Aspects of the thermal phase transition of QCD with small chemical potential


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We propose a new method to investigate the thermal properties of QCD with a small chemical potential (\(\mu\)). The derivatives of the phase transition point with respect to \(\mu\) are computed for 2 flavors of p4-improved staggered fermions. We moreover comment on the complex phase of the fermion determinant in finite-density QCD.

1. Introduction

The study of the quark-gluon plasma in heavy-ion collision experiments is one of the most interesting topics in contemporary physics. To understand these experiments, precise theoretical inputs of the phase transition of QCD are indispensable. Over the last several years, the numerical study of lattice QCD has been successful at zero quark chemical potential (\(\mu\)) and high temperature [1]. In contrast, because the quark determinant is complex at \(\mu \neq 0\) and Monte-Carlo simulation is not applicable directly, study at finite \(\mu\) is still in the stage of development.

Fortunately, the interesting regime for heavy-ion collisions is rather low density, e.g., \(\mu \sim 15\text{MeV (}\mu/T_c \sim 0.1\text{)}\) for RHIC [2]. Therefore, an approach of Taylor expansion may be the most efficient way, by computing the derivatives of physical quantities in terms of \(\mu\) at \(\mu = 0\) to determine these coefficients. Pioneering works in such a framework are given for free energy (quark number susceptibility) [3] and screening mass [4].

In this study, we investigate the transition point (\(\beta_c\)) at finite \(\mu\). A similar study has been done by [5]. In sec. 2, we propose a new method to compute the derivative of the physical quantities with respect to \(\mu\). The result of the second derivative of \(\beta_c\) for 2 flavor QCD is given in sec. 2. Reweighting method for \(\mu\)-direction

Basically most of the attempts to study finite density QCD have been tried by performing simulations at \(\mu = 0\) and reweighting by the identity:

\[
\langle O \rangle_{(\beta, \mu)} = \frac{\langle OW \rangle_{(\beta_0, 0)}}{\langle W \rangle_{(\beta_0, 0)}}, \tag{1}
\]

\[
W = e^{\alpha N_f (\ln \det M(\mu) - \ln \det M(0))} e^{-S_g(\beta) + S_\mu(\beta_0)},
\]

where \(M\) is the fermion matrix, \(S_g\) is the gauge action, \(N_f\) is the number of flavors and \(\alpha\) is 1 or 1/4 for the Wilson or staggered fermion.

The reweighting factor of the gauge part is easy to compute [6]. However, the fermion part is highly non-local and difficult to compute in practice. Here we consider a method which is applicable for small \(\mu\). We perform a Taylor expansion for the fermion part of the reweighting factor around \(\mu = 0\). Similarly, we expand fermionic observables, e.g., the chiral condensate, \(\langle \bar{\psi}\psi \rangle = (1/V)\alpha N_f \langle \text{tr} M^{-1} \rangle\). If we consider up to the \(n\)-th derivative of both the reweighting factor and the fermionic observable, we can calculate the \(n\)-th derivative of the physical quantity with respect to \(\mu\) correctly, which can be easily checked by performing a Taylor expansion of each expectation value of the physical quantity. The derivatives of \(\ln \det M\) and \(\text{tr} M^{-1}\) can be calculated by the random noise method, which enables us to compute on a rather large lattice in comparison with usual studies of finite density QCD.

3. We discuss the problem of the complex phase in sec. 4. Section 5 presents the conclusions.
Table 1

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \beta_c(\bar{\psi}\psi) )</th>
<th>( \frac{d^2 \beta_c}{d\mu^2} )</th>
<th>( \beta_c(\text{Polyakov}) )</th>
<th>( \frac{d^2 \beta_c}{d\mu^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.6463(25)</td>
<td>-1.41(61)</td>
<td>3.6489(27)</td>
<td>-1.71(67)</td>
</tr>
<tr>
<td>0.2</td>
<td>3.7575(19)</td>
<td>-1.51(42)</td>
<td>3.7590(20)</td>
<td>-1.61(36)</td>
</tr>
</tbody>
</table>

Figure 1. Chiral susceptibility at \( m = 0.2 \).

