Superfield Noether Procedure

Marc Magro 1
Laboratoire de Physique, École normale supérieure de Lyon,
46, Allée d’Italie, 69364 Lyon - Cedex 07, France

Ivo Sachs 2
Theoretische Physik, Ludwig-Maximilians Universität,
Theresienstrasse 37, D-80333, München, Germany

Sylvain Wolf 3
Institut de Physique Théorique, BSP, Université de Lausanne
1015 Lausanne, Switzerland

Abstract

We develop a superspace Noether procedure for supersymmetric field theories in
4-dimensions for which an off-shell formulation in ordinary superspace exists. In
this way we obtain an elegant and compact derivation of the various supercurrents
in these theories. We then apply this formalism to compute the central charges for
a variety of effective actions. As a by-product we also obtain a simple derivation
of the anomalous superconformal Ward-identity in $\mathcal{N} = 2$ Yang-Mills theory.
The connection with linearized supergravity is also discussed.

1Marc.Magro@ens-lyon.fr
2ivo@theorie.physik.uni-muenchen.de
3Sylvain.Wolf@ipt.unil.ch
1 Introduction

Noether currents play an important role in any theory with a continuous global symmetry. Moreover, when there is more than one invariance, the various Noether currents themselves form a multiplet for the extended symmetry group. This property is used extensively in supersymmetric theories. Indeed, soon after the first 4-dimensional supersymmetric field theory was proposed by Wess and Zumino [1], the corresponding multiplet of Noether currents, containing the energy momentum tensor, the supersymmetry current and the R-current was constructed [2]. This multiplet structure plays an important role in exploring non-perturbative properties of the quantum theory. So, for example, the rigid multiplet structure of the Noether currents made it possible to obtain the $\beta$-function to all orders for minimally supersymmetric Yang-Mills theory [3, 4, 5]. Furthermore, the multiplet structure of these currents was crucial in explaining higher loop finiteness of theories with extended supersymmetry [6, 7, 8]. Similarly, the supercurrent of $\mathcal{N}=2$ Yang-Mills theory played an important role in deriving the Seiberg-Witten low energy effective action for that theory [9, 10, 11]. On another front, the Noether currents of supersymmetric matter theories can be used to construct linear off-shell supergravities [12].

As with any symmetry, the multiplet structure is best discussed by using a manifestly covariant formalism. For supersymmetric theories, the manifestly covariant formulation is in terms of superfields [13]. However, in much of the existing literature on supercurrents, the approach followed consists of first constructing the multiplet in terms of the component currents and then express the result in terms of superfields. This is partly related to the fact that superfields are not adapted to the canonical formalism in which the Noether currents are usually obtained. This problem can be circumvented by using a variational approach to obtain the Noether currents. However, one typically encounters constrained superfields when formulating realistic supersymmetric theories. Solving these constraints is possible for all $\mathcal{N}=1$ theories but has been worked out in ordinary superspace only for a limited subset of the various theories with extended supersymmetry. Thus, the variational approach is complicated by having to deal with constraint preserving variations.

In spite of these drawbacks, a manifestly covariant derivation of the multiplet of Noether currents is certainly desirable and this is the purpose of the present paper. Concretely, we develop a manifestly covariant Noether procedure (Superfield Noether Procedure) for theories for which an off-shell superfield formulation exists. While this approach may not appear to be very economic to recover the known multiplets of Noether currents in the simplest models, it is rather powerful in generalizing these results to more complicated Lagrangians. In particular, we obtain the supercurrents for a variety of $\mathcal{N}=1$ and $\mathcal{N}=2$ multiplets with arbitrary local action. Such actions arise typically as low energy effective descriptions of the quantum theory. Furthermore, our procedure provides an elegant and economic derivation of the supersymmetry central charge in models with arbitrary Lagrangians, not just holomorphic ones. As another simple application, we will give a simple derivation of the anomalous superconformal Ward-Identity in $\mathcal{N}=2$ Yang-Mills theory [18]. Finally, the superfield Noether procedure provides a tool for a simple construction of linearized supergravities directly at

\footnote{See however [14, 15, 16, 17] for superfield approaches to the Noether currents.}
the superfield level. In particular, we recover various known, linearized $\mathcal{N} = 1$ supergravities \cite{6}, \cite{19}-\cite{23} as well as some $\mathcal{N} = 2$ supergravities \cite{24}-\cite{28} in a simple and uniform manner.

The paper is organized as follows. In section 2, we begin with a review of the super-space diffeomorphism transformations and of the superconformal and super-Poincaré subgroups. In section 3, we describe the general idea underlying the super-Noether procedure developed in this paper. Then, in section 4, we apply this formalism to several $\mathcal{N} = 1$ theories, obtain their supercurrents and compute the corresponding central charges. In section 5, we then extend the formalism to $\mathcal{N} = 2$ theories with off-shell superfield formulation and apply it to the $\mathcal{N} = 2$ vector and tensor multiplets. As a by-product, we obtain a simple derivation of the anomalous superconformal Ward Identity for the vector multiplet. Furthermore, we compute the effective central charge including fermions in the theory as well as the contributions from the non-holomorphic part of the effective action for the vector multiplet. For the tensor multiplet, we derive the supercurrent and discuss the central charge. In section 6, we discuss the construction of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ linearized superfield supergravities using the superfield Noether procedure. Finally, we present the conclusions in the last section.

2 Superdiffeomorphisms

In this paper, we consider field theories formulated on the $\mathcal{N}$-extended superspace described by $z = (x^\mu, \theta^\alpha, \bar{\theta}^\dot{\alpha}) \in \mathbb{R}^{4|4N}$, $i$ being the $SU(N)$ index. Although we are interested in the Noether currents corresponding to the super-Poincaré transformations, it will be useful to view them as a subgroup of the larger group of superdiffeomorphisms. For this we consider the complex chiral- and anti chiral superspaces parametrized by $z_+ = (x_+^\mu, \theta_\alpha)$ and $z_- = (x_-^\mu, \bar{\theta}^{\dot{\alpha}})$. For arbitrary $\mathcal{N}$ this may not be the most natural approach to localise the super-Poincaré transformations. However, in view of the Noether procedure described below it is convenient to localise the super-Poincaré transformations by considering those superdiffeomorphisms which preserve chirality. Such transformations can be described in terms of a superfield $h^{\alpha \dot{\alpha}}$, subject to the constraints

$$\bar{D}_{\dot{i}}^{(\beta} h_i^{\alpha \dot{\alpha})} = 0, \quad D^{(\beta} h_i^{\alpha \dot{\alpha})\dot{\alpha}} = 0.$$ (2.1)

The corresponding transformations of the chiral coordinates are given by

$$\delta x_+^\mu = h_+^\mu(z) + 2i \lambda_+^i(z_+) \sigma^\mu \bar{\theta}^i, \quad \delta \theta_+^i = \lambda_+^i(z_+) ,$$
$$\delta x_-^\mu = h_-^\mu(z) - 2i \theta_-^i \sigma^\mu \bar{\lambda}_{\dot{i}}, \quad \delta \bar{\theta}_{\dot{i}} = \bar{\lambda}_{\dot{i}}(z_-) ,$$ (2.2)

with

$$\lambda_+^i(z_+) = -i 8 \bar{D}_{\dot{i}} h_i^{\alpha \dot{\alpha}}, \quad \bar{\lambda}_{\dot{i}}(z_-) = i 8 D^i h_i^{\alpha \dot{\alpha}}.$$ (2.3)

The representation of (2.1) in terms of the corresponding differential operators acting on chiral and antichiral superfields are then

$$\mathcal{L}_+ = h_+^\mu \partial_\mu + \lambda_+^i D_i^\dot{\alpha} , \quad \mathcal{L}_- = h_-^\mu \partial_\mu + \bar{\lambda}_{\dot{i}} D_\dot{i}^i .$$ (2.4)

---

\footnotetext[5]{We use essentially Wess and Bagger conventions \cite{29}, see appendix C.}

\footnotetext[6]{See e.g. \cite{30} for a comprehensive discussion of superspaces and their isometries.}
These commute with the chirality constraint, \([\bar{D}_{\dot{a}i}, \mathcal{L}_+] = 0\) and \([D^i_{\alpha}, \mathcal{L}_-] = 0\), as a consequence of (2.1).

## 2.1 Superconformal Transformations

An important subgroup of the superdiffeomorphisms containing the super-Poincaré transformations is the superconformal group \([31, 32]\) obtained by imposing, in addition to (2.1), the constraint \(^7\)

\[
h^{\alpha\dot{\alpha}} = h^{\alpha\dot{\alpha}} .
\]

From (2.1) and (2.5), we can now easily extract the Killing equation

\[
\partial_\mu h_\nu + \partial_\nu h_\mu = \frac{1}{2} \eta_{\mu\nu} \partial^\rho h_\rho ,
\]

the general solution of which is [34]:

\[
h^{\alpha\dot{\alpha}} = a^{\alpha\dot{\alpha}} + 4i\varepsilon_1^a \bar{\theta}^{\dot{\alpha}a} + 4i\bar{\varepsilon}_1^\alpha \theta^{\alpha a} - \omega^\alpha_{\beta} x^{\beta}_- + \bar{\omega}^{\dot{\alpha}}_{\dot{\beta}} x^{\dot{\beta}}_+ - 4\eta \bar{\theta}^a \bar{\theta}^{\dot{\alpha}a} - 6i\eta_1^a \bar{\theta}^{\dot{\alpha}a} \bar{\theta}^{\dot{\beta}a} - 4\eta_1 \bar{\theta}^{\dot{\alpha}a} \bar{\theta}^{\dot{\beta}a} (\text{with } (\eta_1)_1 = 0, \eta_1 = 0, \text{ dilations } \kappa, \text{ special conformal transformations } b_{aa}, \text{ and special superconformal transformations } \rho_{\alpha a},)
\]

The different parameters correspond to translations \(a^{\alpha\dot{\alpha}}, \) supersymmetry transformations \(\varepsilon_1^a, \) Lorentz transformations \(\omega^\alpha_{\beta} \) (with \(\omega^\alpha_{\beta} = \omega^\beta_{\alpha} \) and \(\omega^\alpha_{\alpha} = 0\)), \(U(1)_K\) transformations \(\eta, \) \(SU(N)\)-transformations \(\eta_1^a \) (with \((\eta_1^a)_+ = -\eta_1^a \) and \(\eta_1^a = 0\)), dilations \(\kappa, \) special conformal transformations \(b_{aa}, \) and special superconformal transformations \(\rho_{\alpha a} \).

For superconformal transformations there is a simple generalization of the action (2.4) to a general unconstrained superfield given by

\[
\mathcal{L} = \frac{1}{2} (h^\mu + \bar{h}^{\mu}) \partial_\mu + \lambda_i^a D^i_{\alpha a} + \bar{\lambda}_{\dot{i}}^{\dot{a}} \bar{D}^i_{\dot{\alpha} \dot{a}} ,
\]

reducing to (2.4) when acting on chiral or anti-chiral superfields respectively. For later use we also give the algebra between \(\mathcal{L}\) and \(D^i_{\alpha a}:\)

\[
[D^i_{\alpha a}, \mathcal{L}] = \frac{3}{N(4 - N)} ((N - 2)\sigma + 2\bar{\sigma}) D^i_{\alpha a} - i\Lambda^i_{\dot{j}} D^i_{\alpha a} - \Omega^i_{\alpha \beta} D^i_{\beta} ,
\]

where \(\sigma, \Lambda^i_{\dot{j}}\) and \(\Omega^i_{\alpha \beta}\) are defined by

\[
\sigma = \frac{1}{6} \left( D^i_{\alpha a} \lambda_{\dot{a}i} - \frac{1}{2} \partial_{a\dot{a}} h^{\alpha\dot{\alpha}} \right) ,
\]

\[
\Lambda^i_{\dot{j}} = -\frac{i}{4} \left( D^i_{\alpha a} \lambda_{\dot{a}j} + \bar{D}_{\dot{a}j} \bar{\lambda}^{\dot{a}i} - \frac{1}{N} \delta^i_{\dot{j}} \left( D^k_{\alpha a} \lambda^i_{\alpha k} + \bar{D}_{\dot{k}i} \bar{\lambda}^{\dot{i}k} \right) \right) ,
\]

\[
\Omega^i_{\alpha \beta} = \frac{1}{2 - N} \left( D^i_{\alpha a} \lambda_{\beta j} + \frac{1}{2} \partial_{a\dot{a}} h^{\alpha\dot{\alpha}} \right) .
\]

Both \(\sigma\) and \(\Omega^i_{\alpha \beta}\) are chiral and \(\Lambda^i_{\dot{j}}\) is hermitian and traceless, \(\text{i.e.}\)

\[
D_{\dot{a}i} \sigma = 0 , \quad \bar{D}_{\dot{a}i} \Omega^i_{\alpha \beta} = 0 , \quad (\Lambda^i_{\dot{j}})^* = \Lambda^i_{\dot{j}} , \quad \Lambda^i_{\dot{j}} = 0 .
\]

\(^7\)For detailed discussions of the multiplet of superconformal transformations we refer to [32, 16, 33, 30, 18, 17, 34]. We will give generalizations where necessary.
For superconformal transformations the various objects on the r.h.s. of (2.9) can then be expressed in terms of the parameters of the superconformal group as

\[ \lambda^{\alpha}_{i} = \varepsilon^{\alpha}_{i} - \omega^{\alpha}_{\beta} \theta^{\beta}_{i} + i\eta \theta^{\alpha}_{i} + \frac{1}{2} \kappa \theta^{\alpha}_{i} \]

\[ + \theta^{\bar{\alpha}}_{i} b_{\alpha \bar{\beta}} x_{\alpha}^{\bar{\beta}} - \theta^{\bar{\beta}}_{i} \rho^{\bar{\beta}}_{\alpha} \theta^{\alpha}_{j} + \frac{1}{2} \bar{\rho}_{\alpha i} x^{\bar{\alpha}}_{i} , \]

\[ \sigma = \frac{1}{3} (4 - N) \left( \kappa + \theta^{\alpha}_{i} \rho^{\alpha}_{i} + b_{\alpha \bar{\alpha}} x^{\bar{\alpha}}_{i} \right) - \frac{i}{3} N \eta , \]

\[ \Lambda^{i}_{j} = - \frac{3i}{2} \eta^{i} + 4 b_{\alpha \bar{\alpha}} \left( \theta^{\alpha}_{i} \bar{\theta}^{\alpha}_{i} - \frac{1}{N} \delta^{i}_{k} \rho^{\alpha}_{k} \rho^{\alpha}_{i} \right) \]

\[ - i \left( \theta^{\alpha}_{i} \rho^{\alpha}_{i} - \frac{1}{N} \delta^{i}_{k} \rho^{\alpha}_{k} \rho^{\alpha}_{i} \right) - i \left( \bar{\theta}^{\alpha}_{i} \bar{\rho}_{\alpha i} - \frac{1}{N} \delta^{i}_{k} \bar{\theta}^{\alpha}_{k} \bar{\rho}_{\alpha k} \right) , \]

\[ \Omega^{\alpha}_{\beta} = \omega^{\alpha}_{\beta} - b^{\alpha}_{\beta} \frac{x^{\bar{\beta}}}{i} + \rho^{\alpha}_{i} \theta^{\beta}_{i} . \]

2.2 Super-Poincaré Transformations

Finally, the super-Poincaré group is defined as the subgroup of the superconformal group containing the translations, the Lorentz transformations and the supersymmetry transformations. It is characterized by the conditions

\[ \sigma = \bar{\sigma} = \Lambda^{i}_{j} = 0 . \]

As consequences of these conditions, it is easily seen that, for example, \( \partial_{\mu} h^{\mu} = \partial_{\mu} \bar{h}^{\mu} = D^{i}_{a} \lambda^{\alpha}_{i} = \bar{D}_{a i} \lambda^{\alpha}_{i} = 0. \)

3 Superfield Noether Procedure

The purpose of this section is to develop a general formalism to extract the multiplet of Noether currents (supercurrent) associated with the super-Poincaré (superconformal) transformations for an arbitrary theory formulated in terms of \( N \)-extended, unconstrained superfields. For \( N = 1 \) a general procedure to extract a supercurrent was proposed in [14, 35]. Here we elaborate on a superfield Noether procedure given in [16, 17]. The starting point is an abstract supersymmetric theory whose action can be expressed in terms of unconstrained superfields. Let us denote by \( O^{A J} \) a generic, unconstrained superfield, where \( A \) is a collective vector and spinor index and \( J \) transforms in a representation of the \( SU(N) \) internal group. On \( O^{A J} \), the superconformal group acts as

\[ \delta O^{A J}(z) = - \mathcal{L} O^{A J} + (\Omega^{\alpha}_{\beta} S^{\alpha}_{\beta} + \Omega^{\bar{\alpha}}_{\bar{\beta}} S^{\bar{\alpha}}_{\bar{\beta}})_{B}^{A} O^{B J} \]

\[ + i \Lambda^{i}_{j}(R^{i})_{j}^{K} O^{A K} - 2(q \sigma + \bar{q} \bar{\sigma}) O^{A J} , \]

where \( S^{\alpha}_{\beta} \) and \( R^{i} \) are the Lorentz and \( SU(N) \) generators of the representation to which \( O^{A J} \) belongs. The dimension \( d \) and \( U(1)_{R} \)-weight of \( O^{A J} \) are related to \( q \) and \( \bar{q} \) through

\[ d = \frac{4 - N}{3} (q + \bar{q}) , \]

\[ R = - \frac{2N}{3} (q - \bar{q}) . \]
For a global super-Poincaré transformation, we thus have $\delta S[O^{A\bar{A}}]=0$, where $S[O^{A\bar{A}}]$ is the action. As in ordinary Noether procedure, we will then consider a local transformation, by letting the parameters have an arbitrary $x$-dependence. We will implement this by removing the reality condition (2.5) but still maintaining (2.1), i.e. the chirality preserving constraint.\(^8\) Of course, there is always the ambiguity of adding terms to $\delta O^{A\bar{A}}$ which are proportional to derivatives of the parameters of super-Poincaré transformations. By construction, the terms in $\delta S[O^{A\bar{A}}]$ induced by them are of the form “derivatives of the parameters” times “equations of motions” and thus induce in the currents terms which vanish on-shell. We will make use of this freedom when dealing with constrained superfields. Using (3.1), the variation of the action can then always be written as

$$\delta S[O^{A\bar{A}}] = \frac{i}{16} \int d^{4+4N}z \left\{ \left( h^{\dot{\alpha}\dot{\bar{\alpha}}} - \bar{h}^{\dot{\alpha}\dot{\bar{\alpha}}} \right) T_{\alpha\dot{\alpha}} + i \left( h^{\dot{\alpha}\dot{\bar{\alpha}}} + \bar{h}^{\dot{\alpha}\dot{\bar{\alpha}}} \right) K_{\alpha\dot{\alpha}} \right\} \tag{3.3}$$

with $T_{\alpha\dot{\alpha}}$ and $K_{\alpha\dot{\alpha}}$ real and $J$ chiral. However, it is important to note that the superfields $T_{\alpha\dot{\alpha}}, K_{\alpha\dot{\alpha}}$ and $J$ are not unique. Indeed, as a consequence of (2.1) we have the two identities

$$\begin{aligned}
-\frac{1}{2}[D^i_\alpha, D_{\dot{\alpha}\bar{\alpha}}](h + \bar{h})^{\dot{\alpha}\dot{\bar{\alpha}}} - 48i(\sigma - \bar{\sigma}) - i(4 - N)\partial_{\alpha\dot{\alpha}}(h - \bar{h})^{\dot{\alpha}\dot{\bar{\alpha}}} &= 0 , \\
-\frac{1}{2}[D^i_\alpha, D_{\dot{\alpha}\bar{\alpha}}](h - \bar{h})^{\dot{\alpha}\dot{\bar{\alpha}}} - 48i(\sigma + \bar{\sigma}) - i(4 - N)\partial_{\alpha\dot{\alpha}}(h + \bar{h})^{\dot{\alpha}\dot{\bar{\alpha}}} &= 0 ,
\end{aligned} \tag{3.4}$$

which, upon integration by parts lead to the equivalence relation

$$\begin{pmatrix}
T_{\alpha\dot{\alpha}} \\
K_{\alpha\dot{\alpha}} \\
J
\end{pmatrix} \simeq \begin{pmatrix}
T_{\alpha\dot{\alpha}} + \frac{1-N}{2}\partial_{\alpha\dot{\alpha}}(A - \bar{A}) - \frac{N}{4(4-N)}[D^i_\alpha, D_{\dot{\alpha}\bar{\alpha}}](A + \bar{A}) \\
K_{\alpha\dot{\alpha}} + \frac{i}{4}[D^i_\alpha, D_{\dot{\alpha}\bar{\alpha}}](A - \bar{A}) + \frac{N}{2}\partial_{\alpha\dot{\alpha}}(A + \bar{A}) \\
J - \frac{6}{4-N}D^{2N}(2A + (N - 2)\bar{A})
\end{pmatrix}, \tag{3.5}
$$

where $A$ is any unconstrained superfield. This equivalence relation corresponds to the freedom of adding improvement terms to the different Noether currents in the multiplets. Qualitatively this can be understood by noting that the equivalence classes (3.5) are obtained through integration by parts or equivalently, by adding boundary terms. We will explain this correspondence in more detail in the following sections.

