Entropy of the three dimensional Schwarzschild-de Sitter black hole

Y.S. Myung*

Relativity Research Center and School of Computer Aided Science, Inje University, Kimhae 621-749, Korea

Abstract

We study the three dimensional Schwarzschild-de Sitter (SdS$_3$) black hole which corresponds essentially to a conical defect. We compute the mass of the SdS$_3$ black hole from the correct definition of the mass in asymptotically de Sitter space. Then we clarify the relation between the mass, entropy and temperature for this black hole without any ambiguity. Also we establish the SdS$_3$/CFT$_2$-correspondence for the entropy by applying the Cardy formula to a CFT with a central charge $c = 3\ell/2G_3$. Finally we discuss the entropy bounds for the SdS$_3$ black hole.
Recently an accelerating universe has proposed to be a way to interpret the astronomical data of supernova [1–3]. Combining this observation with the need of inflation in the standard cosmology leads to that our universe approaches de Sitter geometries in both the far past and the far future [4–6]. Hence it is very important to study the nature of de Sitter space. However, there are two difficulties in studying de Sitter space: First there is no spatial infinity and second there is no global timelike Killing vector. Thus it is not easy to define the conserved quantities including mass, charge and angular momentum appeared in the de Sitter black holes.

More recently authors in [7] proposed the mass ($M$), temperature ($T$) and entropy ($S$) for the SdS$_3$ black hole. But there is some ambiguity in defining the mass (or energy) of this black hole. They used a relation of $dS/d(-M) = 1/T$ to derive the entropy. We wish to point out that although this provides us the entropy, this is not a correct relation between these quantities. If one defines the mass ($M$) of this black hole properly, one can use the correct relation of $dS/dM = 1/T$ (the first law of the SdS$_3$ black hole thermodynamics) to find the entropy. Furthermore one guesses the SdS/CFT correspondence from the dS/CFT correspondence [8,4], as defined analogously to the AdS/CFT correspondence [9]. So it is important to establish this correspondence by computing both the boundary CFT entropy and bulk SdS entropy. For this purpose, the SdS$_3$ black hole plays the role of a toy model. Also there exist the entropy bounds of $N$-bound and $D$-bound for objects in asymptotically de Sitter space and Bekenstein($B$)-bound for asymptotically flat space [10]. Hence it is interesting to study the relation between these bounds for the SdS$_3$ black hole (a pointlike object) itself.

In this letter, we calculate the mass of the SdS$_3$ black hole using the correct definition of the mass in asymptotically de Sitter space [11]. Then we clarify the relation between the mass, entropy and temperature without any ambiguity. Also we establish the SdS$_3$/CFT$_2$-correspondence for the entropy by applying the Cardy formula to the CFT$_2$ with a central charge $c = 3\ell/2G_3$. Finally we discuss the entropy bounds for the SdS$_3$ black hole.

We start with the d-dimensional Schwarzschild-de Sitter metric in the static coordinates [7]

$$ds^2_d = -(1 - \frac{2m}{r_d-3} - \frac{r^2}{\ell^2})dt^2 + (1 - \frac{2m}{r_d-3} - \frac{r^2}{\ell^2})^{-1}dr^2 + r^2d\Omega_{d-2}^2, \quad (1)$$

where $2m = 16\pi G_d\mathcal{M}/Vol(S^{d-2})$ is a parameter related to the black hole mass [12] and $\ell$ is the curvature radius of de Sitter space. This is solution to the action

$$S_d = \frac{1}{16\pi G_d} \int d^dx \sqrt{-g} [R - 2\Lambda] \quad (2)$$

with the d-dimensional Newtonian constant $G_d$ and the d-dimensional positive cosmological constant $\Lambda = (d-1)(d-2)/2\ell^2$.

