm $\forall B \in \mathcal{V}_{p,0}$. □

Besides, the equality is satisfied if and only if every $A' P = 0$, that is, iff $A' = 0$.

$$\left< A u_\ell, u_{\mathcal{V}_{0,1}} \right>_0 = 0 \iff A' = R_\omega A R_{\bar{\omega}} = 0$$

If we consider only time-like vectors $u_\ell$ and $v_\ell$, then, the operators $R_\omega$ and $R_{\bar{\omega}}$ are invertible. In this case $A' = 0$ implies that $A = R_\omega^{-1} A R_{\bar{\omega}}^{-1} = 0$. This proves the result

**Theorem 3** Let $A \in \mathcal{V}_r$ be any $r$-fold multivector. There exists some set $\{u_\ell, v_\ell\}$ of $2r$ time-like vectors satisfying

$$T\{A\}_{\mu_1 \nu_1 \cdots \mu_r \nu_r} u_\mu v_\nu \cdots u_\mu v_\nu = 0$$

iff $A = 0$. □

The general result including null vectors is the following.

**Theorem 4** Let $A \in \mathcal{V}_r$ be any $r$-fold multivector. There exists some set $\{u_\ell, v_\ell\}$ of $2r$ causal vectors satisfying

$$T\{A\}_{\mu_1 \nu_1 \cdots \mu_r \nu_r} u_\mu v_\nu \cdots u_\mu v_\nu = 0 \text{ iff } \ell A \bar{\ell} = 0,$$

where

$$\ell = \bigotimes_{i=1}^r \ell_i \quad \text{with} \quad \ell_i = \begin{cases} 1 \quad \text{if } u_\ell \text{ is time-like} \\ u_\ell \quad \text{if } u_\ell \text{ is null} \end{cases}$$

$$\bar{\ell} = \bigotimes_{i=1}^r k_i \quad \text{with} \quad k_i = \begin{cases} 1 \quad \text{if } v_\ell \text{ is time-like} \\ v_\ell \quad \text{if } v_\ell \text{ is null} \end{cases}$$

□

**Proof:** First, we take into account that any null vector $\ell$ satisfies $\ell = \lambda e_0 \bar{\ell}$, with $e_0$ a time-like vector and $\lambda \in \mathbb{R}^*$. This enables to define a new $A' = \ell A \bar{\ell}$, following a manipulation analogous to (12). Then, there appears the tensor $T\{A'\}$ contracted with $2r$ time-like vectors, since every null vectors has been replaced by $e_0$. Hence, from theorem 3, the necessary and sufficient condition is $A' = 0$. □
8 Principal Null Directions

A good example which illustrates the possibilities of the \( r \)-fold Clifford formulation is the treatment of the principal null directions of a tensor. In Special and General Relativity, the concept of principal null directions has been mainly used in the algebraic classification of the electromagnetic field and of the Weyl tensor (Petrov classification)\(^{19,20}\), which are respectively a bivector and a double bivector. Let us introduce a general definition for arbitrary \( r \)-fold multivectors.

**Definition 2** A null direction, represented by a null vector \( \ell \), is a principal null direction of an \( r \)-fold multivector \( A \in r\Lambda \), if it satisfies the equation

\[
\ell A \ell = 0 \quad \text{where} \quad \ell = \bigotimes^r \ell
\]

This is equivalent to \((\ell \cdot A) \wedge \ell = 0\). Or written with tensorial indices for a degree–defined \( r \)-fold multivector

\[
\ell^{\mu_1} \ell_{[\rho_1} A_{\rho_2 \cdots \rho_{n_1}]\mu_1 \cdots \mu_r | \rho_{r_2} \cdots \rho_{r_n} | \rho_{r_1}] \ell^{\nu_r} = 0
\]

where in each of the \( r \) blocks we have one contraction with \( \ell \) and the antisymmetrization of the rest of indices with \( \ell \): \( \ell^{\mu_1} \ell_{[\rho_1} A_{\cdots \rho_2 \cdots \rho_{n_1}]\mu_1 \cdots \rho_{r_2} \cdots \rho_{r_n}] \rho_{r_1}] \cdot \square \)

**Theorem 5** A null vector \( \ell \) is a principal null direction of \( A \in r\Lambda \) iff

\[
T\{A\}_{\mu_1 \nu_1 \cdots \mu_r \nu_r} \ell^{\mu_1} \ell^{\nu_1} \cdots \ell^{\mu_r} \ell^{\nu_r} = 0
\]