There are two distinguished sets of multiplets of Noether currents. We denote them by the minimal and the canonical multiplets respectively. To explain this, we first note that vanishing of $\delta S$ for global super-Poincaré transformations implies \(^9\)

$$K_{\alpha\dot{\alpha}} = -\frac{i}{4}[D^i_\alpha, D_{\dot{\alpha}\bar{\alpha}}](X - \bar{X}) - \frac{N}{2}\partial_{\alpha\dot{\alpha}}(X + \bar{X}) , \tag{3.6}$$

\(^8\)One might wonder why this should be the correct way to localize the global transformation. However, it is not too hard to see that removing the reality constraint (2.5) is sufficient to allow for an arbitrary $x$-dependence of the parameters.

\(^9\)For $N = 1$, this was shown in [17]. For $N = 2$, this can be proved following the same argument as in [17]. Note however that the $N = 2$ case is more elaborated than in $N = 1$ as one has to use not only super-Poincaré invariance but also $SU(2)_R$ invariance. The proof of (3.6) for arbitrary $N$ should follow the same pattern.
for some $X$. Hence, starting from a supermultiplet $(T_{a\dot{a}}, K_{a\dot{a}}, J)$ and taking $A = X$, we arrive at the minimal multiplet $^1$ $(T^\text{min}_{a\dot{a}}, K^\text{min}_{a\dot{a}} = 0, J^\text{min})$. In particular, for a superconformal invariant theory, $\delta S = 0$ for a superconformal transformation and hence (3.3) implies that $J^\text{min} = \bar{J}^\text{min} = 0$. It then follows on general grounds that in this case the supercurrent $T_{a\dot{a}}$ contains the traceless (i.e. improved) currents. This is the smallest possible multiplet of Noether currents. If the theory does not have superconformal invariance, the $K_{a\dot{a}} = 0$ multiplet of currents is still a minimal one.

The other multiplet is the canonical multiplet. This multiplet contains the generators of the super-Poincaré transformations and is related to the minimal multiplet by total derivative terms. This multiplet has the further property that it contains among its components central charges.

Thus far the discussion applies to any theory formulated in terms of unconstrained superfields. However, as already mentioned in the introduction, in realistic situations we typically have to deal with constrained superfields. In general, the localization of the super-Poincaré or superconformal transformations in terms of an arbitrary parameter $h^{a\dot{a}}$ subject to (2.1) may not be compatible with the constraints on the superfield. In that case, one may have to impose stronger constraints on $h^{a\dot{a}}$. The purpose of the next sections is to apply and extend the present discussions to the known cases for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ where off-shell superfield formulations exist.

4 $\mathcal{N} = 1$ Supersymmetry

In this section, we will apply the general formalism of section 3 to an arbitrary $\mathcal{N} = 1$ theory formulated in terms of (possibly constrained) $\mathcal{N} = 1$ superfields. Indeed, it turns out that the general structure described above applies directly without modification to any $\mathcal{N} = 1$ theory with an off-shell superfield formulation.

4.1 General Structure

In order to explore the general structure of the current multiplet we begin by noting that the constraint (2.1) can be solved in terms of a spinorial superfield $L^\alpha$ as

$$h^{a\dot{a}} = 2\bar{D}^a L^\alpha, \quad \bar{h}^{a\dot{a}} = -2D^a \bar{L}^{\dot{\alpha}}.$$  \hspace{1cm} (4.1)

From (2.3) and (2.10), we then obtain

$$\lambda^\alpha = -\frac{i}{4} \bar{D}^2 L^\alpha, \quad \Omega_{\alpha\beta} = \frac{i}{8} \bar{D}^2 D_{(a} L_{\beta)}, \quad \sigma = -\frac{i}{24} \bar{D}^2 D^a L_\alpha.$$  \hspace{1cm} (4.2)

For a global superconformal transformation (2.7), we can write $L^\alpha$ as

$$L^\alpha = -\frac{1}{2} \theta^{\alpha\dot{a}} \bar{\theta}_\dot{a} + i\varepsilon^{\alpha\dot{a}} \bar{\theta}^2 - 2i\bar{\varepsilon}_{\dot{a}} \bar{\theta}^a \theta^\alpha = -\frac{1}{2} \omega^{\alpha\beta} x^{\beta\dot{\alpha}} \bar{\theta}_\dot{a} + \frac{1}{2} \bar{\omega}^{\dot{a} \dot{\beta}} x^{\alpha\beta} \bar{\theta}_\dot{a} - \eta \theta^\alpha \bar{\theta}^2$$  \hspace{1cm} (4.3)

$$+ \frac{1}{2} \kappa x^{\alpha\dot{a}} \bar{\theta}_\dot{a} - \frac{i}{2} \chi \theta^\alpha \bar{\theta}^2 + \frac{i}{2} x^{\beta\dot{a}} b_{\beta\dot{a}} \bar{\theta}_\dot{a} + \frac{1}{2} x^{\beta\dot{a}} \rho_{\beta\dot{a}} \bar{\theta}_\dot{a} - \frac{1}{4} \bar{\theta}^2 \bar{\theta}_\dot{a} + \frac{1}{2} \theta^\alpha x^{\alpha\dot{a}}.$$
Using the results in section 3 we can now write the variation of the action under an arbitrary local transformation parametrized by $L^\alpha$ as $^{11}$ [16, 17]

$$\delta S = \int d^8z \left( iS^\alpha R_\alpha + \text{c.c.} \right), \quad \text{with}$$

$$R_\alpha = \bar{D}^\dot{\alpha} T_{a\dot{\alpha}} + i\bar{D}^\dot{\alpha} K_{a\dot{\alpha}} - \frac{1}{6} D_\alpha J,$$

where $K_{a\dot{\alpha}}$ is given by (3.6). Variation with respect to $L^\alpha$ then leads to the supercurrent conservation equation

$$\bar{D}^\dot{\alpha} T_{a\dot{\alpha}} + i\bar{D}^\dot{\alpha} K_{a\dot{\alpha}} - \frac{1}{6} D_\alpha J = 0 .$$

Using (3.6) we can recast the conservation equation (4.5) in a more familiar form

$$\bar{D}^\dot{\alpha} T_{a\dot{\alpha}} + W_\alpha + D_\alpha \tau = 0 ,$$

where

$$W_\alpha = \frac{1}{4} D^2 D_\alpha (X - 2\bar{X}) , \quad \tau = -\frac{1}{6} J + \frac{1}{4} D^2 X .$$

As explained in section 3, the multiplets appearing in (4.5) and (4.6) are not unique. In particular, we can always make the choice $K_{a\dot{\alpha}} = 0$ provided we redefine $T_{a\dot{\alpha}}$ and $J$ as

$$T_{a\dot{\alpha}} \rightarrow \tilde{T}_{a\dot{\alpha}} = T_{a\dot{\alpha}} + \frac{3i}{2} \partial_{a\dot{\alpha}}(X - \bar{X}) - \frac{1}{12} [D_\alpha, \bar{D}_{\dot{\alpha}}](X + \bar{X})$$

and

$$J \rightarrow \tilde{J} = J - 2D^2(2X - \bar{X}) ,$$

where we replaced $\text{min}$ by $\sim$ to avoid clutter. The supercurrent conservation equation then reads

$$R_\alpha = \bar{D}^\dot{\alpha} \tilde{T}_{a\dot{\alpha}} - \frac{1}{6} D_\alpha \tilde{J} = 0 .$$

In particular, if the theory has superconformal invariance, there is a minimal multiplet with $\tilde{J} = 0$. In general, $\tilde{J}$ contains the trace of the energy momentum tensor and the supersymmetry current and is then referred to as the multiplet of anomalies [19, 38].

Choosing the appropriate local parameters contained in $L_\alpha$, we will now extract the different Noether currents from the supercurrent.

### 4.1.1 Supersymmetry Current

We take $L^\alpha = -i\bar{\theta}^\alpha \varepsilon^\alpha(x_+)$ and $\bar{L}^\dot{\alpha} = -2i\bar{\theta}^\dot{\alpha} \theta^\alpha \varepsilon_\alpha(x_-)$. Then

$$\delta S = \int d^4x \varepsilon^\alpha \left( -2i(\sigma^\nu \bar{\sigma}^\mu)_\alpha^\beta D_\beta \partial_\mu \tilde{T}_\nu + \frac{i}{3} \bar{\sigma}^\nu \sigma^\mu_\alpha^\beta \bar{D}_\beta \partial_\mu \tilde{J} \right) ,$$

from which we extract the supersymmetry current

$$j_{\mu a} = \left( 2i(\sigma^\nu \bar{\sigma}_\mu)_\alpha^\beta D_\beta \bar{\sigma}_\nu + \frac{i}{3} \sigma_{\mu a\dot{\alpha}} \bar{D}^\dot{\alpha} \bar{J} + \sigma_{\mu a\alpha} \bar{D}_\beta \bar{D}_\beta (a_1(X + \bar{X}) + a_2(X - \bar{X})) \right) \bigg|,$n

$$= \sigma_{\mu \nu}^\alpha \sigma_\beta^\alpha \bar{D}_\beta \left( -8i\bar{T}_\nu + \partial_\nu(a_1(X + \bar{X}) + a_2(X - \bar{X})) \right) \bigg| ,$$

$^{11}$In this paper, we will take the localization of superconformal transformations even for a theory which is only super-Poincaré invariant. This is because a free superfield, playing the analogous role of $L_\alpha$ but for super-Poincaré transformations, is not known.
where we have used the conservation equation in the second equality. The two complex constants \( a_1 \) and \( a_2 \) introduced here correspond to improvement terms for the supersymmetry current as they are automatically conserved. It is not hard to see that the improvement terms in (4.11) correspond to different choices for \( T_{\alpha\dot{\alpha}} \), \( K_{a\dot{a}} \) and \( J \) in (3.5). Indeed, for a generic choice, the supersymmetry current is given by

\[
j_{\mu\alpha} = 2i(\sigma^\nu\sigma_\mu)\beta D_\beta T_{\nu} - 2i(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{D}^\dot{\alpha}\tau + \frac{i}{2}(\sigma_\mu)_\alpha^\beta D^\beta\bar{D}_\alpha(2\bar{X} - X) . \tag{4.12}\]

A short computation then shows that applying (3.5) with \( A + \bar{A} = a_1(X + \bar{X}) \) and \( A - \bar{A} = a_2(X - \bar{X}) \) on (4.12), reproduces the supersymmetry current (4.11). For \( a_1 = a_2 = 0 \), the supersymmetry current (4.11) is part of the minimal multiplet as defined in section 3. We will give a more precise discussion of the multiplet structure below.

The trace of the supersymmetry current is then easily found to be

\[
(\bar{\sigma}^\mu j_\mu)_{\alpha} = \left(-2i\bar{D}^\alpha\bar{J} + \frac{3}{2}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu D^\alpha(a_1(X + \bar{X}) + a_2(X - \bar{X}))\right) | . \tag{4.13}\]

### 4.1.2 Energy-momentum Tensor

In the same manner, by choosing \( L^\alpha = \bar{\theta}_\alpha^a a^{a\dot{a}}(x_-) \), we get

\[
\delta S = -2 \int d^4x \ a^{a\dot{a}} \left( \partial_{\dot{\alpha}}\bar{D}^\dot{\beta}D^\beta T_{\dot{\nu}\alpha} - \partial_\alpha\bar{D}^\dot{\beta}D^\beta T_{\dot{\nu}\dot{\alpha}} - \partial_{\dot{\beta}}\bar{D}_\alpha D^\beta T_{\dot{\alpha}\beta} + \partial_\alpha\bar{D}_\beta D^\dot{\beta}T_{\dot{\alpha}\dot{\beta}} \right) | , \tag{4.14}\]

from which we extract the symmetric energy-momentum tensor:

\[
T_{\mu\nu} = \left(4\sigma_{(\mu}^{\alpha\dot{\alpha}}[D_\alpha, \bar{D}_{\dot{\alpha}}]T_{\nu)} - 4\eta_{\mu\nu}\sigma^{\rho\alpha\dot{\alpha}}[D_\alpha, \bar{D}_{\dot{\alpha}}]\bar{T}_\rho \right.
\]

\[
\left. + (\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)(b_1(X + \bar{X}) + ib_2(X - \bar{X}))\right) | . \tag{4.15}\]

The two real constants \( b_1 \) and \( b_2 \) introduced here correspond to improvement terms. Its trace is given by:

\[
T_{\mu}^{\mu} = \left(-2(D^2\bar{J} + \bar{D}^2\bar{J}) - 3\Box(b_1(X + \bar{X}) + ib_2(X - \bar{X}))\right) | . \tag{4.16}\]

### 4.1.3 R-current

Finally, choosing \( L^\alpha = -\eta(x)\theta^a\bar{\theta}^\alpha \), we get

\[
\delta S = -\int d^4x \ \eta \left(2\sigma^\alpha_{\dot{\alpha}}\bar{T}_{\alpha\dot{\alpha}} - \frac{i}{6}(D^2\bar{J} - \bar{D}^2\bar{J})\right) | , \tag{4.17}\]

and as expected, this is not automatically a derivative as we are not necessarily dealing with a \( R \)-invariant theory. Indeed, the \( R \)-invariance condition can be written as

\[
D^2\bar{J} - \bar{D}^2\bar{J} = 24i\partial^\mu Z_\mu , \tag{4.18}\]
for some $Z_\mu$. If this condition is satisfied, we can define a conserved $R$-current by

$$j^{(5)}_\mu = -4 \left( \bar{T}_\mu - Z_\mu \right) .$$

(4.19)

The condition (4.18) is solved by

$$\bar{J} = \bar{D}^2 U \quad \text{and} \quad \bar{J} = D^2 U ,$$

(4.20)

for some real superfield $U$. Correspondingly, we have $Z_{\alpha \dot{\alpha}} = \frac{1}{3} [D_\alpha, \dot{D}_{\dot{\alpha}}] U$. Now, applying (3.5) with $A = X + 2U$ we get

$$
\begin{pmatrix}
T_{\alpha \dot{\alpha}} \\
W_\alpha \\
\tau
\end{pmatrix}
\rightarrow
\begin{pmatrix}
T_{\alpha \dot{\alpha}}^{new} = \bar{T}_{\alpha \dot{\alpha}} - \frac{1}{3} [D_\alpha, \dot{D}_{\dot{\alpha}}] U \\
W_\alpha^{new} = \frac{1}{2} \bar{D}^2 D_\alpha U \\
0
\end{pmatrix} ,
$$

(4.21)

where we introduced the label $\textit{new}$ for the multiplet (4.21). This is because (4.21) is, in fact, the new minimal multiplet [20]. We will come back to this below. The conservation equation (4.6) then reads

$$D^{\alpha} T_{\alpha \dot{\alpha}}^{new} + W_\alpha^{new} = 0 ,$$

(4.22)

where $W_\alpha^{new}$ is now a curl ($D^\alpha W_\alpha^{new} + \bar{D}^{\dot{\alpha}} W^{\dot{\alpha}}_{\alpha} = 0$). Therefore

$$\partial^{\alpha \dot{\alpha}} T_{\alpha \dot{\alpha}}^{new} = 0$$

(4.23)

and consequently the conserved $R$-current is given by $T_{\alpha \dot{\alpha}}^{new}$ which agrees with (4.19).

### 4.1.4 Multiplet Structure

Having computed the physically interesting currents, we can now compute their transformation laws under rigid supersymmetry, to exhibit the on-shell multiplet structure of the supercurrent.

We start from the variation of the energy-momentum tensor,

$$\delta_\alpha T_{\mu \nu} = 8 \sigma_{\mu \alpha \beta \gamma} \partial^{\nu} j_\beta$$

$$+ \left( (\partial_\mu \partial_\nu - \eta_{\mu \nu} \Box) D_\alpha ( (b_1 + 2a_1)(X + \bar{X}) + (ib_2 + 2a_2)(X - \bar{X}) ) \right) .$$

(4.24)

It contains not only the derivatives of the supersymmetry current, as expected, but also the derivatives of $D_\alpha X \mid$, which shows that $D_\alpha X \mid$ belongs to the multiplet.

Next we compute the variations of the supersymmetry current,

$$\delta_\alpha j_{\mu \beta} = \sigma_{\mu \alpha \beta} \partial^{\nu} \left( \frac{8}{3} \bar{J} + \frac{1}{2} \bar{D}^2 (a_1 (X + \bar{X}) + a_2 (X - \bar{X})) \right) \right| ,$$

(4.25)

$$\sigma_{\mu \alpha \beta} \delta_\alpha j_{\nu \lambda} = \frac{i}{2} T_{\mu \nu} - 4 \partial_\mu \bar{J}_{\nu}^{(5)} + 4 \eta_{\mu \nu} \partial^{\rho} \bar{J}_{\rho}^{(5)} - 2 i \epsilon_{\mu \nu \rho \lambda} \partial_{\rho} \bar{J}_{\lambda}^{(5)}$$

$$+ \frac{3i}{8} \epsilon_{\mu \nu \rho \lambda} \sigma_{\alpha \dot{\alpha}} \left( [D_\alpha, D_{\dot{\alpha}}] \partial_{\rho} (a_1 (X + X) + a_2 (X - X)) \right) .$$

(4.26)
where we have used the notation
\[
\tilde{j}^{(5)}_{\mu} = \left( T_{\mu} + \frac{1}{16} \sigma^{\alpha\dot{\alpha}}_{\mu} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] (a_1 (X + \bar{X}) + a_2 (X - \bar{X})) \right. \\
\left. + \frac{i}{8} \partial_{\mu} ((b_1 + 2a_1)(X + \bar{X}) + (ib_2 + 2a_2)(X - \bar{X})) \right) .
\] (4.27)

We see that both $X$ and $\tilde{J}$ have to be included in the multiplet. In turn, this implies that all the components of these two superfields have to be included. Note also that the r.h.s. of (4.25) is a total derivative as it should be. For the zero component of the canonical supersymmetry current, this expression is, of course, just the central charge of the SUSY algebra. In $\mathcal{N} = 1$, the central charges are often ignored as they are not compatible with translation invariance. However, they play an important role, for example for domain walls [36, 37].