For the three dimensional Schwarzschild-de Sitter black hole, we have the metric as

$$ds^2_3 = -(1 - 8G_3\mathcal{M} - \frac{r^2}{\ell^2})dt^2 + (1 - 8G_3\mathcal{M} - \frac{r^2}{\ell^2})^{-1}dr^2 + r^2d\phi^2 \quad (3)$$

which can be rewritten by the new parameter $r_c = \sqrt{1 - 8G_3\mathcal{M}}$ as
\[ ds_3^2 = -\left( r_c^2 - \frac{r^2}{\ell^2} \right) dt^2 + \left( r_c^2 - \frac{r^2}{\ell^2} \right)^{-1} dr^2 + r^2 d\phi^2. \] (4)

In the case of \( r_c = 1 (\mathcal{M} = 0) \), we have de Sitter (dS) space. Thus we may identify \( \mathcal{M} \) as the mass of the SdS black hole. However, this attempt leads to a wrong result. We note here that \( 0 < r_c < 1 \) if \( \mathcal{M} \neq 0 \). Hence this describes a conical defect spacetime with deficit angle \( 2\pi(1 - r_c) \), indicating a world with a positive cosmological constant and a pointlike massive object with \( \mathcal{M} \) [7]. In order to obtain a conserved quantity of the mass (\( \mathcal{M} \)), we have to use a proper definition of the mass in an asymptotically de Sitter space as [11]

\[ M = \oint_{\Sigma} d\phi \sqrt{\sigma} N_{\mu} n^{\nu} T_{\mu\nu}. \] (5)

We use this formula on a surface of fixed time and then send time to infinity so it approaches the past(future) infinity at \( I^- (I^+) \). This is so because in a theory of gravity, mass is measure of how much a metric deviates near infinity from its vacuum state. In order to evaluate mass properly, we need an equal time surface \( \Sigma \) of the conical defect spacetime outside the cosmological horizon \( r > \ell r_c \) with

\[ ds_\Sigma^2 = \left( \frac{r^2}{\ell^2} - r_c^2 \right) dt^2 + r^2 d\theta^2. \] (6)

It is noted that for \( r < \ell r_c \), \( t(r) \) is timelike (spacelike) while \( r > \ell r_c \), \( t(r) \) is spacelike (timelike). As \( r \to \infty \), we thus have

\[ M = \frac{1}{8\pi G_3} \oint_{\mathcal{I}^{\pm}} d\theta \frac{r_c^2}{2} = \frac{r_c^2}{8G_3} = \frac{1}{8G_3} - \mathcal{M}. \] (7)

This defines the conserved quantity of mass (energy :\( E \)) for the SdS black hole. If \( \mathcal{M} = 0 \), we recover a mass \( M_{\text{dS}} = 1/8G_3 \) of the pure de Sitter space with a cosmological horizon at \( r_0 = \ell \). If \( \mathcal{M} \neq 0 \), we find a mass of the SdS black hole which is less than that of pure de Sitter space \( M_{\text{dS}} \). Then we ask of why a matter \( \mathcal{M} \) contributes to \( M \) as a negative one. The answer is that even if the matter of a pointlike object has a positive energy, the binding energy to the gravitational de Sitter background can make this a negative one. As a result, the quantity \( \mathcal{M} \) is not considered as the true mass (energy) of the SdS black hole. Authors in [7] used this mass \( \mathcal{M} \) to derive the entropy of the SdS black hole using \( dS/d(-\mathcal{M}) = 1/T \). Although this provides the entropy, this is not a correct expression for the first law of the black hole thermodynamics.

Now we have to use the correct expression to obtain the entropy. The temperature \( T \) of this black hole can be easily found from the fact that considering Eq.(3), the Euclidean green function is periodic in imaginary time (\( \tau \)) with periodicity

\[ \tau \to \tau + i\beta, \beta = \frac{2\pi \ell}{r_c}. \] (8)

From this we find the Hawking temperature

\[ T = \frac{r_c}{2\pi \ell}. \] (9)
Here we consider $M, T, S$ as functions of $r_c : M = M(r_c); T = T(r_c); S = S(r_c)$. Using the correct relation of the black hole thermodynamics

$$dS/dM = 1/T, \quad (10)$$

we obtain the Bekenstein-Hawking entropy

$$S_{sds} = \frac{\pi \ell r_c}{2G_3}. \quad (11)$$

The bulk entropy $S_{ds} = \frac{\pi \ell}{2G_3}$ with $r_c = 1$ for the dS$_3$ space appeared in [13]. We can easily find that for $0 < r_c < 1$, $S_{sds} < S_{ds}$.