\( \square \)

**Proof:** It follows from theorem 4 and definition 2.

This theorem is a generalization of the well-known result\(^{21}\) relating the principal null directions of the Weyl tensor, \( \ell^{\mu} \ell_{[\alpha} C_{\beta \mu \nu]} \ell^{\nu} = 0 \), with the null directions that make the BR superenergy tensor vanish, \( T\{C\}_{\mu \nu \rho \sigma} \ell^{\mu} \ell^{\nu} \ell^{\rho} \ell^{\sigma} = 0 \).

9 Conserved Energy and Superenergy Tensors

We have shown that the basic superenergy tensors \( T\{A\} \) satisfy the DP independently of the generating tensor \( A \). The other important property expected to be satisfied by the energy–momentum tensor of any isolated physical field (or the total energy–momentum) is its local conservation: \( \nabla^\mu T_{\mu \nu} = 0 \). The extension for general superenergy tensors is simply

\[
\nabla^\mu T_{\mu_1 \nu_1 \cdots \mu_r \nu_r} = 0
\]

(13)

Obviously, this property depends on the differential equations governing the physical field. In this section we give a sufficient condition for the generating field \( A \), which guarantees the conservation of its basic superenergy tensor \( T\{A\} \). Moreover, this sufficient condition is satisfied by some physical fields like the electromagnetic field, the Klein-Gordon field and the Riemann and Weyl tensors in Einstein spaces. Consequently, the divergence–free property of the standard electromagnetic and Klein-Gordon energy–momentum tensors, and the Bel and Bel-Robinson superenergy tensors, is unified in a unique procedure. The
case of the current of the Dirac field is also embraced by this condition, if we consider a slightly generalization which will need some discussion. A characteristic feature of the sufficient condition presented here is that its simplest and most natural expression is written using Clifford algebras. However, we will write also its standard tensorial equivalent.

First, let us write equation (13) for the basic superenergy tensor using the $r$-fold Clifford algebra formalism:

$$\nabla \cdot T\{A\}(u) = (-1)^r \frac{1}{2} \left( \langle \nabla A u \overline{A} \rangle_{0,1,\ldots,1} \right)$$

Here some notational conventions are used. The differential operator $\nabla \equiv e^{\mu} \nabla_{\mu}$ is algebraically a vector, hence it multiplies the first block. The overdots are used to indicate which elements must be derived by the operator. In absence of overdots, $\nabla$ affects only the element found immediately at its right–hand side. Notice that the set of vectors $u$ must not be derived, since we want to derive only the tensor $T\{A\}$. Using the Leibniz rule and the cyclic property of the scalar part of a product we obtain:

$$\nabla \cdot T\{A\}(u) = (-1)^r \frac{1}{2} \left( \langle \nabla A u \overline{A} \rangle_{0,1,\ldots,1} - \langle A u \nabla A \overline{A} \rangle_{0,1,\ldots,1} \right) = (-1)^r \left( \langle \nabla A u \overline{A} \rangle_{0,1,\ldots,1} \right) \quad (14)$$

We see here that $\nabla A$ is the unique differentiated quantity which contributes to the divergence. As is evident, $\nabla A = 0$ is a sufficient condition for the superenergy tensor $T\{A\}$ being divergence–free. But there is a more general sufficient condition:

**Theorem 6** Let $A \in \Lambda^r$ be any $r$-fold multivector. If $A$ satisfies the condition

$$\nabla A = \lambda A \quad \text{with} \quad \lambda \in \mathbb{R} \quad (15)$$

then the superenergy tensor $T\{A\}$ is divergence–free. □

*Proof:* From (14) we get

$$\nabla \cdot T\{A\}(u) = (-1)^r \lambda \left( \langle A u \overline{A} \rangle_{0,1,\ldots,1} \right) = 0 \quad \forall A \in \Lambda^r,$$

if we take into account the properties of the Clifford conjugation:

$$\langle A u \overline{A} \rangle_{0,1,\ldots,1} = (-1)^r (-1)^{-1} \langle A u \overline{A} \rangle_{0,1,\ldots,1} = (-1)^{r-1} \langle A u \overline{A} \rangle_{0,1,\ldots,1} = -\langle A u \overline{A} \rangle_{0,1,\ldots,1}$$