It is clear that the size of the multiplet of currents depends on the details of the theory under consideration but also on the choice of improvement terms. If the theory is conformally invariant, then the variation (3.3) must vanish for $h^{\alpha\dot{\alpha}} = \bar{h}^{\alpha\dot{\alpha}}$ and hence it is possible to choose $K_{\alpha\dot{\alpha}} = 0$ and $J = 0$. In this case, the multiplet of Noether currents is contained entirely in $T_{\alpha\dot{\alpha}}$ subject to the conservation equation $D^\dot{\alpha} T_{\alpha\dot{\alpha}} = 0$. This is the improved multiplet with 8 + 8 components first constructed by Ferarra and Zumino [2] for the Wess-Zumino model. The next bigger multiplet is described by $T_{\alpha\dot{\alpha}}$ and $J$. This is the old minimal multiplet [19, 38] with 12 + 12 components. Another possibility is that $\tau$ vanishes but not $W_\alpha$. If $W_\alpha$ is a curl then we obtain the new minimal multiplet [20] with again 12 + 12 components. This multiplet is distinguished in that it contains a conserved $R$-current as explained in subsection 4.1.3. If $W_\alpha$ is not a curl then this multiplet has 16 + 16 components. If we furthermore include $J$, we obtain a multiplet with 20 + 20 components. Finally, for a general $X$, we obtain a multiplet with 28 + 28 components.

4.2 Applications

The purpose of this subsection is to illustrate the use of the general formalism by applying it to some concrete models. As we shall see, the super-Noether procedure is an efficient tool to obtain the various Noether currents and central charges, especially when the action is given in terms of arbitrary prepotentials.

4.2.1 Kähler Sigma Models

As a first application of the general formalism, we compute the supercurrent for the $\mathcal{N} = 1$ sigma model defined in terms of arbitrary real Kähler potential $K(\phi, \bar{\phi})$ and superpotential $\mathcal{W}(\phi)$, where $\phi$ is a chiral scalar field. Such Lagrangians arise as the local part of quantum effective actions for supersymmetric field theories and string theory. The general action is given by
\[
S = \frac{1}{16} \int d^8 z \ K(\phi, \bar{\phi}) - \frac{1}{4} \int d^6 z^+ \ \mathcal{W}(\phi) - \frac{1}{4} \int d^6 z^- \ \mathcal{W}(\bar{\phi}) ,
\] (4.28)

with the corresponding equation of motion
\[
E \equiv \frac{1}{16} \bar{D}^2 K_{\phi} - \frac{1}{4} \mathcal{W}_\phi = 0 ,
\] (4.29)
where a subscript $\phi$ stands for differentiation by $\phi$. According to (3.1), the transformation law for $\phi$ under global super-Poincaré/superconformal transformations is $\delta \phi = - L_\phi - 2q\sigma \phi$, where $q$ is related to the $R$-weight and dimension of $\phi$ through (3.2). Expressing this transformation in terms of $L^\alpha$, we have

$$\delta \phi = \frac{i}{4} D^2 \left( L^\alpha D_\alpha \phi + \frac{1}{3} q D^\alpha L_\alpha \phi \right). \quad (4.30)$$

Substitution into (4.28) then leads to the variation of the form (4.4) with

$$R_\alpha = 2D_\alpha \phi \mathcal{E} - \frac{2}{3} q D_\alpha (\phi \mathcal{E}), \quad (4.31)$$

from which we get

$$\check{T}_{\alpha\dot{\alpha}} = \frac{1}{12} D_\alpha \phi D_\dot{\alpha} \bar{\phi} K_{\phi\bar{\phi}} - \frac{i}{6} \partial_{\alpha\dot{\alpha}} \phi K_\phi + \frac{i}{6} \partial_{\alpha\dot{\alpha}} \bar{\phi} K_\bar{\phi},$$

$$\check{J} = - \frac{i}{4} D^2 (K - q \phi K_\bar{\phi}) + 3 \mathcal{W} - q \phi \mathcal{W}. \quad (4.32)$$

Referring to the discussion in subsection 4.1.3, we infer that the existence of a conserved $R$-current requires that $\check{J}, \check{\mathcal{J}}$ can be written in the form (4.20). This, in turn implies

$$3 \mathcal{W} - q \phi \mathcal{W} = 0 \quad \text{and} \quad K(\phi, \bar{\phi}) = H(\phi \bar{\phi}). \quad (4.33)$$

for some function $H(\phi \bar{\phi})$. Superconformal invariance requires $\check{J} = 0$ which implies the same condition as in (4.33) on $\mathcal{W}$, but furthermore that $K - q \phi K_\bar{\phi} = 0$.

Next we compute the supersymmetry current. According to (4.11), we have

$$j_{\mu\alpha} = \left( \sigma_{\mu} \gamma^\alpha D_\beta \left( \frac{1}{6} \sigma_\nu \gamma^\gamma D_\gamma \bar{\phi} D_\gamma \bar{\phi} K_{\phi\bar{\phi}} - \frac{2i}{3} \partial_\nu \phi K_\phi + \frac{2i}{3} \partial_\nu \bar{\phi} K_{\bar{\phi}} \right) 
+ \partial_\nu (a_1 (X + \bar{X}) + a_2 (X - \bar{X})) \right). \quad (4.34)$$

Finally, we compute the central charge for this model. For this, we first need to extract the canonical supersymmetry current. The canonical current can be characterized by the absence of space-time derivatives on fermions. That is, the second term in (4.34) must be canceled by the improvement terms, i.e. $a_1 (X + \bar{X}) + a_2 (X - \bar{X}) = \frac{2i}{3} K$. Thus,

$$j_{\mu\alpha}^{\text{can}} = \sigma_{\mu} \gamma^\alpha D_\beta \left( \frac{1}{6} \sigma_\nu \gamma^\gamma D_\gamma \bar{\phi} D_\gamma \bar{\phi} K_{\phi\bar{\phi}} + \frac{4i}{3} \partial_\nu \phi K_{\bar{\phi}} \right), \quad (4.35)$$

where “can” labels the canonical Noether current. Using (4.25), we then read off the result for the central charge $Z_{(\alpha\beta)} = \int d^3 x \, \delta (\alpha j_{\beta}^{\text{can}})^0$ as

$$Z_{(\alpha\beta)} = \int d^3 x \, c_{alpha} \partial_i \left( - \frac{4i}{3} \check{J} + \frac{i}{3} D^2 K \right) = -3 \int d^3 x \, \sigma_{\alpha\beta}^{0i} \partial_i \mathcal{W}. \quad (4.36)$$

This is in agreement with the result found in [36] but without the “ambiguous term” which was due to the fact that in [36] the minimal rather than the canonical multiplet was used in the computation of the central charge. For the canonical multiplet, which is the correct multiplet to use to compare with the canonical formalism, these extra terms are absent as it should be.
4.2.2 Supersymmetric QED

We now discuss the supercurrent of $\mathcal{N} = 1$ QED in our formalism. This will serve as a useful preparation for dealing with the constrained superfield of $\mathcal{N} = 2$ Yang-Mills discussed in the next section.

The $\mathcal{N} = 1$ gauge multiplet is described by a curl superfield $W_\alpha$ satisfying the constraints

$$\bar{D}^\dot{\alpha} W_\alpha = 0, \quad D_\alpha \bar{W}_{\dot{\alpha}} = 0, \quad D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}_{\dot{\alpha}}.$$  \hspace{1cm} (4.37)

These constraints are solved in terms of a real prepotential $V$ by

$$W_\alpha = -\frac{1}{4} D^2 D_\alpha V.$$  \hspace{1cm} (4.38)

$W_\alpha$ is invariant under the gauge transformations:

$$\delta_g V = i(\Lambda - \bar{\Lambda}) , \quad \text{with} \quad D_\alpha \bar{\Lambda} = 0, \quad \bar{D}_{\dot{\alpha}} \Lambda = 0,$$  \hspace{1cm} (4.39)

and the free action is given by

$$S_W = \frac{1}{4} \int d^6 z \ W^\alpha W_\alpha + \text{c.c.}.$$  \hspace{1cm} (4.40)

To couple this $\mathcal{N} = 1$ gauge multiplet to a chiral multiplet $\phi$, we first recall its gauge transformations

$$\delta_g \phi = -ig \Lambda \phi , \quad \delta_g \bar{\phi} = ig \bar{\phi} \bar{\Lambda}.$$  \hspace{1cm} (4.41)

The corresponding invariant action is then given by

$$S_\phi = \frac{1}{16} \int d^8 z \ (\bar{\phi} e^{gV} \phi).$$  \hspace{1cm} (4.42)

For global superconformal transformations, we have (3.1)

$$\delta V = -\mathcal{L} V - 2(q \sigma + \bar{q} \bar{\sigma}) V,$$  \hspace{1cm} (4.43)

with $q = \bar{q}$ since $V$ is real. The gauge potential $A_\mu = \frac{1}{2} \sigma_\mu^{\alpha\dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] V$ is of dimension 1 and hence $V$ has to be of dimension 0, which implies that we take $q = 0$. As explained in [17] the gauge covariant localization of (4.43) reads

$$\delta V = -\mathcal{L} V - \frac{i}{2} (h - \bar{h})^{\alpha\dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] V,$$  \hspace{1cm} (4.44)

up to gauge transformations. In particular, it is convenient to add to (4.44) a pure gauge term of the form

$$-\frac{i}{4} \bar{D}^2 (L^\alpha D_\alpha V) + \frac{i}{4} D^2 (\bar{L}_{\dot{\alpha}} \bar{D}_{\dot{\alpha}} V).$$  \hspace{1cm} (4.45)

Combining the two contributions we end up with

$$\delta V = i (L^\alpha W_\alpha - \bar{L}_{\dot{\alpha}} \bar{W}_{\dot{\alpha}}).$$  \hspace{1cm} (4.46)
As we have added a gauge transformation with parameter \( \Lambda = -\frac{1}{4} \bar{D}^2(L^\alpha D_\alpha V) \) to the transformation law for \( V \), we have to add the corresponding modification to the transformation law for \( \phi \). Thus, (4.30) is modified into

\[
\delta \phi = \frac{i}{4} \bar{D}^2 \left( L^\alpha (D_\alpha \phi + g D_\alpha V \phi) + \frac{1}{3} g D^\alpha L_\alpha \phi \right)
\]

\[
= \frac{i}{4} \bar{D}^2 \left( L^\alpha \nabla_\alpha \phi + \frac{1}{3} g D^\alpha L_\alpha \phi \right),
\]

(4.47)

where \( \nabla_\alpha \phi = D_\alpha \phi + g D_\alpha V \phi \) is the gauge covariant derivative for \( \phi \) [39, 16]. Alternatively, (4.47) can be written as

\[
\delta \phi = \left( \frac{1}{2} h^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} - \chi^\alpha \nabla_\alpha - 2 q \sigma \right) \phi,
\]

(4.48)

with the gauge covariant space-time derivative \( \nabla_\alpha \phi \).

In order to extract the supercurrent for the combined theory \( S = S_W + S_\phi \) we first define \( \Phi = e^{2gV} \phi \) so that the invariant matter action takes the same form as for the Wess-Zumino model (4.28) with \( K = \Phi \bar{\Phi} \) and \( \mathcal{W} = 0 \): \( S_\phi = \frac{1}{16} \int d^8 z \ \Phi \bar{\Phi} \). This suggests that the supercurrent is just the one for the Wess-Zumino model, (4.32), but with derivatives replaced by covariant ones [16]:

\[
\bar{T}_{\alpha\dot{\alpha}}^\phi = \frac{1}{12} \nabla_\alpha \Phi \nabla_\dot{\alpha} \bar{\Phi} - \frac{i}{2} \Phi \nabla_{\alpha\dot{\alpha}} \bar{\Phi},
\]

\[
\bar{J}^\phi = \frac{1}{4} (q - 1) \nabla^2 \left( \Phi \bar{\Phi} \right).
\]

(4.51)

Plugging this into (4.9) then shows that this is the correct answer, \( i.e. \) we recover (4.49). Thus, the gauge invariant supercurrent for the \( \mathcal{N} = 1 \) abelian YM theory coupled to a

\[\text{For a general gauge covariant expression } X \text{ transforming as } \delta_q X = ig(e\Lambda - \bar{e}\Lambda)X, \text{ the gauge covariant derivatives are defined as } \nabla_\alpha X = D_\alpha X - egD_\alpha V X \text{ and } \nabla_\dot{\alpha} X = \bar{D}_\dot{\alpha} X - \bar{e}g\bar{D}_\dot{\alpha} V X. \text{ Note that we still have } \nabla_\dot{\alpha} \phi = 0, \text{ } i.e. \phi \text{ is gauge covariantly chiral. The covariant derivative } \nabla_{\alpha\dot{\alpha}} \text{ acts on } X \text{ as } \nabla_{\alpha\dot{\alpha}} X = \partial_{\alpha\dot{\alpha}} X - \frac{i}{2}g(e\bar{D}_\dot{\alpha} D_\alpha V + \bar{e}D_\dot{\alpha} \bar{D}_\alpha V)X.\]
scalar multiplet is
\[
\tilde{T}_{\alpha\dot{\alpha}} = -\frac{1}{8} W_{\alpha} \bar{W}_{\dot{\alpha}} + \frac{1}{12} \nabla_{\alpha} \Phi \nabla_{\dot{\alpha}} \bar{\Phi} - \frac{i}{6} \bar{\Phi} \nabla_{\alpha\dot{\alpha}} \Phi ,
\]
\[
\tilde{J} = \frac{1}{4} (q - 1) \nabla^2 \left( \Phi \bar{\Phi} \right) .
\] (4.52)

### 4.2.3 Tensor Multiplet

The last application we consider is the tensor multiplet [40] described by a superfield $G$ constrained by:
\[
D^2 G = 0 , \quad \bar{D}^2 G = 0 , \quad G = \bar{G} .
\] (4.53)

Again, this will serve as a preparation for the $\mathcal{N} = 2$ tensor discussed in section 5. We consider a general action $S = \int d^8 z \, \mathcal{F}(G)$ depending on an arbitrary function $\mathcal{F}$. There are two distinguished cases. The first one is the free field action $\mathcal{F}(G) = G^2$ which is $R$-invariant for a suitable choice of the $R$-weight of $G$, but is not conformal invariant for any choice of the conformal weight. The second model has $\mathcal{F}(G) = G \log G$ [41]. This model is superconformal invariant for a suitable choice of the conformal weight of $G$.

The constraints (4.53) are solved by
\[
G = D^a \phi_\alpha + \bar{D}_\dot{\alpha} \bar{\phi}^\dot{\alpha} ,
\] (4.54)

where $\phi_\alpha$ is an unconstrained chiral spinor superfield. Moreover, as expected, there is a gauge freedom of the form $\delta G = \bar{D}^2 D_\alpha K$, with $K = \bar{K}$. In analogy with the vector multiplet in the last subsection, we find for the gauge covariant localization of the global symmetry transformation of $\phi_\alpha$
\[
\delta \phi_\alpha = -\mathcal{L}_{\phi_\alpha} - 3 \sigma \phi_\alpha + \Omega_{\alpha} \beta \phi_\beta - \frac{i}{8} \bar{D}^2 \left( (h_{\alpha\dot{\alpha}} - \bar{h}_{\alpha\dot{\alpha}}) \bar{\phi}^\dot{\alpha} \right) + \frac{i}{4} \bar{D}^2 D_\alpha \left( L^3 \phi_\beta + \bar{L}^\beta \bar{\phi}^\beta \right)
= -\frac{i}{4} \bar{D}^2 \left( L_\alpha G \right) ,
\] (4.55)
corresponding to $q = \frac{3}{8}$, which is the only choice which leads to a gauge invariant transformation for $G$. The equation of motion for $G$ reads $E_\alpha = -\bar{D}^2 D_\alpha \mathcal{F}'$, where the prime denotes, as usual, the derivative of $\mathcal{F}$ w.r.t. $G$. The computation of the supercurrent conservation equation is now straightforward and we end up with
\[
R_\alpha = -2G E_\alpha = 2G \bar{D}^2 D_\alpha \mathcal{F}' = -2 \bar{D}^\dot{\alpha} \left( G \bar{D}_\dot{\alpha} D_\alpha \mathcal{F}' - \bar{D}_\dot{\alpha} G D_\alpha \mathcal{F}' \right) .
\] (4.56)

It is then just a matter of separating the real part from the imaginary one to get
\[
T_{\alpha\dot{\alpha}} = -2D_\alpha G \bar{D}_{\dot{\alpha}} G \mathcal{F}'' + G [D_\alpha, \bar{D}_{\dot{\alpha}}] \mathcal{F}' ,
K_{\alpha\dot{\alpha}} = 2G \partial_{\alpha\dot{\alpha}} \mathcal{F}' = -2 \partial_{\alpha\dot{\alpha}} (\mathcal{F} - G \mathcal{F}') ,
J = 0 .
\] (4.57)

From these objects, we finally obtain the minimal multiplet:
\[
\tilde{T}_{\alpha\dot{\alpha}} = -\frac{1}{3} D_\alpha G \bar{D}_{\dot{\alpha}} G (\mathcal{F}'' - 2G \mathcal{F}'''') + \frac{1}{3} G \mathcal{F}''' [D_\alpha, \bar{D}_{\dot{\alpha}}] G ,
\tilde{J} = 4 \bar{D}^2 \left( G \mathcal{F}' - \mathcal{F} \right) .
\] (4.58)
We immediately see that $\tilde{J}$ fulfills the $R$-invariance condition (4.18) with $U = 4(GF' - F)$ and, correspondingly, $Z_\mu = \frac{4}{3}\sigma_\mu^{\dot{a}}[D_\alpha, \bar{D}_{\dot{\alpha}}](F - GF')$. Thus, the model is $R$-invariant for any function $^13 F(G)$.

For superconformal invariance, we have to impose $\tilde{J} = 0$, which leads to the differential equation $\bar{D}^2(F - GF') = 0$ whose solutions are $\mathcal{F}(G) = G$ which is not physically interesting and $\mathcal{F}(G) = G \log G$. Thus, the action $S = \int d^8z \ (G \log G)$ is indeed superconformal invariant.

Finally, we end this section by computing the central charge for this model. The canonical supersymmetry current is first constructed following the same argument as in subsection 4.2.1, i.e. by imposing that there is no derivative on the fermions. This uniquely fixes $a_1 (X + \bar{X}) + a_2 (X - \bar{X}) = -\frac{44}{3}(GF' - F)$ and the canonical supersymmetry current reads $^14$ 

\[ j_{\mu a}^{\text{can}} = -8i\sigma_\beta^{\dot{a}} D_\alpha \left( \mathcal{F}' D_\beta G \bar{D}_{\dot{\beta}} G \right). \] 

(4.59)

Finally, using (4.25) shows that the central charge for this model identically vanishes.

5 $\mathcal{N} = 2$ Supersymmetry

In this section, we will explain how the general procedure developed in section 3 has to be amended in order to deal with the various constrained superfields in $\mathcal{N} = 2$ supersymmetry. Concretely the challenge is to find the appropriate localizations of the global symmetry transformations compatible with the constraints. A novelty arising in $\mathcal{N} = 2$ is that, contrary to $\mathcal{N} = 1$, there is room to replace (2.1) by a stronger constraint while still keeping the global symmetry transformations in the parameter superfield $h^{\alpha \dot{\alpha}}$. As we shall see, this is crucial to construct localizations of the global super-Poincaré (superconformal) transformations consistent with the constraints. This is the subject of the next subsection. In the second part, we will then illustrate the formalism by considering two concrete models, that is, the $\mathcal{N} = 2$ vector and tensor multiplets respectively.