Also the dS$_3$/CFT$_2$ correspondence was proposed by using the AdS$_3$/CFT$_2$ correspondence [8,4,9]. Now we are in a position to discuss the SdS$_3$/CFT$_2$ correspondence. This can be regarded as an extended version of the dS$_3$/CFT$_2$ correspondence. For this purpose, we use the Cardy formula together with the central charge. From the anomalous transformation law of the stress-energy tensor $T_{++} = -\frac{\ell}{16\pi G_3} \partial_+^3 \xi^+ (T_{--} = -\frac{\ell}{16\pi G_3} \partial_3 \xi^-)$ in a two-dimensional conformal theory at $\mathcal{I}^\pm$, we can read off the central charge $c$ [8,11]

$$c = \frac{3\ell}{2G_3}. \quad (12)$$

The eigenvalues of the conformal generators $L_0$ and $\bar{L}_0$ in static coordinates are given by [11]

$$L_0 = \bar{L}_0 = \frac{M\ell}{2}. \quad (13)$$

The Cardy formula for the asymptotic density of states of a unitary CFT is

$$S_{CFT} = 2\pi \sqrt{\frac{cL_0}{6}} + 2\pi \sqrt{\frac{c\bar{L}_0}{6}} \quad (14)$$

which gives us the boundary entropy at $\mathcal{I}^\pm$. This exactly leads to the bulk entropy Eq.(11) for the SdS$_3$ black hole as

$$S_{CFT} = 4\pi \ell \sqrt{\frac{M}{8G_3}} = \frac{\pi \ell r_c}{2G_3} = S_{sds}. \quad (15)$$

This establish the SdS$_3$/CFT$_2$ correspondence for the entropy.

Finally we discuss the entropy bounds for the SdS$_3$ black hole (a pointlike object with $\mathcal{M}$) which is located at $r = 0$ [10,12]. This corresponds to a very small, dilute object in asymptotically de Sitter space. The $N$-bound means that the entropy of an object in asymptotically de Sitter space is bounded by the de Sitter entropy : $S_{sds}^N \leq S_{ds}$. Further, applying the Geroch process to the cosmological horizon $r = r_c \ell$ leads to the entropy of an object in asymptotically de Sitter space is bounded by the difference of the entropies in pure de Sitter space and in asymptotically de Sitter space. This is called $D$-bound : $S_{sds}^D \leq S_{ds} - S_{sds}$. In addition, for a system of volume $V$ (linear size $R$) with the limited self-gravity, the total entropy of the system satisfies the Bekenstein bound ($B$-bound) :
$S_M^D \leq 2\pi R\mathcal{E}$. Here $\mathcal{E}$ is the energy of the system. This is the flat space bound and thus has nothing intrinsically to do with the de Sitter bounds like $N$-bound and $D$-bound.

Introducing the gravitational radius $r_g = 8G_3\mathcal{M}$ in Eq.(3), this corresponds a well-defined quantity in asymptotically de Sitter space. Here we have a relation $r_g = (r_0 - \ell \ell r_c)(r_0 + \ell r_c)/r_0^2$. In this case of $r_g << 1$ (a very small and dilute object localized at $r = 0$), we have $r_c \approx 1 - 4G_3\mathcal{M}$. Then $D$-bound reduces to $S_M^D \leq 2\pi r_0\mathcal{M}$. Replacing $r_0 \approx r_c$ by $R$ and $\mathcal{M}$ by $\mathcal{E}$, $D$-bound leads to $B$-bound : $S_M^D \rightarrow S_M^B \leq 2\pi R\mathcal{E}$. This shows an example for the relation between $D$-bound and $B$-bound : for dilute spherical systems, $D$-bound in asymptotically de Sitter space coincides with $B$-bound in flat space [10].

In conclusion, we calculate the mass of the SdS$_3$ black hole using the correct definition of the mass in asymptotically de Sitter space [11]. Then we use the first law of thermodynamics which is the relation between the mass, entropy and temperature to obtain the bulk Bekenstein-Hawking entropy. Also we establish the SdS$_3$/CFT$_2$-correspondence for the entropy by applying the Cardy formula to the CFT$_2$ with a central charge $c = 3\ell/2G_3$. Finally we show that $D$-bound coincides with $B$-bound in flat space.

ACKNOWLEDGEMENT

We thank H.W. Lee for helpful discussions. This work was supported in part by the Brain Korea 21 Program of Ministry of Education, Project No. D-1123 and KOSEF, Project No. 2000-1-11200-001-3.
REFERENCES