In order to express condition (15) in index tensorial notation we need to split $A$ into its different degrees for the first block:

$$A = \sum_k A[k] \quad \text{where} \quad A[k] = \frac{1}{k!} A_{[\mu_1 \ldots \mu_k]} e^{\mu_1 \cdots \mu_k} \in \Lambda_k \otimes \Lambda \otimes \cdots \otimes \Lambda$$

and then to collect the contributions to different degrees coming from exterior and inner products:

$$\nabla A = \lambda A \quad \Leftrightarrow \quad \nabla \wedge A[k-1] + \nabla \cdot A[k+1] = \lambda A[k] \quad \forall k = 0, \ldots, d \quad (16)$$
where

\[(\nabla \wedge A_{[k-1]})_{\mu_1 \mu_2 \cdots \mu_k} = k \nabla_{[\mu_1} A_{\mu_2 \cdots \mu_k]C} \quad \text{and} \quad (\nabla A_{[k+1]})_{\mu_1 \mu_2 \cdots \mu_k} = \frac{1}{k+1} \nabla^\nu A_{\nu \mu_1 \mu_2 \cdots \mu_k C}\]

This sufficient condition is satisfied by diverse physical fields, so that the vanishing of the divergence of their energy or superenergy tensors follows from theorem 6:

- The source–free electromagnetic field. The full set of Maxwell equations for the Faraday bivector take the form \(\nabla F = 0\), and the standard energy–momentum tensor is \(T\{F\}\). Then, it immediately follows: \(\nabla \cdot T\{F\} = 0\).

- The Klein-Gordon field:

\[\nabla^2 \phi = m^2 \phi\]

Its standard energy–momentum tensor is generated by \((\nabla + m)\phi\in \Lambda:\)

\[T\{(\nabla + m)\phi\}_{\mu\nu} = T\{\nabla \phi\}_{\mu\nu} + T\{m\phi\}_{\mu\nu} = -\frac{1}{2} \langle e_\mu \nabla \phi e_\nu \nabla \phi \rangle_0 - \frac{1}{2} \langle e_\mu m\phi e_\nu m\phi \rangle_0\]

\[= \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \nabla_\rho \phi \nabla^\rho \phi g_{\mu\nu} - \frac{1}{2} m^2 \phi^2 g_{\mu\nu}\]

We can easily check that it satisfies condition (15).

\[\nabla (\nabla + m)\phi = m(\nabla + m)\phi \quad \Rightarrow \quad \nabla \cdot T\{(\nabla + m)\phi\} = 0\quad (17)\]

- The Weyl conformal and the Riemann tensor in Einstein spaces. Einstein spaces satisfy \(\nabla R = \nabla C = 0\). For \(d = 4\), These conditions follow easily from the differential Bianchi identity, \(\nabla \wedge R = 0\), and the Lanzos identity, \(R = - \ast R\ast\) (where \(\ast R\ast\) is the double Hodge dual of the Riemann). For \(d > 4\) the equations also hold, but they are a bit more difficult to show. Therefore, we obtain:

\[\nabla C = \nabla R = 0 \quad \Rightarrow \quad \nabla \cdot T\{C\} = \nabla \cdot T\{R\} = 0\]

which says that the generalization of the Bel-Robinson, \(T\{C\}\), and the Bel, \(T\{R\}\), superenergy tensors are divergence-free in Einstein spaces of any dimension.

**Definition 3** An \(r\)-fold multivector differential operator \(D \in \Lambda\) is a differential operator which algebraically belongs to \(\Lambda\). \(D\) will be said to be constant if it is formed by any combination of constant multivectors and the differential operator \(\nabla\). \(\square\)

Some examples of constant 2-fold multivector differential operators are: \(1 \otimes (\nabla + m)\), \(\nabla \otimes \nabla\), and \(B \otimes (v \wedge \nabla)\), with \(m, B, v \in \Lambda\) constant.