5.1 $\mathcal{N} = 2$ Superconformal Group

In analogy with the $\mathcal{N} = 1$ case, the constraints (2.1) can be solved in terms of an unconstrained $\mathcal{N} = 2$ superfield $L_i^\alpha$ as

\[ h^{\alpha \dot{\alpha}} = -\frac{2}{3} \bar{D}^{\dot{a} \dot{a} i} L_i^\alpha \quad \text{and} \quad \bar{h}^{\alpha \dot{\alpha}} = -\frac{2}{3} D^{a \dot{a} i} \bar{L}_i^{\dot{\alpha}}. \] 

(5.1)

In the global limit, $h^{\alpha \dot{\alpha}} = \bar{h}^{\alpha \dot{\alpha}}$, the identity $D_{\alpha i} D_{\dot{\beta} j} h^{\gamma \dot{\gamma}} = D_{\alpha i} D_{\dot{\beta} j} h^{\gamma \dot{\gamma}} = 0$ holds [34], and a straightforward computation shows that a suitable $L_i^\alpha$ is given by $L_i^\alpha = -\frac{1}{12} \bar{D}^{\dot{a} \dot{a} i} h^{\alpha \dot{\alpha}}$. However, contrary to the $\mathcal{N} = 1$ case, it is possible to replace $L_i^\alpha$ by a constrained superfield without losing the global transformations. Indeed we can write

\[ L_i^\alpha \equiv D^{\alpha \dot{a} j} L_{j i}^{\dot{a}} \quad \text{and} \quad \bar{L}_i^{\dot{\alpha}} \equiv -\bar{D}^{\dot{a} \dot{a} j} \bar{L}_{j i}^{\dot{a}} \] 

(5.2)

\[^{13}\text{This is expected as } G, \text{ being real, is not charged under } R\text{-symmetry.}\]

\[^{14}\text{After adding an equation of motion.}\]
where \( L_{ij} \) is symmetric. The interested reader can verify that all global transformations are correctly parametrized by

\[
L_{ij} = -\frac{1}{18} \theta^a \bar{\theta}^{\dot{a}} \left[ a_{\alpha\dot{\alpha}} - \omega_{\alpha\beta} x^\beta + \bar{\omega}_{\dot{\alpha}\dot{\beta}} x^\dot{\beta} + \kappa x_{\alpha\dot{\alpha}} \right] \\
- \frac{1}{9} \delta^{(i} \theta_{j)\dot{a}} \bar{\theta}^a + \frac{1}{18} \eta \theta_{ij} \bar{\theta}^4 + i \frac{1}{8} \eta (i k \theta_{j}) \bar{\theta}^4 \\
- \frac{1}{18} \theta^a \bar{\theta}^{\dot{a}} c_{\alpha j} \bar{\theta}^a + \frac{1}{36} \eta \theta_{ij} \bar{\theta}^4 x_{+\alpha} \bar{\theta}^a \\
- \frac{1}{144} \theta^a \bar{\theta}^{\dot{a}} c_{i k} \bar{\theta}^a + \frac{1}{36} \eta \theta_{ij} \bar{\theta}^4 x_{+\alpha} \bar{\theta}^a.
\]

(5.3)

This allows us to define a new scalar superfield \( H \) which, like \( h^{\alpha\dot{\alpha}} \), is real for superconformal transformations. Indeed we can take

\[
H \equiv \bar{D}^i L_{ij}, \quad \bar{H} \equiv D^i \bar{L}_{ij}.
\]

(5.4)

For a global superconformal transformation we then have

\[
H = \bar{H} = -\frac{1}{2} \theta^a \bar{\theta}^{\dot{a}} \left[ a_{\alpha\dot{\alpha}} - \omega_{\alpha\beta} x^\beta + \bar{\omega}_{\dot{\alpha}\dot{\beta}} x^\dot{\beta} + \kappa x_{\alpha\dot{\alpha}} \right] \\
+ \frac{2i}{3} \delta_{\alpha j} \theta^a \bar{\theta}^{\dot{a}} + \frac{3}{4} \eta \theta_{ij} \bar{\theta}^4 - \frac{3i}{2} \eta (i k \theta_{j}) \bar{\theta}^4 \\
- \frac{1}{2} \theta^a \bar{\theta}^{\dot{a}} c_{i k} \bar{\theta}^a + \frac{1}{3} \bar{\theta}^{\dot{a}} \theta_{ij} \bar{\theta}^4 x_{+\alpha} \bar{\theta}^a.
\]

(5.5)

Furthermore, \( h^{\alpha\dot{\alpha}} \) can be expressed in terms of \( H \) as

\[
h^{\alpha\dot{\alpha}} = \frac{1}{2} [D^a, \bar{D}^{\dot{a}}] H \quad \text{and} \quad \bar{h}^{\alpha\dot{\alpha}} = \frac{1}{2} [D^{\dot{a}}, \bar{D}^a] H.
\]

(5.6)

In terms of \( L_{ij} \), we have

\[
\lambda^a = \frac{i}{24} \bar{D}^4 D^a D^b L_{ij} \quad \text{and} \quad \sigma = \frac{i}{144} \bar{D}^4 D^a D^b L_{ij}.
\]

(5.7)

According to the prescription given in section 3, we localize the parameters of the global symmetry transformations by relaxing the reality condition \( h^{\alpha\dot{\alpha}} = \bar{h}^{\alpha\dot{\alpha}} \) but by maintaining the chirality preserving condition (2.1). More precisely, we consider only those \( h^{\alpha\dot{\alpha}} \) that can be written in terms of \( L_{ij} \) where \( L_{ij} \) is an arbitrary symmetric superfield. It is not hard to see that this is sufficient to allow for an arbitrary \( x \)-dependence of the parameters of the symmetry transformations.

### 5.2 \( \mathcal{N} = 2 \) Vector Multiplet

As a first application of the general formalism we consider the Abelian vector multiplet [42]. This multiplet plays an important role for the low energy effective description of non-Abelian \( \mathcal{N} = 2 \) Yang-Mills theory [9]. The vector multiplet is described by a

15We checked that for super-Poincaré transformations it is even possible to integrate further by \( L_{ij} = D^a_{(i} A_{j)} \). However, as this does not seem to play a role for the Superfield Noether Procedure, we do not explore this possibility further in the present work.
chiral superfield $\mathcal{A}$ of $R$-weight -2 and dimension 1, subject to the Bianchi constraint $D^i \mathcal{A} = \bar{D}^i \bar{\mathcal{A}}$. This constraint can be solved by $\mathcal{A} = D^4 D^{\bar{i}} V_{ij}$ where the prepotential $V_{ij}$ is a real superfield.

The global superconformal transformation of $\mathcal{A}$ is given by (3.1) with $q = \frac{3}{2}$ (and, of course, $\bar{q} = 0$ since $\mathcal{A}$ is chiral). Although the corresponding local transformation, i.e. when $L_{ij}$ is free, does preserve the chirality of $\mathcal{A}$, it does not preserve the Bianchi constraint. Nevertheless, we can proceed by complete analogy with the $\mathcal{N} = 1$ case: this suggests to generalize the transformation (4.46) of the $\mathcal{N} = 1$ prepotential $V^{ij}$ to

$$\delta V_{ij} \equiv -\frac{i}{48} (AL_{ij} - \bar{A}L_{ij}).$$

Using the definition of $\mathcal{A}$ and the constraints it satisfies, we then compute the transformation of $\mathcal{A}$. It leads to

$$48i \delta \mathcal{A} = D^4 (D^{\bar{i}} AL_{ij}) + 2D^4 (D^{a} L_{ij} D^{\bar{a}} \mathcal{A}) + \mathcal{A} D^4 D^{\bar{i}} L_{ij} - D^4 (\bar{\mathcal{A}} D^{\bar{i}} L_{ij}). \quad (5.9)$$

The Bianchi constraint enables us to write the first term in this variation as $\bar{D}^4 (\bar{D}^{a} L_{ij} D^{\bar{a}} \mathcal{A}) = \bar{D}^4 (\bar{\mathcal{A}} D^{\bar{i}} L_{ij})$. Then, using the definition of $H$ and $\bar{H}$ we end up with

$$\delta \mathcal{A} = -\frac{i}{24} \bar{D}^4 \left( D^{a} L_{ij} D^{\bar{a}} \mathcal{A} \right) - \frac{i}{48} D^4 \bar{\mathcal{A}} D^4 L_{ij} - \frac{i}{48} \left[ (H - \bar{H}) \bar{\mathcal{A}} \right]. \quad (5.10)$$

Therefore, this transformation is a suitable one as it reduces to the transformation (3.1) in the superconformal limit $H = \bar{H}$ and furthermore preserves the Bianchi constraint.

### 5.2.1 Variation of the holomorphic Action

As for the $\mathcal{N} = 1$ chiral multiplet, the most general local action for the vector multiplet consists of holomorphic and a non-holomorphic terms. We first consider the holomorphic part, referring non-holomorphic terms to subsection 5.2.8.

The general holomorphic action is given by

$$S[\mathcal{A}] \equiv \frac{1}{4\pi} \text{Im} \int d^8 z \mathcal{F} (\mathcal{A}). \quad (5.11)$$

The classical Yang-Mills action corresponds to $\mathcal{F} (\mathcal{A}) \equiv \tau \mathcal{A}^2$ with $\tau \equiv \frac{g}{2\pi} + i \frac{4\pi}{g^2}$ where $g$ is the coupling constant and $\theta$ the vacuum-angle. For convenience, we also introduce the dual superfield $\mathcal{A}_D \equiv \mathcal{F}' (\mathcal{A})$. The equations of motion are then $D^{\bar{i}} \bar{\mathcal{A}}_D = D^{\bar{i}} \bar{\mathcal{A}} D_{\alpha}^i \mathcal{A}_D$.

It is now straightforward to compute the variation of the action (5.11) under the transformation (5.10). Indeed, as

$$\delta \mathcal{F} = -\frac{i}{24} \bar{D}^4 \left( D^{a} L_{ij} D^{\bar{a}} \mathcal{F} \right) - 3\mathcal{A}_D \mathcal{A}_D - \frac{i}{48} \bar{D}^4 \left[ (H - \bar{H}) \bar{\mathcal{A}} \mathcal{A}_D \right], \quad (5.12)$$

the variation of the action is

$$\delta S = \frac{1}{4\pi} \text{Im} \left\{ -\frac{i}{24} \int d^{12} z \ D^{a} L_{ij} D^{\bar{a}} \mathcal{F} - 3 \int d^8 z + \sigma \mathcal{A}_D \mathcal{A}_D - \frac{i}{48} \int d^{12} z \ (H - \bar{H}) \bar{\mathcal{A}} \mathcal{A}_D \right\}. \quad (5.13)$$
Then, we successively integrate by parts the first term, write it as an integral on the chiral superspace, use the chirality of $\mathcal{F}(\mathcal{A})$ and the relation (5.7) between $\sigma$ and $L_{ij}$ to obtain:

$$
\delta S = \frac{1}{4\pi} \text{Im} \left\{ 6 \int d^8 z_+ \sigma (\mathcal{F} - \frac{1}{2} \mathcal{A} \mathcal{A}_D) - i \frac{1}{48} \int d^8 z_+ (H - \bar{H}) \mathcal{A} \mathcal{A}_D \right\}.
$$

(5.14)

This means that the variation of the action can be written as

$$
\delta S = i \int d^{12} z (H - \bar{H}) T - 144 i \int d^8 z_+ \sigma J + 144 i \int d^8 z_+ \bar{\sigma} \bar{J}
$$

(5.15)

with

$$
T = - \frac{i}{384 \pi} (\mathcal{A} \mathcal{A}_D - \mathcal{A} \mathcal{A}_D), \quad J = \frac{1}{192 \pi} (\mathcal{F} - \frac{1}{2} \mathcal{A} \mathcal{A}_D).
$$

(5.16)

This result deserves some comments:

1) Contrary to the general situation of section 3, it is not possible to obtain the variation of the action (5.11) in terms of $h^{\alpha \dot{\alpha}}$ due to the constraints on $\mathcal{A}$.

2) Nevertheless, the invariance of the action under super-Poincaré transformations is explicit as $H = \bar{H}$ and $\sigma = \bar{\sigma} = 0$ for these transformations.

Moreover, as in $\mathcal{N} = 1$, the conservation equations are obtained from (5.15) by expressing $H$, $\bar{H}$, $\sigma$ and $\bar{\sigma}$ in terms of the free parameters $L_{ij}$ and $\bar{L}_{ij}$. This leads to

$$
D^{ij} T = - i \bar{D}^{ij} \bar{J}.
$$

(5.17)

3) Contrary to the situation in $\mathcal{N} = 1$, there is no freedom in the definition of $T$ and $J$ i.e. they are uniquely determined in terms of the constrained superfield $\mathcal{A}$.

4) It is now clear from 3) that the theory is superconformal invariant if and only if $J = 0$. Hence, in analogy with the $\mathcal{N} = 1$ case, $J$ is the superconformal anomaly and therefore our method provides a simple derivation of the anomalous superconformal ‘Ward identity’ first derived in [18].

### 5.2.2 Projections of Currents

To determine the various Noether currents [43, 6, 15, 44] one proceeds in complete analogy with the $\mathcal{N} = 1$ case. Therefore we give only the results of this computation.\textsuperscript{16}

**Supersymmetry Current**

We take $L_{ij} = - \frac{i}{9} \varepsilon^\alpha (x^+) \theta_{\alpha ij} \theta^4$ and $\bar{L}_{ij} = \frac{2i}{9} \varepsilon^\alpha (x^-) \theta_{3k} \bar{\theta}\bar{\theta}_{ij}$ and get

$$
J^a_i = 192 \left[ i \sigma'^\mu_{\alpha \dot{\alpha}} \bar{D}^a_{\dot{\alpha}} D^{ij} T - 3i \sigma'^{\mu \beta \dot{\alpha}} \bar{D}^a_{\beta} D_{\alpha \dot{\alpha}} T - 12 \partial^\mu D^i_{\alpha} T + \alpha \sigma'^{\mu \beta \dot{\alpha}} \partial_{\nu} D^i_{\alpha} T \right],
$$

(5.18)

where the term parametrized by $a$ corresponds to an improvement term. Note that the first term in the r.h.s. of (5.18) is equal on-shell to $192 \sigma'^\mu \bar{D}^{3ai} \bar{J}$. As the second term in the r.h.s of (5.18) is traceless, the trace of $J^a_i$ is simply given by

$$
(\sigma'^\mu J^i_\mu)^{ai} = 192 \left[ - 4 \bar{D}^{3ai} \bar{J} - \frac{3}{2} (8 + a) \partial^{\dot{\alpha}} D^i_{\alpha} T \right].
$$

(5.19)

Therefore, this trace vanishes when $J = 0$ (superconformal case) and $a = -8$.

\textsuperscript{16}Here we use the results (C.7) of appendix C.2.
One way to fix the value of $a$ corresponding to the canonical current is to go to components in the particular case of classical Yang-Mills theory, where $F(A)$ is quadratic. However, it is also possible to determine this value at the superspace level, for any $F$, by using the algebra satisfied by the currents. This will be explained in subsection 5.2.7. Both methods agree and lead to $a = -24$.

**Stress-energy tensor** With $L_{ij} = -\frac{1}{18} \theta^i_{\alpha} \bar{\theta}^j_{\dot{\alpha}} a_{\alpha\dot{\alpha}} (x^+)$, one finds

\[
T_{\mu\nu} = -24 \left[ \frac{1}{2} \eta^{\mu\nu} \{ D \bar{\psi}, \bar{D}_{ij} \} T - \frac{3}{2} \bar{\sigma}^{\mu} \bar{\psi} \{ \bar{D}_{\alpha\beta}, \bar{D}_{\dot{\alpha}\dot{\beta}} \} T \right. \\
+ 48 \eta^{\mu\nu} \Box T - b(\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu) T] \right].
\]

(5.20)

Again, on-shell the first term in the r.h.s. of (5.20) is equal to $-12i \eta^{\mu\nu} (D^4 J - \bar{D}^4 \bar{J})$. Next we determine also the value of $b$ for the two interesting cases. The second term in the r.h.s. of (5.20) is traceless and thus we immediately have

\[
T_\mu^\mu = -48i(D^4 J - \bar{D}^4 \bar{J}) + 72(b - 64) \Box T.
\]

(5.21)

Therefore, this trace vanishes again when $J = 0$ and $b = 64$.

The value of $b$ corresponding to the canonical stress-energy tensor can only be fixed at the components level. For this purpose, we consider classical Yang-Mills theory. All the fields being decoupled, it is enough to determine the contribution from the scalar field $\phi \equiv A|$, to the Hamiltonian density $T^{00}$. So, suppose that $F(A) \propto A^2$. It follows then from (5.16) that $T = A \dot{A}$, up to some global factor, and that $J = 0$. Consider then the different terms of (5.20).

- As $J = 0$, the first term vanishes on-shell.
- For the second term, when all the fields except $\phi$ are set to zero, we have:

\[
\{ D_{\alpha\beta}, \bar{D}_{\dot{\alpha}\dot{\beta}} \} T = AD_{\alpha\beta} \bar{D}_{\dot{\alpha}\dot{\beta}} A| + c.c. = 16 A \partial_{(\alpha}(\dot{\alpha})_{\dot{\beta})} A| + c.c.
\]

(5.22)

where we have used the chirality of $A$ and the relation (C.5) of appendix C. Using the identity $\bar{\sigma}^{\mu} \bar{\psi} \partial_{(\alpha} \partial_{\dot{\alpha})_{\dot{\beta}}} = -\eta^{\mu\nu} \Box + 4 \partial^\mu \partial^\nu$, the second term gives a contribution:

\[
24 \phi \left( -\eta^{\mu\nu} \Box \phi + 4 \partial^\mu \partial^\nu \phi \right) + c.c. = 96 \phi \partial^\mu \partial^\nu \phi + c.c.
\]

(5.23)

where we have used the equation of motion $\Box \phi = 0$.

- The contribution of the last terms is simply: $(b - 48) \eta^{\mu\nu} \Box (\phi \phi) - b \partial^\mu \partial^\nu (\phi \phi)$.

Therefore, on-shell,

\[
T^{\mu\nu} \propto 2(b - 48) \eta^{\mu\nu} \partial^\rho \phi \partial_\rho \phi + 96 \phi \partial^\rho \partial^\nu \phi + 96 \phi \partial^\mu \partial^\nu \phi - b \partial^\mu \partial^\nu (\phi \phi).
\]

(5.24)

The Hamiltonian density is then

\[
T^{00} \propto -(2b - 96) \left[ -|\partial_0 \phi|^2 + |\partial_i \phi|^2 \right] + (96 - b) \partial^0 \partial^i (\phi \phi) - 192 \partial^0 \phi \partial^i \phi
\]

\[
T^{00} \propto -(2b - 96) |\partial_0 \phi|^2 + (2b - 96 - 192) |\partial_0 \phi|^2 + (96 - b) \partial^0 \partial^0 (\phi \phi)
\]

(5.25)

where we have used $\eta^{00} \equiv -1$. Therefore, $T^{00}$ corresponds to the Hamiltonian density if $-(2b - 96) = 2b - 96 - 192$ and $96 - b = 0$ i.e. when $b = 96$.