Moreover, we should notice that the odd order derivatives of \( \ln \det M \) and \( \text{tr} M^{-1} \) are pure imaginary and the even order derivatives are real at \( \mu = 0 \). This property is proved using the identities:

\[ M^\dagger(\mu) = \gamma_5 M(-\mu)\gamma_5, \quad \text{and} \quad \frac{d^n M^\dagger}{d\mu^n}(\mu) = (-1)^n \gamma_5 \frac{d M}{d\mu}(-\mu)\gamma_5. \]

From this property, we can explicitly confirm that, if the operator we compute is real at \( \mu = 0 \), e.g., \( \langle \bar{\psi}\psi \rangle \), the first derivative of the expectation value is zero at \( \mu = 0 \), as we expect from the symmetry under the change from \( \mu \) to \(-\mu\). We moreover can easily estimate the phase factor of the fermion determinant from this feature, which is discussed in sec. 4.

3. Results for \( N_f = 2 \) improved staggered

We calculate the second derivative of \( \beta_c \) with respect to \( \mu \). We employ a combination of the Symanzik improved gauge and the p4-improved staggered fermion actions [7]. It is known that this action makes the discretization error of pressure small at high temperature, and \( T_c \) obtained by this action is consistent with that using improved Wilson fermions [8]. To include the finite density effect, we multiply the hopping terms proceeding \( n \)-steps in positive and negative temporal directions by \( e^{\mu n} \) and \( e^{-\mu n} \) respectively.

We investigate the transition points at \( m = 0.1 \) and 0.2. The corresponding pseudo-scalar and vector meson mass ratios are \( m_{PS}/m_V \approx 0.7 \) and 0.85. We compute Polyakov loop, chiral condensate, and their susceptibilities. The simulations are performed on a \( 16^4 \times 4 \) lattice at \( \beta = 3.64-3.65 \) with a total of 45000 trajectories for \( m = 0.1 \), and \( \beta = 3.74-3.80 \) with 76700 (Polyakov) or 69000 (\( \bar{\psi}\psi \)) trajectories for \( m = 0.2 \). 10 sets of Z(2) noise vectors are used for each trajectory to compute the reweighting factor up to the second order. We plot, in Fig.1, the chiral susceptibility at \( m = 0.2 \). This figure shows that the peak position becomes lower as \( \mu \) increases. Figures 2 show the transition point determined by the peak position of the Polyakov loop susceptibility and the chiral susceptibility as a function of \( \mu^2 \). Because the first derivative is zero, as we discussed before, we fit the data for \( \beta_c \) by a straight line in \( \mu^2 \), fixing \( \beta_c \) at \( \mu = 0 \), in a range, \( \mu^2 \leq 0.005 \) and 0.01 for \( m = 0.1 \) and 0.2, respectively, in which the phase problem is not serious (see sec. 4). We then obtain the results in Table 1. Quark mass dependence of \( \frac{d^2 \beta_c}{d\mu^2} \) is not visible within the error. We denote the interesting value of \( \mu \) for RHIC in Figs.2. The shift of \( \beta_c \) from \( \mu = 0 \) is
Consequently the phase problem starts to appear at $\mu \sim 0.07(0.1)$, i.e., $\mu_{\text{phys}}/T_c \sim 0.3(0.4)$ for $m = 0.1(0.2)$, since the phase problem arises if the phase fluctuation becomes larger than O(1). We notice that the value of $\mu$ for which the phase fluctuations becomes significant decreases as the quark mass increases. Figure 3 shows the average of the phase factor $e^{i\theta}$. We can see that the average becomes small around those values of $\mu$.

5. Conclusions

We propose a new method based on Taylor expansion to investigate thermal properties of finite density QCD. We computed the chiral susceptibility and the Polyakov loop susceptibility for 2 flavors of p4-improved staggered fermions, and found that $\beta_c$ and $T_c$ become smaller as $\mu$ increases. Our results suggest that the discrepancy of $T_c$ from $\mu = 0$ is small in the interesting region for heavy-ion collisions. We also estimated the complex phase of the fermion determinant for a $16^3 \times 4$ lattice. We found that the sign problem is not serious in the range of $\mu_{\text{phys}}/T_c < 0.3 - 0.4$ for $m = 0.1 - 0.2$, which covers the regime of the heavy-ion collisions at RHIC.

REFERENCES

5. Z. Fodor and S.D. Katz, hep-lat/0104011; hep-lat/0106002.