Consider now that we have an \(r\)-fold multivector \(A \in \Lambda\) satisfying condition (15) with \(\lambda\) constant. Let \(D \in \Lambda\) be a constant \(s\)-fold multivector differential operator. We can then construct the \(r'\)-fold multivector

\[A' \equiv \dot{A} \dot{D} \in \Lambda\]

where \(D\) differentiates \(A\). A simple example is provided by \(A' = \dot{A}(1 \otimes \dot{\nabla}) = \dot{A} \otimes \dot{\nabla} \equiv \nabla_\mu A \otimes e^\mu\). Then, the \(r'\)-fold multivector \(A'\) also satisfies condition (15), except for terms...
involving the non-commutativity of derivatives: $\nabla_{[\mu} \nabla_{\nu]}$, i.e. involving terms linear in the Riemann tensor or its derivatives:

$$\nabla(\dot{A} \dot{D}) = (\nabla A) \dot{D} + O(R) = \lambda(\dot{A} \dot{D}) + O(R)$$  \hfill (18)

Hence, for flat space-times we get the result:

**Theorem 7** Let $A \in \Lambda^r$ be any $r$-fold multivector on a flat space-time $\mathcal{M}$, satisfying $\nabla A = \lambda A$, with $\lambda$ constant. Then, for any constant $s$-fold multivector differential operator $\mathcal{D}$, $A' = \dot{A} \dot{D}$ also satisfies $\nabla A' = \lambda A'$. □

**Proof:** Trivial from (18).

Thus, from theorem 6 the superenergy tensor $T\{A'\}$ will be divergence–free.

**Theorem 8** Let $A \in \Lambda^r$ be any $r$-fold multivector on a flat space-time $\mathcal{M}$, satisfying $\nabla A = 0$, then the $(r+1)$-fold multivector $A' = \nabla \otimes A$ also satisfies $\nabla A' = 0$. □

**Proof:** On a flat space-time, the Clifford square of the operator nabla is algebraically a scalar $\nabla^2 \in \Lambda_0$. Thus, it easily follows:

$$\nabla A' = \nabla^2 \otimes A = 1 \otimes \nabla^2 A = 1 \otimes \nabla(\nabla A) = 0$$

Thus, from theorem 6, the derived tensors $A'$ obtained in any of these two theorems, will generate a divergence–free superenergy tensor $T\{A'\}$. Examples of this constructions are provided by the superenergy tensors proposed by Senovilla\textsuperscript{3} and previously introduced in\textsuperscript{22,23}, for the electromagnetic and Klein-Gordon fields:

- For the source–free electromagnetic field the rank-4 superenergy tensor which has been considered\textsuperscript{22,3} is the full symmetrization of $T\{\nabla \otimes F\}$. From theorems 8 and 6 it follows that the tensor $T\{\nabla \otimes F\}$ is divergence–free for flat space-times:

$$\nabla F = 0 \quad \Rightarrow \quad \nabla(\nabla \otimes F) = 0 \quad \Rightarrow \quad \nabla \cdot T\{\nabla \otimes F\} = 0$$

However, in the divergence of the symmetrized tensor there also contributes the divergence of the tensor $T\{\hat{F} \otimes \hat{\nabla}\}$. This term also vanishes in flat space-times, as follows from theorems 7 and 6:

$$\nabla F = 0 \quad \Rightarrow \quad \nabla(\hat{F} \otimes \hat{\nabla}) = 0 \quad \Rightarrow \quad \nabla \cdot T\{\hat{F} \otimes \hat{\nabla}\} = 0$$

Thus, the symmetrized electromagnetic superenergy tensor is divergence–free.

- For the Klein-Gordon field the rank-4 superenergy tensor\textsuperscript{23,3} is $T\{(\nabla + m) \otimes (\nabla + m) \phi\}$. From (17) we know that $(\nabla + m) \phi$ satisfies condition (15) with $\lambda = m$ constant. Therefore, applying theorem 7, for flat space-times we get:

$$\nabla((\nabla + m) \otimes (\nabla + m) \phi) = m((\nabla + m) \otimes (\nabla + m) \phi)$$

$$\Rightarrow \quad \nabla \cdot T\{(\nabla + m) \otimes (\nabla + m) \phi\} = 0$$

Let us introduce now a generalization of the sufficient condition (15), which we will apply to the current of the Dirac field. We will generalize (15) in two steps. The first consists in allowing the scalar $\lambda$ to be a more general multivector.
**Theorem 9** Let \( A \in \Lambda \) be any r-fold multivector. If \( A \) satisfies the condition
\[
\nabla A = BA , \quad \text{with} \quad B \in \Lambda \quad \text{and} \quad B - \overline{B} = 0 ,
\]
then the superenergy tensor \( T\{A\} \) is divergence-free. \( \square \)