Note that in full generality, improvement terms proportional to $J$ could be added to the supersymmetry current and to the stress-energy tensor above. However, we do not consider such a possibility here.
We obtain
\[ R_{ij} = -864i\sigma^{\alpha\dot{\alpha}}_{\dot{a}\dot{a}} [D_{ij}^\alpha, \bar{D}^\dot{\alpha}] T]. \] (5.26)

### 5.2.3 Supersymmetry Transformations of the Currents

The next few subsections are devoted to identifying the super multiplets of Noether currents \([43, 6, 15, 44]\) for a given choice of improvement terms. For this, we first give the supersymmetry transformations of the conserved currents. The result is:

\[
\delta_{\alpha} R_{\mu jk} = \left( \frac{3}{2} \varepsilon_{ij} [J_{\alpha k}^\mu - (24 + a)\sigma_{\alpha}^{\mu\nu} \beta \partial_\nu D_{\beta k}^\mu] T \right),
\]
(5.27)

\[
\delta_{\dot{\alpha}} J^{\dot{j} \dot{\alpha}} = \varepsilon^{\dot{i}} \sigma_{\nu \alpha \dot{\alpha}} [-2i T^{\mu\nu} + 48i(b - 48 + 2a)(\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu) T] + \varepsilon^{\dot{i}} \left\{ \frac{1}{4} [a + 24] [\sigma_{\alpha \dot{\alpha}}^{\mu\nu} \partial_\nu R^\mu_{\gamma} - \partial_\alpha R^\mu T] - \frac{i a}{4} \varepsilon^{\mu\nu\rho\tau} \sigma_{\tau \alpha \dot{\alpha}} \partial_\nu R^\mu_{\rho} \right\}
\]
(5.28)

\[
\delta_{\alpha} J^{\mu}_{\beta j} = \partial_\mu \left\{ 24i \varepsilon_{\alpha \beta} \varepsilon_{\dot{j} \dot{\alpha}} Z^\mu + 96(a + 24) \left[ \varepsilon_{\dot{i} \dot{j}} \sigma_{(\alpha}^{\mu\nu} \gamma \partial_\beta T + \sigma_{\alpha \beta}^{\mu\nu} D_{\dot{j} \dot{\alpha}} T \right] \right\},
\]
(5.29)

\[
\delta_{\alpha} T^{\mu \nu} = \sigma^{\mu \nu}_{\alpha \beta} \partial_\nu J^{\mu}_{\beta j} + 24(b - 48 + 2a)(\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu) D_{\alpha} T \] . (5.30)

where we have introduced

\[ R^\mu \equiv -48\sigma^{\mu}_{\alpha \dot{a}} [D^{\alpha \dot{a}}, \bar{D}^\dot{\alpha}] T]. \] (5.31)

It follows from (C.6) that \( \partial^\mu R^\mu = -8iD^{\dot{j} \dot{\alpha}} \bar{D}^\dot{\alpha} T \), or, using the equation of conservation (5.17), that

\[ \partial^\mu R^\mu = 8(D^4 J + \bar{D}^4 \bar{J}) \] . (5.32)

In addition we introduced

\[ Z^{\mu \nu} \equiv 96 \left[ \sigma^{\mu \nu}_{\dot{a} \dot{\alpha}} \bar{D}^\dot{\alpha} \bar{J} - i(\frac{1}{2} - \frac{a}{48})\sigma^{\mu \nu}_{\alpha \dot{a}} D^\alpha \bar{T} \right]. \] (5.33)

We give in appendix A.1 some details for the computation of the variation (5.29) which has special interest because it is related to the central charge of the supersymmetry algebra.

### 5.2.4 Multiplet Structure: General Discussion

Let us now determine with which other components of \( T, J \), and \( \bar{J} \), the above Noether currents form a multiplet. For this we first note that, contrary to the situation in \( N = 1 \), for a \( N = 2 \) superconformal theory, the conserved \( R \) and \( SU(2)_R \) and the traceless stress-energy tensor alone can not form a multiplet with the traceless supersymmetry currents. This is so because the number of bosonic components, \( i.e. 3 + 9 + 5 = 17 \) differs from the 16 fermionic components of \( J^\mu_{\alpha \dot{a}} \). Hence the improved multiplet contains other components than the Noether currents above.
The following discussion closely follows that for \( N = 1 \). We start with the variation (5.30) of the stress-energy tensor \( T^{\mu\nu} \). Two cases have to be considered.

- \( b - 48 + 2a = 0 \): we consider then the variation (5.28) of \( J_{a}^{\mu} \). As \( b - 48 + 2a = 0 \), we conclude that \( \mathcal{R}^{\mu} \) belongs to the multiplet. However, we then have

\[
\delta_{a} \mathcal{R}^{\mu} = 16i \left[ \frac{1}{192} J_{a}^{\mu} + 2\sigma_{a}^{\mu \beta} \partial_{\beta} \hat{\mathcal{J}} - (a + 24)\sigma_{a}^{\mu \beta} \partial_{\beta} D_{3} T \right].
\]  

(5.34)

Using the relation (5.19) for the trace of \( J_{a}^{\mu} \), we can eliminate \( \hat{D}_{i}^{3a} \hat{\mathcal{J}} \) to obtain

\[
\delta_{a} \mathcal{R}^{\mu} = 16i \left[ \frac{1}{192} \left[ \frac{3}{2} J_{a}^{\mu} + \sigma_{a}^{\mu \beta} J_{\nu \beta} \right] - \frac{3}{4} (8 + a) \partial^{\mu} D_{a} T \left. \right| + \frac{1}{2} (a - 24)\sigma_{a}^{\mu \beta} \partial_{\beta} D_{3} T \right].
\]  

(5.35)

This shows that \( \chi_{a} \equiv D_{a} T \) also belongs to the multiplet. We then continue with the variations of \( \chi_{a} \). Defining \( t \equiv T \), and \( u_{a \beta} \equiv D_{a \beta} T \), we find

\[
\delta_{a} \chi_{\beta j} = \frac{1}{2} \varepsilon_{ij} u_{a \beta} - \frac{i}{2} \varepsilon_{ij} \partial_{a} \hat{\mathcal{J}},
\]  

(5.36)

\[
\delta_{a} \chi_{\alpha j} = \frac{i}{3456} R_{a \alpha j} + \frac{1}{384} \varepsilon_{ij} \partial_{a} \hat{\mathcal{J}} \varepsilon_{\alpha \beta} + i \varepsilon_{ij} \partial_{a} t.
\]  

(5.37)

From the first variation (5.36) we conclude that \( \hat{D}_{ij} \hat{\mathcal{J}} \) and therefore all the components of \( \hat{\mathcal{J}} \) belong to the multiplet. The second variation (5.37) shows that \( t = T \) belongs to the multiplet. As \( t \) is the lowest component of \( T \), this shows that the multiplet contains \( T \). On the other hand, as \( T, \mathcal{J} \) and \( \hat{\mathcal{J}} \) are the only superfields present, we have shown that, for any value of \( a \) and \( b \) such that \( b - 48 + 2a = 0 \), the multiplet of currents corresponds to \( T, \mathcal{J} \) and \( \hat{\mathcal{J}} \) constrained by the equations of conservation (5.17).

- \( b - 48 + 2a \neq 0 \): in that case, we proceed as follows: the variation (5.30) of \( T^{\mu\nu} \) shows that \( \chi_{a} \) belongs to the multiplet of currents. However, we consider then the variation (5.37) of \( \chi_{a} \). As \( \mathcal{R}^{\mu} \) and \( t \) are real, this shows that both of them belong to the multiplet. Therefore, the conclusion is the same as in the preceding case.

Let us now choose a convenient set of independent components of \( T, \mathcal{J} \) and \( \hat{\mathcal{J}} \). We take \( R_{ij}^{\mu}, J_{ai}^{\mu}, T_{\mu \nu}^{i}, \mathcal{R}^{\mu}, Z_{\mu \nu}, t, \chi_{ai}, u_{a \beta}, \hat{\mathcal{J}}, \hat{\mathcal{D}}_{ai} \hat{\mathcal{J}}, \) \( \hat{D}_{ij} \hat{\mathcal{J}} \). Indeed, first, \( \hat{D}_{\alpha \beta} \hat{\mathcal{J}} \) is the anti self-dual part of \( Z_{\mu \nu} \) as given by the equation (5.33). Secondly, \( \hat{D}^{3} \hat{\mathcal{J}} \) and \( \hat{D}^{4} \hat{\mathcal{J}} \) are related to other components via the 'trace' equations (5.19), (5.21) and (5.32). In order to count the number of components of that multiplet, we need to distinguish between the cases where \( \mathcal{J} \neq 0 \) and \( \mathcal{J} = 0 \) respectively.

### 5.2.5 Multiplet Structure: Case \( \mathcal{J} \neq 0 \)

In the general case, the theory is neither \( R \)-invariant nor conformal invariant. For any value of \( a \) and \( b \), we have the following number of components.

\[
\mathcal{R}^{\mu}(4), \quad R_{ij}^{\mu}(9), \quad J_{ai}^{\mu}(-24), \quad T_{\mu \nu}^{i}(6), \quad Z_{\mu \nu}(6),
\]  

(5.38)

\[
t(1), \quad \chi_{ai}(-8), \quad u_{a \beta}(6), \quad \hat{\mathcal{J}}(2), \quad \hat{\mathcal{D}}_{ai} \hat{\mathcal{J}}(-8), \quad \hat{D}_{ij} \hat{\mathcal{J}}(6).
\]
This forms a \((40 + 40)\) multiplet. The algebra satisfied by these components is summarized in subsection A.2 of appendix A.

Let us examine the conditions for \(R\)-invariance. As \(H = \bar{H}\) for superconformal transformations, it follows from the variation (5.15) of the action, and from the specific value (5.3) of \(L_{ij}\) for \(U(1)_R\) transformations, that the condition is that there exists \(r^\mu\) such that 

\[-96(D^4 J + \bar{D}^4 \bar{J}) = \partial_\mu r^\mu.\]

Considering local \(U(1)_R\) transformations, we find that the conserved \(R\)-current is in that case \(\mathcal{R}^\mu + r^\mu\). However, as in \(\mathcal{N} = 1\) and as a consequence of our general discussion above, this is \(\mathcal{R}^\mu\) itself rather than the \(R\)-current that is in the multiplet. This is to be expected as the supersymmetry current and stress-energy tensor are not traceless. Therefore, the number of components of the multiplet is again \((40 + 40)\).

**5.2.6 Conformal Case \(J = 0\)**

When \(J = 0\), the theory is conformal invariant and the equation of conservation is \(D^\dot{\mu} T = 0\). It is also \(R\)-invariant and \(\mathcal{R}^\mu\) is the conserved \(R\)-current. Let us now discuss how the multiplet structure depends on the improvement terms in the conformal case. We consider first the improved multiplet \([6, 15, 44]\) which corresponds to the values \(a = -8\) and \(b = 64\). It has the components

\[\mathcal{R}^\mu(3), \quad R^\mu_{ij}(9), \quad J^\mu_{\alpha i}(-16), \quad T^{\mu\nu}(5), \quad u_{\alpha\beta}(6), \quad t(1), \quad \chi_{\alpha i}(-8).\] (5.39)

Its dimension is \((24 + 24)\). The corresponding transformations of these components are given in \([44, 45]\) and in subsection A.3 of appendix A.

Let us now turn to the canonical multiplet. This corresponds to \(a = -24\) and \(b = 96\). The difference with the improved multiplet is that here \(\chi_{\alpha i}\) and \(J^\mu_{\alpha i}\) are not independent. More precisely, when \(J = 0\) and for \(a = -24\), equation (5.19) becomes \((\bar{\sigma}^\alpha J^\mu_{\alpha i}) = 4608 \bar{\sigma}^\alpha \chi_{\alpha i}^4\). Thus, the fermionic components of the canonical multiplet are as follows: There are the 8 components of \(\chi_{\alpha i}\). For \(J^\mu_{\alpha i}\) we have only to count its traceless part, i.e., 16 components, as the trace is contained in \(\chi_{\alpha i}\). Hence we recover the 24 fermionic components of \(T\).

To summarize, for a conformal theory, the canonical multiplet contains the bosonic components \(\mathcal{R}^\mu(3), R^\mu_{ij}(9), (T^{\mu\nu}, t)(6), Z^{\mu\nu}(6)\) and the fermionic components \((J^\mu_{\alpha i}, \chi_{\alpha i})(-24)\). Thus the canonical multiplet is also a \((24 + 24)\) multiplet.

**5.2.7 Central Charge**

For classical \(\mathcal{N} = 2\) Yang-Mills theory, the central charge was first computed in \([46]\) by explicit evaluation of the anticommutator of the supersymmetry charges. The quantum corrected effective central charge of \(\mathcal{N} = 2\) supersymmetry is important because it determines the mass of the BPS states in the quantum theory. Indeed, the central charge formula for low-energy effective action of \(\mathcal{N} = 2\) Yang-Mills theory which was assumed in \([9]\) contains the seeds of the duality properties of this model \([10, 11]\). That this assumption is correct was proved in components in \([47, 48]\). In general, the complete computation of central charges for effective theories in components is a rather magnificent task. On the other hand, we have seen in subsection 4.2 that the Superfield Noether Procedure leads to a simple computation even for complicated actions. This
method naturally extends to the $\mathcal{N} = 2$ case and as we shall now show it leads to a simple and efficient computation of the effective central charge. In order to have equivalence with the canonical computation [46, 47, 48] it is, however, important to work with the canonical multiplet. The algebra of the $\mathcal{N} = 2$ supersymmetry Noether charges is

$$\{Q_{\alpha i}, Q_{\beta j}\} = \frac{1}{4} \varepsilon_{ij} \varepsilon_{\alpha \beta} Z + \sigma^{0k}_{\alpha \beta} \Lambda_{k(ij)}. \quad (5.40)$$

As stated in subsection 5.2.2, the canonical supersymmetry current corresponds to eq. (5.18) with $a = -24$. We give now the postponed alternative determination of this value. This is based on the comparison of the variation (5.29) of the supersymmetry current and the supersymmetry algebra. Indeed, on one hand, as $\{Q_{\alpha i}, Q_{\beta j}\} = \int d^3 x \delta_{\alpha i} J^0_{\beta j}$ we have, in superspace,

$$\{Q_{\alpha i}, Q_{\beta j}\} = \int d^3 x D_{\alpha i} J^0_{\beta j}. \quad (5.41)$$

Therefore, the algebra can be read from the result (5.29) i.e.

$$\delta_{\alpha i} J^0_{\beta j} = \partial_k \left\{ 24 i \varepsilon_{\alpha \beta} \varepsilon_{ij} Z^{0k} + 96(a + 24) \left[ \varepsilon_{ij} \sigma^{0k \gamma}_{(\alpha \beta)} \gamma_T + \sigma^{0k}_{\alpha \beta} D_{ij} T \right] \right\}. \quad (5.42)$$

On the other hand, the Poisson algebra (5.40) of the charges allows only terms having the same symmetry properties in the $SU(2)_R$ and spinor indices. It is then clear that the r.h.s. of equation (5.42) has the required symmetry only when $a = -24$. Note that this implies the vanishing of $\Lambda_{k(ij)}$.

According to the previous discussion, the central charge $Z$ is then given by:

$$Z = \int d^3 x D^{0\alpha i} J^0_{\alpha i} = 96 \int d^3 x \partial_i Z^{0i}, \quad (5.43)$$

where $Z^{0i}$ is given by eq.(5.33), i.e.

$$Z^{0i} = 96i \left[ \bar{\sigma}^{0i}_{\alpha \beta} \bar{D}^{\dot{\alpha} \dot{\beta}} \bar{J} - i \sigma^{0i}_{\alpha \beta} D^{\alpha \beta} T \right] |. \quad (5.44)$$

Note that this result is universal in the sense that this formula for the center is valid for any theory whose variation of the action takes the form (5.15).

Using the particular expressions (5.16) of $T$ and $\bar{J}$, we end up with

$$Z^{0i} = \frac{i}{4\pi} \bar{A} \left\{ \sigma^{0i}_{\alpha \beta} D^{\alpha \beta} A_D - \bar{\sigma}^{0i}_{\alpha \beta} \bar{D}^{\dot{\alpha} \dot{\beta}} \bar{A}_D \right\} - \frac{i}{4\pi} \bar{A}_D \left\{ \sigma^{0i}_{\alpha \beta} D^{\alpha \beta} A - \bar{\sigma}^{0i}_{\alpha \beta} \bar{D}^{\dot{\alpha} \dot{\beta}} \bar{A} \right\}. \quad (5.45)$$

This result, which is the expression of the central charge in superspace exhibits manifestly the duality between $A$ and $A_D$. However, to complete the calculation we need to identify the r.h.s. of (5.45) with the electric and magnetic charges.

To identify the magnetic charge we use that$^{17}$ $D_{\alpha \beta} A | \equiv \frac{1}{2\sqrt{3}} F_{\alpha \beta}$, where $F_{\alpha \beta} \equiv F_{\mu \nu} \sigma^{\mu \nu}_{\alpha \beta}$ and $B^i \equiv \frac{1}{2} \varepsilon^{ijk} F_{jk}$. It follows then that

$$B^i = -i \sqrt{3} \left[ \sigma^{0i}_{\alpha \beta} D^{\alpha \beta} A - \bar{\sigma}^{0i}_{\alpha \beta} \bar{D}^{\dot{\alpha} \dot{\beta}} \bar{A} \right]. \quad (5.46)$$

$^{17}$For classical Yang–Mills theory where $F(A) = i \frac{2}{\sqrt{3}} A^2$, this definition leads to the usual normalization $-\frac{1}{12} F^{\mu \nu} F_{\mu \nu}$. 

24
Next we determine the conjugate momentum $\Pi^i$ of the gauge field. In general, it is not straightforward to determine the conjugate momentum of a field in superspace. However, we proceed by using that $\Pi^i$ can be extracted from the Gauss law $\partial_i \Pi^i = 0$ which is an equation of motion. More precisely, the gauge field appears in the action only via the field strength (in superspace language, this is reflected by the fact that the superfield $A$ is a gauge invariant object). Therefore, we have the sequence of equalities:

$$\partial_i \Pi^i = \partial_t \frac{\partial L}{\partial (\partial_0 A_i)} = -\partial_t \frac{\partial L}{\partial (\partial_0 A_0)} = \frac{\delta S}{\delta A_0},$$

which is just expressing the fact that Gauss law is equivalent to $\frac{\delta S}{\delta A_0}$. To obtain Gauss law in superspace we then proceed in two steps. First we compute the fundamental derivative

$$\frac{\delta}{\delta A_0(y)}(D_{\alpha\beta}A|[x]) = -\frac{1}{\sqrt{3}}\sigma_{\alpha\beta}^0 \partial_x \delta(x - y).$$

Then we identify the terms in the action containing $D_{\alpha\beta}A$. Now, since

$$D^4F(A) = F' D^4A + 3D^{\alpha i} F' D_{\alpha i}A + \frac{3}{2} D^{\alpha i} F' D_{\alpha j}A - \frac{3}{2} D^{\alpha i} F' D_{\alpha i}A - D^{\alpha i} F' D_{\alpha i}A,$$

the terms we are looking for are

$$\frac{3}{8\pi} \text{Im} \int d^4 x \ D^{\alpha\beta} F' D_{\alpha\beta}A |.$$ (5.50)

Differentiating with respect to $A_0$ and using (5.48) we then end up with

$$\frac{\delta S}{\delta A_0} = -\frac{\sqrt{3}}{8\pi} \partial_i \text{Im} \left[ \sigma_{\alpha\beta}^0 \left( D^{\alpha\beta} F' + F'' D_{\alpha\beta}A \right) \right].$$ (5.51)

Thus we have

$$\Pi^i = -\frac{\sqrt{3}}{8\pi} \text{Im} \left[ \sigma_{\alpha\beta}^0 \left( D^{\alpha\beta} F' + F'' D_{\alpha\beta}A \right) \right].$$ (5.52)

We now consider the phase space for which $\phi \equiv |A|$ goes to a constant at infinity and where the fermions $D^{\alpha i}A$ decrease sufficiently fast enough such that

$$F'' D^{\alpha\beta}A = D^{\alpha\beta} F' | - F'' D^{\alpha i} A D_{\alpha i} | \xrightarrow{|x| \to \infty} D^{\alpha\beta} F'.$$

Therefore, we have, at infinity, $\Pi^i = -\frac{\sqrt{3}}{4\pi} \text{Im} \sigma_{\alpha\beta}^0 D^{\alpha\beta} A_D |$, i.e.

$$\Pi^i = \frac{\sqrt{3}}{8\pi} \left[ \sigma_{\alpha\beta}^0 D^{\alpha\beta} A_D - 2 \bar{D}_{\alpha \beta} A_D \right].$$ (5.53)

Finally, combining the expressions (5.46) and (5.54) respectively for the magnetic and electric fields, the result for the central charge (5.45) can be rewritten as

$$Z_{\alpha i} \xrightarrow{|x| \to \infty} \frac{2}{\sqrt{3}} \left( \bar{A} \Pi^i + \frac{1}{8\pi} \bar{A}_D B^i \right),$$

which corresponds to the result found in [47].
5.2.8 Non-holomorphic Action

We now consider a non-holomorphic, local action for the vector multiplet

\[ S[A] \equiv \int d^{12}z \, \mathcal{H}(A, \bar{A}). \] (5.56)

To compute the corresponding \( T \) and \( J \), we start from the following form of (5.10):

\[ \delta A = \frac{1}{2} \bar{h}^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} A + \lambda^{ai} D_{ai} A - 3 \sigma A - \frac{i}{48} \bar{D}^4 \left( (H - \bar{H}) \dot{A} \right). \] (5.57)

The contributions from the last two terms to \( T \) and \( J \) are immediately obtained. Let us so concentrate on the first two terms. We use the relations (2.3) to obtain

\[ \delta S = \int d^{12}z \left\{ \bar{h}^{\alpha\dot{\alpha}} \left( \frac{1}{2} \mathcal{H}_A \partial_{\alpha\dot{\alpha}} A + \frac{i}{8} \bar{D}_4 D_{ai} \mathcal{H} \right) + \bar{h}^{\alpha\dot{\alpha}} \left( \frac{1}{2} \mathcal{H}_A \partial_{\alpha\dot{\alpha}} A - \frac{i}{8} D^{ai} \bar{D}_4 \mathcal{H} \right) \right\} \]

\[ = \int d^{12}z \left\{ \frac{1}{2} \left( \bar{h}^{\alpha\dot{\alpha}} - \bar{h}^{\alpha\dot{\alpha}} \right) \left( \frac{1}{2} \mathcal{H}_A \partial_{\alpha\dot{\alpha}} A - \frac{i}{8} D^{ai} \bar{D}_4 \mathcal{H} \right) \right\} \]

\[ = \int d^{12}z \left\{ \frac{1}{4} \mathcal{H}_A D_{ai} \bar{D}_4 \mathcal{H} \right\} . \] (5.58)

We conclude by using the relation (5.6) between \( h^{\alpha\dot{\alpha}} \) and \( H \) and by integrating by parts. Adding up all the contributions finally leads to

\[ T = -\frac{1}{48} \left[ \bar{A} \bar{D}^4 H_A + A D^4 \bar{H}_A + 3 [D^{ai}, \bar{D}_4] (\mathcal{H}_A D_{ai} \bar{A} D_{aj} \dot{A}) \right], \] (5.59)

\[ J = -\frac{i}{48} A D^4 \bar{H}_A. \] (5.60)

Of course, the expression of the conserved currents and the structure of their multiplet is the same as in the holomorphic case since these properties were derived in a generic way.

We finish this section by showing that the non-holomorphic part of the action does not contribute to the center. We denote generically by a subscript \( h \), holomorphic terms and by a subscript \( n.h. \), non-holomorphic ones.

Among the diverse terms that appear in the expression (5.44) for the center \( Z_{n.h.}^{\alpha i} \), we concentrate on the ones which fall off slowest at infinity. Let us take the example of

\[ \bar{D}^{\alpha\beta} \bar{J} \propto \bar{D}^{\alpha\beta} (\bar{A} \bar{D}^4 H_A). \]

Remember that the dimension of \( A \) is \(-1\). Furthermore, as \( A \) goes to a constant at infinity, all the derivatives of \( H \) share this property. Therefore, the term that decreases the least is the one where as less as possible derivatives act on \( H \) namely \( \bar{A} H_{A\dot{A}} (\bar{D}^{\alpha\beta} \bar{D}^4 A) \). Clearly, this term decreases sufficiently fast enough at infinity, and thus, this holds similarly for all the other terms. As a consequence, there is no contribution to the center from \( \bar{J} \) and, for the same reasons, from \( T \) such that

\[ Z_{n.h.}^{\alpha i} \left[ \frac{1}{x} \right] \rightarrow 0. \]

However, in the same way, by computing the Gauss law, it is easy to prove that at infinity, the non-holomorphic terms do not contribute to \( \Pi^i \). Thus, in a theory where both holomorphic and non-holomorphic terms are present, we have, on
one hand, $\Pi^i = \Pi^i_{h.i} + \Pi^i_{h.n.h.}$, and on the other hand $Z^{0i} = Z^{0i}_{h}$. Therefore,

\[ Z^{0i} \xrightarrow{|x| \to \infty} \frac{2}{\sqrt{3}} \left( \mathcal{A} \Pi^i_{h.i} + \frac{1}{8\pi} \mathcal{A}DB^i \right) \left| \mathcal{A} \Pi^i_{h} + \frac{1}{8\pi} \mathcal{A}DB^i \right| \]  

This proves that the central charge is entirely determined by the holomorphic part of the action.

### 5.3 $\mathcal{N} = 2$ Tensor Multiplet

The second example of a $\mathcal{N} = 2$ theory with off-shell superfield formulation we consider is the tensor multiplet [49, 50]. This multiplet is described by a chiral field $\Phi$ of dimension 1 and $R$-weight $-2$ and classical action

\[ S[\Phi] \equiv \int d^4x d^8\theta \Phi \bar{\Phi} + 24 \int d^4 x^4 \theta \Phi (m^2 - \square) \Phi + 24 \int d^4 x^4 \bar{\theta} \bar{\Phi} (m^2 - \square) \bar{\Phi}. \]  

(5.62)

For $m = 0$, this action is invariant under the gauge transformation $\delta \Phi \equiv \bar{D}^4 D^i F^i$ where $K_{ij}$ is real. The invariant “field strength” tensor is given by

\[ F^i_{ij} \equiv i(D_i \Phi - D_j \Phi) \]  

(5.63)

which is the analogue of eq.(4.54) for the $\mathcal{N} = 1$ tensor. It satisfies the following properties:

\[ \bar{D}^4_{\dot{\alpha}} F^i_{\dot{\alpha}j} = 0 \quad \text{(completely symmetric)}, \]

\[ \bar{D}^4_{\dot{\alpha}} F^i_{\dot{\alpha}k} = \frac{2}{3} \varepsilon_{i[j} \bar{D}^4_{\dot{\alpha}} F_{k]} \dot{i} , \]

\[ \bar{D}^4_{\dot{\alpha}} F^i_{\dot{\alpha}k} = \frac{1}{3} \varepsilon_{i[k\dot{\alpha}]j} \bar{D}^4_{\dot{\alpha}} F^i_{\dot{\alpha}k}. \]  

(5.64)

Finally, $D_i F^i$ is chiral and so is $D_i F^i$ as a consequence of (5.64).

The equation of motion can be written in the two equivalent forms

\[ \bar{D}^4 \Phi - 48 (\square - m^2) \Phi = 0 \quad \Leftrightarrow \quad i \bar{D}^4 \Phi = 48 m^2 \Phi = 0. \]  

(5.65)

#### 5.3.1 Supercurrent

The variation of $\Phi$ under superconformal transformations is given by eq.(3.1) with $q = \frac{3}{2}$ (and $\bar{q} = 0$), i.e.

\[ \delta \Phi = -\frac{i}{48} \bar{D}^4 D^i \bar{(L_i \Phi)} + \frac{i}{48} \bar{D}^4 (L_i D^i \Phi). \]  

(5.66)

As in $\mathcal{N} = 1$, it is convenient to add to the local corresponding transformation the term $-\frac{i}{48} \bar{D}^4 \left((H - \bar{H}) \Phi \right)$ that vanishes in the global limit. This leads to

\[ \delta \Phi = \frac{1}{48} \bar{D}^4 (L_i F^i) - \frac{i}{48} \bar{D}^4 D^i (L_i \Phi + \bar{L}_i \Phi), \]  

(5.67)

which is of the same form as for the vector multiplet. The advantage of this form is that, for $m = 0$, the second term is in fact a gauge transformation and can be ignored.
To continue we then treat the variation of the mass term, \( S_m \), and of the gauge invariant part, \( S_g \) separately. The action \( S_m \) is similar to the \( \mathcal{N} = 2 \) classical Yang-Mills action, i.e. the action (5.11) with \( \mathcal{F}(A) \equiv i192\pi m^2 A^2 \). It follows then from (5.15)-(5.16) that

\[
\delta S_m = -2im^2 \int (H - \bar{H})\Phi\bar{\Phi} .
\]

(5.68)

\( S_g \) being gauge invariant, we can compute its variation by taking only into account the first term in the variation given by eq.(5.67). This leads to

\[
\delta S_g = \frac{i}{48} \int L_{ij} F^{ij} \bar{D}^{kl} F_{kl} + c.c.
\]

(5.69)

Naively, one might expect that this variation should be proportional to \( H - \bar{H} \) and \( \sigma \). However, it turns out that it is impossible to write it in this form. Nevertheless, it can be brought into the suggestive form

\[
\delta S_g = \frac{i}{48} \left[ \int (H - \bar{H}) F^{ij} F_{ij} + \frac{3}{10} (H - \bar{H})_{ijkl} F^{ij} F^{kl} \right]
\]

(5.70)

where

\[
H_{ijkl} \equiv \bar{D}_{(ij} L_{kl)} \quad \text{(completely symmetric)}.
\]

(5.71)

This result relies on the identity

\[
F_{ij} \bar{D}^{kl} F_{kl} = \bar{D}_{(ij} (F^{kl} F_{kl}) + \frac{3}{10} \bar{D}^{kl} (F_{(ij} F_{kl)}),
\]

(5.72)

which in turn can be obtained from the properties (5.64) of the field strength. More precisely, one shows that

\[
\bar{D}_{ij} (F^{kl} F_{kl}) = \frac{2}{3} F_{ij} \bar{D}^{kl} F_{kl} - \frac{4}{9} \bar{D}^{k} F_{(ik} \bar{D}^{i} F_{j)l},
\]

\[
\bar{D}^{kl} (F_{(ij} F_{kl)}) = \frac{10}{9} F_{ij} \bar{D}^{kl} F_{kl} + \frac{40}{27} \bar{D}^{k} F_{(ik} \bar{D}^{i} F_{j)l}.
\]

(5.73)

We then get the relation (5.72) as a consequence of (5.73).

Taking the sum of (5.68) and of (5.70), we end up with the variation of the total action for the tensor multiplet

\[
\delta S = \frac{i}{48} \int d^4 z \ (H - \bar{H}) (F^{ij} F_{ij} - 96m^2 \Phi\bar{\Phi}) + \frac{3}{10} (H - \bar{H})_{ijkl} F^{ij} F^{kl} .
\]

(5.74)

Therefore we define

\[
T \equiv \frac{1}{48} (F^{ij} F_{ij} - 96m^2 \Phi\bar{\Phi}) \quad \text{and} \quad T^{ijkl} \equiv \frac{1}{160} F^{(ij} F^{kl)} ,
\]

(5.75)

where \( T^{ijkl} \) is completely symmetric. Indeed, it follows from the analysis of subsection 5.1, and in particular from the expression (5.3) of \( L_{ij} \) for superconformal transformations, that for \( \mathcal{N} = 2 \) super-Poincaré and \( U(1)_R \) transformations, \( H \) and \( H_{ijkl} \) are
both real. Thus, eq.(5.74) makes the invariance of the action under these transformations explicit and the corresponding conserved currents are expressed as components of $T$ and $T^{ijkl}$ in analogy with the previous examples. For the $SU(2)_R$-transformations and the corresponding current, the situation is more subtle. We refer the proof for this case to the next subsection. Finally, if we define $\tau^{ij} \equiv -\bar{D}_{kl}T^{ijkl}$, the conservation equation following from (5.74) can be written as

$$D^{ij}T = \tau^{ij} = -\bar{D}_{kl}T^{ijkl}.$$  

(5.76)

This form of the conservation equation, which was proposed in [51], is similar to that proposed in [52] except from the fact that here $\tau^{ij}$ is not real.

## 5.3.2 Noether Currents

In this section, we identify the contributions from $T^{ijkl}$ to the various conserved currents. The currents for the tensor multiplet are then simply the sum of the terms given respectively in equations (5.18), (5.20), (5.26), (5.31) and of the terms from $T^{ijkl}$ given explicitly below.

### Supersymmetry Current

A short computation gives the following contribution of $T^{ijkl}$ to $J_{\mu}^{ij}$:

$$768\sigma_{a\dot{a}}^\mu \bar{D}_{a}^\dot{a} D_{k} T^{ijkl} = -768\sigma_{a\dot{a}}^\mu \bar{D}_{a}^\dot{a} D^{ij}T$$

(5.77)

where we have used the equation of conservation. This means that $J^{ij\mu}_{\dot{a}}$ can be expressed in terms of $T$ only. This is due to the fact that, for supersymmetry transformations, it is possible to localize $L_{ij}$ such that $H_{ijkl} = \bar{H}_{ijkl}$ is still satisfied.\textsuperscript{18}

### Stress-energy Tensor

For global translations, it follows from (5.3) that $H_{ijkl} = \bar{H}_{ijkl} = 0$. It is immediate to see that this also holds for the local $L_{ij}$ taken in subsection 5.2.2. Thus, $T^{ijkl}$ does not contribute to the stress-energy tensor.

### $R$-current

Again, a short computation indicates that there is no contribution of $T^{ijkl}$ to the $R$-current.

### $SU(2)_R$ Invariance and associated Current

Contrary to the $\mathcal{N} = 2$ super-Poincaré and $U(1)_R$ transformations, for global $SU(2)_R$ transformations, we do have $H = \bar{H}$ but $H_{ijkl} \neq \bar{H}_{ijkl}$. More precisely, it follows from (5.3) that

$$H_{ijkl} - \bar{H}_{ijkl} = -\frac{3i}{2} \eta_{m(l}[\theta^{m} j \bar{\theta}_{kl]} + \bar{\theta}^{m] j \theta_{kl}]).$$

(5.78)

It is nevertheless still possible to express the corresponding $SU(2)_R$ current in terms of $T$ and of $T_{ijkl}$. To see this, we consider first global transformations and concentrate on the term giving problem i.e. the one proportional to $(H - \bar{H})_{ijkl}$. Its contribution

\textsuperscript{18}This choice of $L_{ij}$ differs from the one made in subsection 5.2.2 by terms proportional to the derivative of $\varepsilon^{a}_\mu$. The fact that the corresponding currents are equal on-shell is thus an explicit example of the general remark given in section 3.
to the variation of the action under $SU(2)_R$ transformations after integration on the Grassman variables is
\[
\frac{27i}{20} \int d^4 x \, \eta_{mi} [D^m_j \bar{D}_{kl} + \bar{D}^m_j D_{kl}] T^{ijkl}. \tag{5.79}
\]

However, we prove in appendix B that
\[
(D^{(m}_j \bar{D}_{kl}) + \bar{D}^{(m}_j D_{kl}) T^{ijkl} = -8i \partial_{\alpha\bar{\alpha}} [D^{\alpha}_{k} \bar{D}_{l}] T^{mkl}. \tag{5.80}
\]
Thus, as $\eta_{ij}$ is symmetric, this proves the invariance of the action and gives the contribution of $T^{ijkl}$ to the $SU(2)_R$ current $R^{ij}$:
\[
1728i \sigma_{\alpha\bar{\alpha}}^{\mu k} \bar{D}^{\bar{\alpha}} T^{ijkl}. \tag{5.81}
\]

### 5.3.3 Central Charge

We end this section with the computation of the center for the $\mathcal{N} = 2$ tensor multiplet. For this we take the variation of the supersymmetry current, $J_{\mu i}^{\alpha} = 192 [3i \sigma_{\alpha\bar{\alpha}}^{\mu} \bar{D}^{\bar{\alpha}} T - 3i \sigma^{\mu \alpha \bar{\beta}} \bar{D}_{\alpha} D_{\beta} T - 12 \partial^{\mu} \bar{D}^{1}_{\alpha} T + a \sigma^{\mu \alpha \bar{\beta}} \partial_{\alpha} \bar{D}^{1}_{\beta} T] |. \tag{5.82}

Note that as a consequence of the conservation equation (5.76), we have $D^{(3}_{\alpha} T = 0$ on-shell. This identity is useful for the following reason. As explained for the vector, to fix the value of the coefficient $a$ corresponding to the canonical case and to determine the center $Z$, we have to compute the symmetric or antisymmetric part of $D^{i}_{\alpha} J_{0}^{\alpha} \bar{\beta}$ respectively. However, the corresponding variation of the first term in the r.h.s. of (5.82) is proportional to $D^{i}_{\alpha} \bar{D}^{\bar{\alpha}} D_{ij} T \propto \bar{D}^{\bar{\alpha}} D^{3}_{ij} T = 0$. Therefore, we can ignore this term for our purpose.

Let us now consider the variation of the second term in the r.h.s. of (5.82). Using the algebraic identity (A.3) and the fact that $D^{3}_{ij} T$ vanishes on-shell, we get
\[
D^{i}_{\alpha} [3i \sigma^{\bar{\alpha} \gamma} \bar{D}_{\bar{\alpha}} D_{\beta} T] = -24 \left[ \sigma^{\mu \alpha \gamma} D^{\bar{\beta}} T - 1/2 \partial^{\beta} D_{\alpha} T \right]. \tag{5.83}
\]

Finally, the variations of the last two terms in the r.h.s. of (5.82) can be easily read from the results (A.6) and (A.8). Here we give the result. First, as in the case of the vector multiplet, the canonical supersymmetry current corresponds to $a = -24$. Then, we get $Z = 96 \int d^3 x \partial_{i} Z^{0i}$ with
\[
Z^{0i} = 96 \sigma_{\alpha\bar{\alpha}}^{0i} D^{\alpha\beta} T |. \tag{5.84}
\]
That is we recover the universal formula (5.44) with $J = 0$. Using the expression (5.75) of $T$ we then end up with
\[
Z^{0i} = 2 \sigma_{\alpha\bar{\alpha}}^{0i} D^{\alpha\beta} (F^{ij} F_{ij} - 96 m^2 \Phi \Phi) |. \tag{5.85}
\]

However,
\[
D_{\alpha\beta} F_{ij} = i (D_{\alpha\beta} D_{ij} \Phi - D_{\alpha\bar{\beta}} \bar{D}_{ij} \bar{\Phi}) = 8 \partial_{(\alpha\beta)} \partial_{(i} \bar{D}_{j)} \bar{\Phi} = 0 \tag{5.86}
\]
where we have used the fact that $D_{\alpha\beta}D_{ij} = 0$, the result \((C.4)\) of appendix \(C.2\) and the anti-chirality of $\Phi$. As a consequence, the first term in the center (5.85) receives contribution only from $(DF)^2$ i.e. from fermions. Therefore, two cases have to be considered.

- $m \neq 0$: in that case all the fields decrease sufficiently fast enough at infinity such that the center vanishes.
- $m = 0$: as there are only contributions from the fermions that decrease sufficiently fast enough at infinity, the center vanishes also in that case.

### 6 Linearized Supergravity

It is well known that the on-shell multiplet of Noether currents of supersymmetric matter theories can be used to construct off-shell Supergravities \([12]\), at least at the linearized level. The purpose of this section is to discuss some elements of constructing supergravities directly on the superfield level using the Superfield Noether Procedure developed in this paper.