**Proof:** Taking expression (14) we get
\[
(-1)^r 2 \nabla \cdot T\{A\}(\underline{u}) = \langle BA \underline{u} \overline{A} \rangle_{0,1,\ldots,1} - \langle A \underline{u} \overline{B} \overline{A} \rangle_{0,1,\ldots,1} = \langle (B - \overline{B})A \underline{u} \overline{A} \rangle_{0,1,\ldots,1}
\]

Observe that the result of theorem 7 is trivially generalized for the condition (19).

The second step consists in allowing to appear factors multiplying \( A \) from both sides: \( \nabla A = BAC \). But in this case, the divergence of the superenergy tensor \( T\{A\} \) depends on the commutativity of the right-hand factor \( C \) with the vectors to which \( T\{A\} \) is applied. Hence, the result is not the vanishing of the divergence, but of the divergence applied to some collection of \( r \) vectors: \( \nabla \cdot T\{A\}(\underline{u}) = 0 \). For this reason, the result can be applied to the current of a Dirac field \( j = -2T\{\Psi\}(e_0) \).

**Theorem 10** Let \( A \in \Lambda \) be any r-fold multivector and \( \underline{u} \in \otimes^r \Lambda_1 \) any r-fold vector. If they satisfy the conditions
\[
\nabla A = \sum_i B_i AC_i , \quad \text{with} \quad B_i \in \Lambda , \ C_i \in \Lambda \quad (20)
\]
and \( B_i = s_i \overline{B_i} , \ C_i \underline{u} = s_i \underline{u} \overline{C_i} , \) where \( s_i = \pm 1 \)
then the divergence of the superenergy tensor contracted with \( \underline{u} \) vanishes: \( \nabla T\{A\}(\underline{u}) = 0 \). \( \square \)

**Proof:** Taking expression (14) we get
\[
(-1)^r 2 \nabla \cdot T\{A\}(\underline{u}) = \sum_i \langle B_i AC_i \underline{u} \overline{A} \rangle_{0,1,\ldots,1} - \langle A \underline{u} \overline{C_i} \overline{A} \overline{B_i} \rangle_{0,1,\ldots,1}
\]
\[
= \sum_i \langle B_i A (C_i \underline{u}) \overline{A} \rangle_{0,1,\ldots,1} - \langle \overline{B_i} A (\underline{u} \overline{C_i} \overline{A}) \rangle_{0,1,\ldots,1}
\]

**Corollary 1** If \( \underline{u} \) is constant then \( T\{A\}(\underline{u}) \) is divergence-free. \( \square \)

**Corollary 2** If \( T\{A\} \) is completely symmetric: \( T\{A\}_{\mu_1 \nu_1 \ldots \mu_r \nu_r} = T\{A\}_{(\mu_1 \nu_1 \ldots \mu_r \nu_r)} \), and \( \underline{u} \) is a collection of Killing vectors, then \( T\{A\}(\underline{u}) \) is divergence–free. \( \square \)

The Hestenes’ real multivector formulation of the Dirac equation is \( \uparrow^{17} \)
\[
\nabla \Psi = m \Psi e_{012} + qA \Psi e_{12}
\]
where \( \Psi \in \Omega^1_{3,1} \) is the operator spinor described in section 6, and \( A \) is the electromagnetic vector potential. Observe that, although the Dirac-Hestenes equation is usually written for the signature 1–3, it can be also formulated for the signature 3–1 \( \uparrow^{24} \). We can easily see that \( \Psi \) satisfies the conditions of the theorem 10, with \( B_1 = m, \ C_1 = e_{012}, \ B_2 = qA \) and \( C_2 = e_{12} \), for \( j = -2T\{\Psi\}(e_0) \):
\[
\begin{align*}
m &= \frac{m}{e_{012} e_0 = e_0 e_{012}} \\
qA &= -\frac{qA}{e_{12} e_0 = -e_0 e_{12}}
\end{align*}
\]
\[
\Rightarrow \quad \nabla \cdot T\{\psi\}(e_0) = 0
\]
Since $e_0$ is an inertial observer in special relativity, then the current is conserved:

$$\nabla \cdot j = 0$$

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