Coupling the matter to gravity means gauging the super-Poincaré transformations. As explained in subsection 2.2, these transformations are parametrized by $h^{\alpha\dot{\alpha}} = \bar{h}^{\dot{\alpha}\alpha} = 0$ and

$$\sigma = \bar{\sigma} = \Lambda^j_1 = 0.$$  

(6.1)

On the other hand, we recall from (3.3) and (3.6) that the variation of the matter action under an arbitrary local transformation is of the general form

$$\delta S[O^{A\bar{J}}] = \frac{i}{16} \int d^{4+4N} z \left\{ \left( h^{\alpha\dot{\alpha}} - \bar{h}^{\dot{\alpha}\alpha} \right) T_{\alpha\dot{\alpha}} + i \left( h^{\alpha\dot{\alpha}} + \bar{h}^{\dot{\alpha}\alpha} \right) K_{\alpha\dot{\alpha}} \right\} - \frac{1}{2} \int d^{4+2N} z_+ \sigma J - \frac{1}{2} \int d^{4+2N} z_- \bar{\sigma} \bar{J},$$  

(6.2)

where

$$K_{\alpha\dot{\alpha}} = -\frac{i}{4} [D_i^\alpha, \bar{D}_{i\dot{\alpha}}](X - \bar{X}) - \frac{N}{2} \partial_{\alpha\dot{\alpha}}(X + \bar{X}).$$  

(6.3)

To construct an invariant action at the linear level, one follows the standard procedure of coupling the currents to potentials. Concretely we introduce the real superpotentials $H^{\alpha\dot{\alpha}}, B, C$, as well as a chiral potential $\Omega$ and add the terms

$$\frac{i}{16} \int d^{4+4N} z \left\{ H^{\alpha\dot{\alpha}} T_{\alpha\dot{\alpha}} + B(X + \bar{X}) + C(X - \bar{X}) \right\} - \frac{1}{2} \int d^{2+2N} z_+ \Omega J - \frac{1}{2} \int d^{2+2N} z_- \bar{\Omega} \bar{J},$$  

(6.4)

to the action. The supergravity potentials must then transform like

$$\delta H^{\alpha\dot{\alpha}} \equiv -(h^{\alpha\dot{\alpha}} - \bar{h}^{\dot{\alpha}\alpha}), \quad \delta B \equiv -\frac{N}{2} \partial_{\alpha\dot{\alpha}}(h^{\alpha\dot{\alpha}} + \bar{h}^{\dot{\alpha}\alpha}), \quad \delta C \equiv -\frac{1}{4} [D_i^\alpha, \bar{D}_{i\dot{\alpha}}](h^{\alpha\dot{\alpha}} + \bar{h}^{\dot{\alpha}\alpha}),$$  

$$\delta \Omega \equiv -\sigma.$$  

(6.5)

By construction, the action obtained in this way has a larger invariance than the superdiffeomorphisms. In particular, it is invariant under linearized Weyl transformations

$$h^{\alpha\dot{\alpha}} \equiv \frac{12}{N} \theta^\alpha_i \bar{\theta}^{\dot{\alpha}i} \sigma(x^+), \quad \bar{h}^{\dot{\alpha}\alpha} \equiv -i \frac{12}{N} \theta^\alpha_i \bar{\theta}^{\dot{\alpha}i} \bar{\sigma}(x^-).$$  

(6.6)
Indeed, $h^{\alpha \dot{\alpha}}$ satisfies the chirality preserving constraint $\bar{D}^{(\dot{\beta}} h_{\alpha \dot{\alpha})} = 0$. On the other hand, the metric $g^{\mu \nu}$, which is proportional to $\bar{\sigma}^{(\mu \dot{\alpha} \nu \dot{\alpha})} \beta \{[D^i, \bar{D}_i] H_{\beta \beta}\}$, transforms as required for a Weyl transformation. Therefore, $\Omega$ ensures that the gauged action is Weyl invariant. In order to restrict the gauge group to the superdiffeomorphisms alone, $\Omega$ has to be set to a fixed value. Thus $\Omega$ is a compensator.

6.1 Improvement Terms

From the point of view of linear supergravity, the supersymmetry current and the stress-energy tensor are obtained by variation of the action (6.4) with respect to the gravitino and the metric respectively. We now explain how to obtain the various conserved currents differing by improvement terms. The procedure is analogous to that used in the Weyl gauging [53] to understand the relation between non-minimal coupling and improvement terms for the stress-energy tensor of non-supersymmetric theories. The key point is that the divergence of the gauge potential for the scale symmetry transforms like the Ricci scalar $R$ under diffeomorphisms and Weyl transformations.

We start by explaining how this works here at the component level and show how the Weyl gauging is done at the level of the multiplets within the superfield Noether procedure. For simplicity we first consider a generic $\mathcal{N} = 1$ theory.

**Components** We can identify the component in $B$ and $C$ transforming like the Ricci scalar $R$ by noting that

$$\delta(D^2 \bar{D}^2 B) = -96i\Box \sigma, \quad \delta(D^2 \bar{D}^2 C) = 96i\Box \sigma.$$  \hspace{1cm} (6.7)

Therefore, the highest component, $d$, of $B$ (and similarly for $C$) is a scalar that transforms like $R$, which, at the linear level, is given by $R = \partial_\mu \partial_\nu g^{\mu \nu} - \Box g^{\mu \nu}$. Therefore, the component $d$ can either be considered as independent of the metric or to be $d' + b_1 R$. It is then clear that variation of (6.4) with respect to the metric will give an improvement term of the form $b_1(\eta^{\mu \nu} - \partial^\mu \partial^\nu)(X + \bar{X})$ for $T^{\mu \nu}$, which is what we were looking for. We can repeat the same procedure with the second highest component of $B$ (and $C$) to improve the supersymmetry current.

**Superfield** In order to see how this procedure lifts to superspace, we recall the identities (3.4) satisfied by the parameter superfield $h^{\alpha \dot{\alpha}}$. Take now a representative triplet of currents, $(T, X, J)$ say, and the corresponding linearized action (6.4). Now, due to (3.4), the variations of these potentials are not independent. Indeed we have

$$\delta B + \frac{1}{12} [D_\alpha, \bar{D}_{\dot{\alpha}}] \delta H^{\alpha \dot{\alpha}} + 8i(\delta \Omega + \delta \bar{\Omega}) = 0,$$

$$\delta C + \frac{3}{2} i \partial_{\alpha \dot{\alpha}} \delta H^{\alpha \dot{\alpha}} + 24i(\delta \Omega - \delta \bar{\Omega}) = 0. \hspace{1cm} (6.8)$$

In particular, if we replace $C$ in (6.4) by $\frac{3}{2} i \partial_{\alpha \dot{\alpha}} H^{\alpha \dot{\alpha}} - 24i(\Omega - \bar{\Omega})$, then the total action obtained in this way is also invariant. The effect of this substitution is, as in the component approach, to relate the top components of $C$ to the metric and gravitino, but this time in a supersymmetric way.

In order to make the equivalence with adding improvement terms explicit, we take two superfields $U$ and $V$ and rewrite (6.4) as

$$\frac{i}{16} \int d^8 z \{ H^{\alpha \dot{\alpha}} T_{\alpha \dot{\alpha}} + B(X + \bar{X} + U) + C(X - \bar{X} + V) - BU - CV \}$$

32
Then, using (6.8), we replace (6.9) by

\[
\frac{i}{16} \int d^6z \left\{ H^{\alpha\dot{\alpha}} T_{\alpha\dot{\alpha}} + B(X + \bar{X} + U) + C(X - \bar{X} + V) \right. \\
- \left. \left[ \frac{1}{12} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] H^{\alpha\dot{\alpha}} - 8i(\Omega + \bar{\Omega}) \right] U - \left[ \frac{3i}{2} \partial_{\alpha\dot{\alpha}} H^{\alpha\dot{\alpha}} - 24i(\Omega - \bar{\Omega}) \right] V \right\} \\
- \frac{1}{2} \int d^6z_+ \Omega J + \frac{1}{2} \int d^6z_- \bar{\Omega} \bar{J}. \tag{6.10}
\]

The total action obtained in this way is, of course, also invariant under superdiffeomorphisms and Weyl transformations. Integrating by parts and regrouping the terms in \( \Omega \) and \( \bar{\Omega} \), we end up with

\[
\frac{i}{16} \int d^6z \left\{ H^{\alpha\dot{\alpha}} \left[ T_{\alpha\dot{\alpha}} + \frac{1}{12} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] U - \frac{3i}{2} \partial_{\alpha\dot{\alpha}} V \right] \\
+ B(X + \bar{X} + U) + C(X - \bar{X} + V) \right\} \\
- \frac{1}{2} \int d^6z_+ \Omega \left( J + D^2 U + 3D^2 V \right) \\
- \frac{1}{2} \int d^6z_- \bar{\Omega} \left( \bar{J} + D^2 U - 3D^2 V \right). \tag{6.11}
\]

Therefore, starting with the coupling (6.4) and the representative \( (T, X, J) \), we have constructed a different coupling (6.11) associated with another representative \( (T', X', J') \). This shows explicitly how the equivalence relation (3.5) relates different supergravities to each other. In particular, if the matter action is conformal, we can obtain the conformal \( N = 1 \) supergravity [19, 38] in this way.

**Comparison with the Literature** Let us now see how we can recover the existing results in the literature [6, 19, 20, 21, 22] on the various linearized \( N = 1 \) supergravities from our formalism. For this, we first recall the conservation equation (4.6)

\[
\bar{D}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} + W_\alpha + D_\alpha \tau = 0 \tag{6.12}
\]

with

\[
W_\alpha \equiv \frac{1}{4} \bar{D}^2 D_\alpha (X - 2\bar{X}), \quad \tau \equiv \frac{1}{4} \bar{D}^2 X - \frac{1}{6} J. \tag{6.13}
\]

Let us now furthermore impose the restriction that \( X \) is purely imaginary, \( X = -\bar{X} \). In this case we can compare the conservation equation (6.12) with eq. (3.4) in [22]. The case \( n = -\frac{1}{3} \) (old minimal) in [22] then corresponds to \( W_\alpha = 0 \) in (6.12), whereas \( n = 0 \) (new minimal) in [22] corresponds to \( \tau = 0 \). For all other real \( n \), eq.(3.4) in [22], with \( \lambda_\beta = D_\beta \Gamma \), can be written as

\[
\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = \frac{i}{6} \bar{D}^2 D_\alpha (\Gamma + \bar{\Gamma}) - \frac{i}{3n + 1} D_\alpha \bar{D}^2 \bar{\Gamma}. \tag{6.14}
\]

This then agrees with (6.12) provided we identify

\[
\frac{i}{6} (\bar{\Gamma} + \Gamma) = -\frac{3}{2} X \quad \text{and} \quad \frac{i}{3(3n + 1)} \bar{D}^2 \bar{\Gamma} = \tau = -\frac{1}{6} J + \frac{1}{2} D^2 X. \tag{6.15}
\]

33
For a general $W_\alpha$ ($X$ and $\bar{X}$ unrelated) and $\tau = 0$, the multiplet has $(16 + 16)$ components. If furthermore $\tau \neq 0$, we count $(20 + 20)$ components, which agrees with the non minimal supergravity.

Finally if we impose no relation between the various improvement terms, we showed in subsection 4.1.4 that we have to count all components of $X$. So, we end up with $(28 + 28)$ components, which agrees with the number of components of the flexible supergravity proposed in [23].

### 6.2 $\mathcal{N} = 2$ Supergravity

To obtain a linearized $\mathcal{N} = 2$ supergravity we can start with the $\mathcal{N} = 2$ vector multiplet discussed in subsection 5.2. In analogy with the $\mathcal{N} = 1$ case, we introduce the potentials $\Lambda, \Omega$ and a coupling

$$\int d^{12}z \, \Lambda T + \int d^8 z^+ \, \Omega \mathcal{J} + \int d^8 z^- \, \bar{\Omega} \bar{\mathcal{J}}. \quad (6.16)$$

The corresponding transformations are

$$\delta \Lambda \equiv -i (H - \bar{H}), \quad \delta \Omega \equiv 144i \sigma. \quad (6.17)$$

Note that the Weyl multiplet $\Lambda$, which was derived in [52] within harmonic superspace, arises here in ordinary superspace as a direct consequence of the localized transformation (5.10) of the vector multiplet $A$.

To see how the improvement terms are traced to supergravity couplings we again follow the same path as for $\mathcal{N} = 1$. Concentrating on the stress-energy tensor we notice that the improvement term is a double derivative of lowest component in the supercurrent $T$. Thus the highest component of $\Lambda$ transforms like the Ricci scalar and therefore we are free to relate it to the metric. What is different here is that there is no relation of the type (6.8) between the variation of the different superpotentials. However, we can nevertheless implement the substitution at the superspace level by expressing $\Lambda$ in terms of the unconstrained spinor superfield $\psi_{\alpha i}$ of Poincaré supergravity [49] as

$$\Lambda \equiv D^{\alpha i} \psi_{\alpha i} + \bar{D}_{\dot{\alpha} \dot{i}} \bar{\psi}^{\dot{\alpha} \dot{i}}. \quad (6.18)$$

This has indeed the effect of relating the highest component of $\Lambda$ to the metric.

### Conclusions

In this paper, we developed a method to construct the various multiplets of Noether currents directly at the superfield level (Superfield Noether Procedure). This formalism is useful in view of a unified treatment of those supersymmetric theories for which an off-shell superfield formulation exists. In particular, it produces a manifestly supersymmetric treatment of the various improvement terms interpolating between canonical and improved Noether currents.

A prominent feature of this formulation is that the various algebraic manipulations are independent of the complexity of the action for a given supermultiplet. This makes this approach particularly suited for dealing with (quantum) effective actions of supersymmetric theories. As a specific application we obtained an efficient algorithm
to compute the supersymmetry central charge for an arbitrary local action of a given
off-shell superfield. As another application we gave a systematic derivation of the
supercurrent of the $\mathcal{N}=2$ tensor multiplet as well as the multiplet of canonical Noether
currents of $\mathcal{N}=2$ Yang-Mills theory. As a by-product we then also found a simple
derivation of the anomalous superconformal Ward-Identity for the effective action of
that theory.

Of course, by its very nature the application of our procedure is limited to those
theories for which an off-shell superfield formulation exists. This is an obvious limita-
tion when dealing with models with extended supersymmetry. In view of this, it would
be interesting to generalize our formalism to harmonic superspace [54].

Acknowledgements

We acknowledge helpful discussions with F. Delduc, P. Howe, E. Ivanov, S. Kuzenko and
E. Sokatchev. This work has been supported by the TMR contract FMRX-CT96-0012
of the European Union, the ACI 2078-CDR-2 program of Ministère de la Recherche
and by the DFG-Stringtheorie Schwerpunktsprogramm SPP 1096.

A Multiplet of Currents for the $\mathcal{N}=2$ Vector

In this appendix, we first give some details how to obtain the variation (5.29) of the
supersymmetry current for the $\mathcal{N}=2$ vector multiplet. Then, we summarize the
multiplet structure in the general case and in the improved case.

A.1 Variation of the Supersymmetry Current

We are interested in the computation of $D_{\alpha i}J^\mu_{\beta j}$ where $J^\mu_{\alpha i}$ is given by eq.(5.18) i.e.

$$J^\mu_{\alpha i} = 192 \left[ i\sigma^\mu_{\alpha a} \bar{D}^a_\dot{\alpha} \bar{D}^a_\dot{\beta} \bar{J} T - 3i\bar{\sigma}^{\mu \dot{\alpha} \dot{\beta}} \bar{D}^\dot{\alpha}_\alpha D_{\alpha \beta} T - 12i\bar{D}^a_\dot{\alpha} T + a\sigma^\mu_{\alpha \beta} \partial_\nu D^\dot{\alpha}_\beta T \right]. \quad (A.1)$$

The method consists of course of decomposing all the terms into the symmetric and
antisymmetric parts with respect to the $SU(2)_R$ and spinor indices.

For the first term in the r.h.s. of (A.1), we use its equivalent form in terms of $\bar{J}$ and the chirality of $\bar{J}$ to get:

$$D_{\alpha i} \left[ i\sigma^\mu_{\beta \dot{\alpha}} \bar{D}^\dot{\beta}_k D^k_\dot{\gamma} T \right] = 3i\bar{\epsilon}_{\dot{\gamma} \gamma} \sigma^\mu_{\alpha \dot{\beta}} \partial_{\nu} D_{\alpha \beta} T - 3i\bar{\epsilon}_{\gamma \gamma} \sigma^\mu_{\alpha \dot{\beta}} \partial_{\nu} D_{\alpha \beta} T - 6i\sigma^\mu_{\alpha \beta} \partial_{\nu} \bar{D}_{\dot{\gamma} \dot{\gamma}} T.$$  

(A.2)

For the second term in the r.h.s. of (A.1), we use the relation

$$D_{\alpha i} \bar{D}_{i\dot{\gamma}} D_{\beta \gamma} T = -2i\bar{\epsilon}_{\dot{\gamma} \gamma} \partial_{\alpha \dot{\beta}} D_{\beta \gamma} T + \frac{2}{3}\bar{\epsilon}_{\alpha (\beta} \bar{D}_{i\dot{\gamma})}$$

(A.3)

$$= -2i\bar{\epsilon}_{\dot{\gamma} \gamma} \partial_{\alpha \dot{\beta}} D_{\beta \gamma} T - 2\bar{\epsilon}_{\dot{\gamma} \gamma} \epsilon_{\alpha (\beta} \partial_{\gamma) \dot{\gamma}} \bar{D}_{\dot{\alpha} \dot{\beta}} T - 2\bar{\epsilon}_{\alpha (\beta} \partial_{\gamma) \dot{\gamma}} \bar{D}_{\dot{\alpha} \dot{\beta}} T.$$  

(A.4)
where we have used the equation of conservation (5.17). This enables to get:

\[
D_{a} [ -3i \bar{\sigma}^{\mu \nu \gamma} D_{\alpha \gamma} D_{\beta \gamma} T ] = \varepsilon_{i j} \varepsilon_{\alpha \beta} \left[ 6 \sigma^{\mu \nu}_{\gamma \delta} D^{\gamma \delta} T + 9i \bar{\sigma}^{\mu \nu}_{\alpha \beta} D^{\alpha \beta} \bar{J} \right] \\
+ \varepsilon_{i j} \left[ 6 \partial^{\mu} D_{\alpha \beta} T + 12 \sigma^{\mu \nu}_{(\alpha} \gamma \partial_{\nu} D_{\beta) \gamma} T + 3i \sigma^{\mu \nu}_{(\alpha \dot{\alpha}} \sigma^{\nu \dot{\beta})}_{\beta} \partial_{\nu} D^{\dot{\alpha} \dot{\beta}} \bar{J} \right] \\
- 9i \varepsilon_{\alpha \beta} \partial^{\mu} D_{i j} \bar{J} - 6i \sigma^{\mu \nu}_{\alpha \beta} \partial_{\nu} D_{i j} \bar{J} . \tag{A.5}
\]

For the third and the last terms in the r.h.s. of (A.1) we immediately get:

\[
D_{a} [ -12 \partial^{\mu} D_{\beta \gamma} T ] = -6 \varepsilon_{i j} \partial^{\mu} D_{\alpha \beta} T - 6 \varepsilon_{\alpha \beta} \partial^{\mu} D_{i j} T \\
= -6 \varepsilon_{i j} \partial^{\mu} D_{\alpha \beta} T + 6i \varepsilon_{\alpha \beta} \partial^{\mu} D_{i j} \bar{J} , \tag{A.6}
\]

\[
D_{a} \left[ a \sigma^{\mu \nu}_{\alpha \beta} \partial_{\nu} D_{i j} T \right] = \frac{a}{2} \partial_{\nu} \left[ -\frac{1}{2} \varepsilon_{i j} \varepsilon_{\alpha \beta} \sigma^{\nu \rho}_{\gamma \delta} D^{\gamma \delta} T + \varepsilon_{i j} \sigma^{\mu \nu}_{(\alpha} \gamma D_{\beta) \gamma} T + \sigma^{\mu \nu}_{\alpha \beta} D_{i j} T \right] \tag{A.8}
\]

Finally, taking the sum of (A.2), (A.5), (A.7) and of (A.8) leads to the result (5.29).

### A.2 Multiplet in the General Case

The multiplet is formed of

\[
\begin{align*}
t &= T |, \\
\chi_{ai} &= D_{ai} T |, \\
u_{a \beta} &= D_{a \beta} T |, \\
R^{\mu} &= -48 \sigma^{\mu \alpha}_{\dot{\alpha}} [D^{ai}, D^{\dot{a}i}] T |, \\
R_{i j}^{\mu} &= -864 i \sigma^{\gamma}_{\dot{\alpha} \dot{\beta}} [D^{i a}, D^{j a}] T |, \\
J_{a}^{i j} &= 192 \left[ i \sigma^{\gamma}_{\dot{\alpha} \dot{\beta}} D^{\gamma \dot{a}} D^{\gamma \dot{b}} T | - 3i \bar{\sigma}^{\gamma \dot{a} \dot{b}} \bar{D}_{a} D_{a} T | - 12 \partial^{\mu} \chi^{i j} + a \sigma^{\mu \nu}_{\alpha \beta} \partial_{\nu} \chi^{i j} \right] , \\
T^{\mu \nu} &= -24 \left[ \frac{1}{2} \eta^{\mu \nu} \{ D^{i j}, D^{i j} \} T | - \frac{3}{2} \sigma^{\mu \alpha}_{\dot{\alpha}} \sigma^{\nu \beta}_{\dot{\beta}} \{ D_{a \beta}, \bar{D}_{a \dot{\beta}} \} T | \\
& \quad + 48 \eta^{\mu \nu} t - b ( \eta^{\mu \nu} - \partial^{\mu} \partial^{\nu} ) t \right] , \\
Z^{\mu \nu} &= 96 \left[ \bar{\sigma}_{a \beta}^{\mu \nu} \bar{D}^{a \beta} \bar{J} | - i ( \frac{1}{2} - \frac{a}{48} ) \sigma^{\mu \nu}_{a \alpha} u_{a \beta} \right] , \\
\bar{J} |, \\
\bar{D}_{a i} \bar{J} |, \\
\bar{D}_{i j} \bar{J} |.
\end{align*}
\]

The transformations of these components are

\[
\begin{align*}
\delta_{a i} t &= \chi_{ai} , \\
\delta_{ai} \chi_{\beta j} &= \frac{1}{2} \varepsilon_{ij} u_{a \beta} - \frac{i}{2} \varepsilon_{a \beta} D_{ij} \bar{J} , \\
\tilde{\delta}_{ai} \chi_{\alpha j} &= \frac{i}{3456} R_{a a i j} + \frac{1}{384} \varepsilon_{ij} R_{a a} + i \varepsilon_{ij} \partial_{a \dot{a}} t , \\
\delta_{ai} u_{\beta \gamma} &= 4 \varepsilon_{(\beta \dot{\beta}) (\gamma \dot{\gamma})} D^{a \beta} \bar{J} ,
\end{align*}
\]
The transformation properties of these components are given by:

\[\delta_{\alpha i}u_{\alpha \beta} = -\frac{i}{1152}\sigma_{\mu(\alpha \beta)}J_{\beta i}^\mu + \frac{i}{12}(a - 24)\partial(\alpha \beta)\chi_{\alpha i},\]

\[\delta_{\alpha i}R_{j k}^\mu = \frac{3}{2}\varepsilon_{ij}\left[J_{\alpha k}^\mu - (24 + a)\sigma^\mu_{\alpha \beta}\partial_\nu\chi_{\beta i}\right],\]

\[\delta_{\alpha i}\mathcal{R}^\mu = 16i\left\{\frac{1}{192}\left[\frac{3}{2}J_{\alpha i}^\mu + \sigma^\mu_{\alpha \beta}J_{\beta i}^\mu\right] - \frac{3}{4}(8 + a)\partial^\mu\chi_{\alpha i} + \frac{1}{2}(a - 24)\sigma^\mu_{\alpha \beta}\partial_\nu\chi_{\beta i}\right\},\]

\[\delta_{\alpha i}J_{\beta i}^\mu = \varepsilon_{ij}\sigma_{\nu a\beta} - 2i\varepsilon^{T\mu\nu} + 48i(b - 48 + 2a)(\eta^{\mu\nu\square} - \partial^\mu\partial^\nu)t\]

\[+ \varepsilon_{ij}\left\{\frac{1}{4}[a + 24]\left[\sigma^\mu_{\alpha a}\partial^\nu\mathcal{R}_\rho - \partial_{\alpha a}\mathcal{R}^\mu\right] - \frac{i\alpha}{4}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\tau a\alpha}\partial_\nu\mathcal{R}_\rho\right\},\]

\[+ \frac{i}{36}(a - 24)\partial_{\alpha a}\mathcal{R}^\mu_{ij} - \frac{1}{36}(a + 24)\varepsilon^{\mu\nu\rho\sigma}\sigma_{\tau a\alpha}\partial_\nu\mathcal{R}_\rho^\mu,\]

\[\delta_{\alpha i}J_{\beta j}^\mu = \partial_\nu\left\{24i\varepsilon_{ij}\varepsilon_{\mu\nu}\chi^\mu_{\beta i} + 96(a + 24)\left[\varepsilon_{ij}\sigma^{\mu\nu}(\chi_{\beta i}) - i\sigma^{\mu\nu}\sigma_{\alpha \beta}\right]\right\},\]

\[\delta_{\alpha i}\mathcal{T}^{\mu\nu} = \sigma^{\mu\nu}\partial_\rho J_{\beta j}^\rho + 24(b - 48 + 2a)(\eta^{\mu\nu\square} - \partial^\mu\partial^\nu)\chi_{\alpha i},\]

\[\delta_{\alpha i}Z_{\mu\nu} = -96i(1 + \frac{a}{24})(\sigma^{\mu\nu}\partial^\rho - \sigma^{\nu\rho}\partial^\mu)\mathcal{D}_i\mathcal{J} + 96(3 - \frac{a}{24})\varepsilon^{\mu\nu\rho\sigma}\sigma_{\tau a\alpha}\partial_\rho\mathcal{D}_i\mathcal{J},\]

\[\mathcal{J}_{\alpha i} = 0,\]

\[\mathcal{J}_{\alpha i} = \mathcal{D}_{\alpha i},\]

\[\delta_{\alpha i}(\mathcal{D}_{\beta j}\mathcal{J}) = -2i\varepsilon_{ij}\partial_{\alpha a}\mathcal{J},\]

\[\delta_{\alpha i}(\mathcal{D}_{\beta j}\mathcal{J}) = -\frac{1}{2}\varepsilon_{\alpha j}\mathcal{D}_{\beta j},\]

\[\delta_{\alpha i}(\mathcal{D}_{j k}\mathcal{J}) = -4i\varepsilon_{ij}\mathcal{D}_{\alpha a}\mathcal{D}_{k},\]

\[\delta_{\alpha i}(\mathcal{D}_{j k}\mathcal{J}) = \frac{1}{6}\varepsilon_{ij}\left[\frac{1}{192}(\mathcal{J}_{\mu k})\mathcal{J} - \frac{3}{2}(8 + a)\partial_{\alpha a}\mathcal{J}_{k}\right].\]

### A.3 Improved Multiplet

The multiplet of improved currents is composed of:

\[t = T,\]

\[\chi_{\alpha i} = D_{\alpha i}T,\]

\[u_{\alpha \beta} = D_{\alpha \beta}T,\]

\[\mathcal{R}^\mu = -48\sigma_{\alpha a}(D_{\alpha i}, \mathcal{D}_{\mathcal{J}^i})T,\]

\[R_{ij}^\mu = -84i\sigma_{\alpha a}(D_{\alpha a}, \mathcal{D}_{\mathcal{J}^i})T,\]

\[J_{\mu i} = 192\left[-12\partial^\mu\chi_{\alpha i} - 3i\partial^\mu\partial^\nu - 8\sigma_{\mu \beta}^\nu\partial_\nu\chi_{\beta i}\right],\]

\[T_{\mu \nu} = -24\left[\frac{3}{2}\sigma_{\mu \alpha \beta}^\nu\partial_{\alpha \beta}^\nu\mathcal{D}_{\alpha \beta}T - 64\eta^{\mu \nu\square} - 64(\eta^{\mu \nu\square} - \partial^\mu\partial^\nu)t\right].\]

The transformation properties of these components are given by:

\[\delta_{\alpha i}t = \chi_{\alpha i},\]

\[\delta_{\alpha i}\chi_{\beta j} = \frac{1}{2}\varepsilon_{ij}\sigma_{\alpha \beta}.\]
\[ \delta_{ab} \chi_j = \frac{i}{3456} R_{a\dot{a}j} + \frac{1}{384} \varepsilon_{ij} R_{a\dot{a}} + \frac{i}{5} \varepsilon_{ij} \partial_{a\dot{a}} t, \]
\[ \delta_{ab} \mathcal{F}_{ij} = 0, \]
\[ \delta_{ab} \mathcal{F}_{ij} = -i \frac{2}{1152} \sigma_{\mu(\beta \dot{a} \nu \dot{b})} - \frac{8i}{3} \partial_{(\beta \dot{a}} \chi_{\nu \dot{b})}, \]
\[ \delta_{ab} \mathcal{R}_\gamma = 16 \mu \left[ \frac{1}{192} J_{\mu}^a - 16 \sigma_{\mu \nu}^a \partial_\nu \chi^a \right], \]
\[ \delta_{ab} \mathcal{R}_{ijk} = \frac{3}{2} \varepsilon_{ij} \left[ J_{\mu}^a \right] - 16 \sigma_{\mu \nu}^a \partial_\nu \chi^a \right], \]
\[ \delta_{ab} J_{ij}^a = \varepsilon_{ij} \left[-2i \sigma_{\nu a b} T_{\mu}^{\nu} - 4 \partial_{a b} \mathcal{R}_\mu + 2i \varepsilon_{\mu \nu \rho \sigma} \sigma_{\tau a b} \partial_\rho \mathcal{R}_\nu \right], \]
\[ \delta_{ab} J_{ij}^a = \frac{8i}{9} \partial_{a b} \mathcal{R}_{\mu \nu} - 4i \varepsilon_{\mu \nu \rho \sigma} \sigma_{\tau a b} \partial_\rho \mathcal{R}_{\nu} \], \]
\[ \delta_{ab} T^{\mu \nu} = \sigma^{\mu \nu \rho \sigma \delta} \partial_\rho \mathcal{R}_{\nu}^{ij}. \]

**B SU(2)_R Invariance for the \( \mathcal{N} = 2 \) Tensor Multiplet**

In this appendix we sketch the proof of the identity (5.80)

\[ (D^{(m)}_i D_k + \bar{D}^{(m)}_i D_{kl}) T^{ijkl} = -8i \partial_{a b} [D^a_{kl}, \bar{D}^b_{ij}] T^{imkl}. \] (B.1)

The starting point is to notice that

\[ T^{ijkl} = \frac{1}{160} F_{ijkl} = D^{ijkl} + \text{c.c.} \text{ with } X^{ij} \equiv \frac{i}{160} \Phi F^{ij}. \] (B.2)

As \( \Phi \) is chiral, \( X^{ij} \) has clearly the same properties (5.64) than \( F^{ij} \). The first step consists in writing \( D^{(m)}_i D_k = [D^{(m)}_i, D_k] + D_k D^{(m)}_i \). After some algebra we find

\[ [D^{(m)}_i, \bar{D}_k] D^{ijkl} = 4i \partial_{a b} \bar{D}_k D_i D^{ijkl}, \] (B.3)

\[ \bar{D}_k D_k D^{ijkl} = \frac{5}{36} \varepsilon_{ijkl} D^{ijkl} + \frac{5}{18} D^{ijkl} D^{ijkl}. \] (B.4)

On the other hand,

\[ \bar{D}^{(m)}_i D_k D^{ijkl} = \frac{5}{18} \varepsilon_{ijkl} D^{ijkl} + \frac{5}{9} D^{ijkl} D^{ijkl}. \] (B.5)

Adding (B.3), (B.4) and (B.5), we find

\[ (D^{(m)}_i D_k + \bar{D}^{(m)}_i D_{kl}) D^{ijkl} = \frac{5}{12} \varepsilon_{ijkl} D^{ijkl} + \frac{5}{6} \bar{D}^{ijkl} D^{ijkl} \] (B.6)

For the second step, we use the fact that

\[ \bar{D}^{(m)}_i D^{ijkl} = [\bar{D}^{(m)}_i, D^{ijkl}] \]

since the other term vanishes due to the properties (5.64) of \( X^{ij} \). Again, we find after some algebra:

\[ [\bar{D}^{(m)}_i, D^{ijkl}] = 48 \bar{D}^{(m)}_i D^{ijkl} + \frac{32i}{5} \partial_{a b} \bar{D}^{(m)}_i D^{ijkl}. \]
However, a direct computation shows also that
\[ \partial_{\alpha\dot{\alpha}} [D_k^\alpha, \bar{D}_l^{\dot{\alpha}}] D^{(mi} X^{kl)} = \frac{20i}{3} D^{(mj} X^{lj)} - \frac{40}{27} \partial_{\alpha\dot{\alpha}} D^{2\alpha(i \bar{D}_j^{\dot{\alpha}} X^{mj)}. \]

Thus we have proved that
\[ \bar{D}^{(mj} D^4 X^{lj)} = -\frac{36i}{5} \partial_{\alpha\dot{\alpha}} [D_k^\alpha, \bar{D}_l^{\dot{\alpha}}] D^{(mi} X^{kl)}. \tag{B.7} \]

To conclude, we put the results (B.6) and (B.7) together. This leads to the final relation:
\[ \left(D^m_j \bar{D}_{kl} + \bar{D}^m_j D_{kl}\right) D^{(ij} X^{kl)} = \frac{5}{12} \varepsilon^{mi} \bar{D}^{jk} D^4 X_{jk} - 8i \partial_{\alpha\dot{\alpha}} [D_k^\alpha, \bar{D}_l^{\dot{\alpha}}] D^{(mi} X^{kl)}, \]
from which we get (B.1) as a consequence.

C Conventions and Identities for the SUSY Algebra

The conventions used in this paper are essentially those of Wess and Bagger [29]. The conventions about covariant derivatives and their algebra in \( N = 1 \) as well as in extended supersymmetry are exposed in this appendix.

The following definitions are valid for any \( N \). The spinorial derivatives are defined by
\[ \partial^i_\alpha \theta_\dot{\alpha} = \frac{\partial \theta_\dot{\alpha}}{\partial \alpha^i} = \delta_\dot{\beta}^\dot{\alpha} \delta_i^\beta, \quad \bar{\partial}^\dot{i}_{\dot{\alpha}} \dot{\theta}^\dot{\alpha} = \frac{\partial \dot{\theta}^\dot{\alpha}}{\partial \alpha^i} = \delta_\dot{\alpha}^\dot{\beta} \delta_{\dot{\beta}}^i. \tag{C.1} \]

The covariant derivative are then defined as
\[ D^i_{\alpha} = \partial^i_{\alpha} + i \bar{\theta}^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}}, \quad \bar{D}_{\dot{i}} = -\bar{\partial}_{\dot{i}} + i \theta^\alpha \partial_{\alpha \dot{\alpha}}. \tag{C.2} \]

Finally, the algebra following from these definitions is
\[ \{D^i_{\alpha}, \bar{D}_{\dot{i}}\} = -2i \delta^i_j \partial_{\alpha \dot{\alpha}}, \quad \{D^i_{\alpha}, D^j_{\beta}\} = 0. \tag{C.3} \]

We furthermore use the following usual definition of \( x^\mu_\pm \): \( x^{\alpha\dot{\alpha}} \equiv x^{\alpha\dot{\alpha}} \pm 2i \theta^{\alpha\dot{\alpha}} \overline{\theta}_i^\dot{\alpha} \). The main properties of \( x^\mu_\pm \) with respect to the covariant derivatives are:
\[ \bar{D}_{\alpha} x^\mu_+ = 0, \quad D^i_{\alpha} x^\mu_- = 0, \]
\[ D_{\alpha} x^{\mu}_{+\beta} = 4i \varepsilon_{\alpha \beta} \bar{\theta}^{\dot{\beta} \dot{i}}, \quad \bar{D}_{\dot{\alpha}} x^{\mu}_{-\beta \dot{\beta}} = -4i \varepsilon_{\dot{\alpha} \dot{\beta}} \theta_{\dot{\beta} \dot{i}}. \]

The different integration measures are then defined by:
\[ \int d^{4+4N} z \equiv \int d^4 x D^{2N} \bar{D}^{2N}, \quad \int d^{4+2N} z^+ \equiv \int d^4 x D^{2N}, \quad \int d^{4+2N} z^- \equiv \int d^4 x \bar{D}^{2N}, \]
where \( D^{2N} \) is defined below for the specific values \( N = 1, 2 \).
C.1 \( \mathcal{N} = 1 \) Identities

We give there a list of useful identities for \( \mathcal{N} = 1 \) covariant derivatives following from the definitions and conventions given above.

Definition of \( D^2 \):
\[
D^2 \equiv D^\alpha D_\alpha , \quad \bar{D}^2 \equiv \bar{D}_\dot{\alpha} \bar{D}^{\dot{\alpha}} .
\]

Products of covariant derivatives:
\[
D^\alpha \bar{D}^2 D_\alpha = \bar{D}_\dot{\alpha} D^2 \bar{D}^{\dot{\alpha}} , \\
D^\alpha \bar{D}_\dot{\alpha} D_\alpha = -\frac{1}{2} \bar{D}_\dot{\alpha} D^2 - \frac{1}{2} D^2 \bar{D}_\dot{\alpha} , \\
\bar{D}_\dot{\alpha} D^\alpha \bar{D}^{\dot{\alpha}} = -\frac{1}{2} D^\alpha \bar{D}^2 - \frac{1}{2} \bar{D}^2 D^\alpha , \\
\left[ D^\alpha , \bar{D}^{\dot{\alpha}} \right] \left[ D_\alpha , \bar{D}_\dot{\alpha} \right] = 2 \{ D^2 , \bar{D}^2 \} - 24 \Box .
\]

Algebra of covariant derivatives:
\[
\left[ D_\alpha , \bar{D}^2 \right] = -4i \partial_\alpha \bar{D}^{\dot{\alpha}} , \quad \left[ \bar{D}_\dot{\alpha} , D^2 \right] = 4i D^\alpha \partial_{\dot{\alpha}} , \\
\left[ D^2 , \bar{D}^2 \right] = -4i \left[ D^\alpha , \bar{D}^{\dot{\alpha}} \right] \partial_{\alpha \dot{\alpha}} .
\]

C.2 \( \mathcal{N} = 2 \) Identities

Raising and lowering of \( SU(2) \) indices and Fierz formula:
\[
a^i = \varepsilon^{ij} a_j , \quad a_i = \varepsilon_{ij} a^j , \\
\text{with } \varepsilon^{ij} = -\varepsilon_{ij} , \quad \varepsilon^{12} = 1 , \quad \varepsilon_{ij} = -\varepsilon^{ij} , \quad \varepsilon^{ij} \varepsilon_{jk} = \delta^i_k .
\]
\[
a_i b^j - a^i b_j = \varepsilon_{ij} a^k b_k , \quad a^i b^j - a^j b^i = -\varepsilon^{ij} a^k b_k .
\]

Definitions of some products of \( \theta^i \)’s:
\[
\theta^2 : \quad \theta^{ij} \equiv \theta^a_i \theta^a_j , \quad \theta^{\alpha \beta} \equiv \theta^a_i \theta^b_j , \quad \bar{\theta}^{ij} \equiv \bar{\theta}_a^i \bar{\theta}_a^j , \quad \bar{\theta}^{\dot{\alpha} \dot{\beta}} \equiv \bar{\theta}_a^i \bar{\theta}_a^j .
\]
\[
\theta^3 : \quad \theta^{a_i}_a \equiv \theta^a_i \theta_a^j = -\theta^j_1 \theta_a^j , \quad \bar{\theta}^{a_i}_\dot{a} \equiv -\bar{\theta}_a^i \bar{\theta}_a^j = \bar{\theta}_a^i \bar{\theta}_a^j .
\]
\[
\theta^4 : \quad \theta^4 \equiv \bar{\theta}^{ij} \theta_{ij} = -\theta^{a \beta} \theta_{a \beta} , \quad \bar{\theta}^4 \equiv \bar{\theta}^{ij} \bar{\theta}_{ij} = -\bar{\theta}^{\dot{a} \dot{\beta}} \bar{\theta}_{\dot{a} \dot{\beta}} .
\]

Properties of the products of \( \theta^i \)’s:
\[
\theta_{a i} \theta_{b j} = -\frac{2}{3} \varepsilon_a \theta_{b j} , \quad \theta_{a i} \theta_{b j} = \frac{2}{3} \varepsilon_b \theta_{a j} , \quad \bar{\theta}_{a i} \bar{\theta}_{\dot{b} j} = \frac{2}{3} \varepsilon_a \bar{\theta}_{\dot{b} j} , \quad \bar{\theta}_{a i} \bar{\theta}_{\dot{b} j} = \frac{2}{3} \varepsilon_b \bar{\theta}_{a j} ,
\]
\[
\theta^{a i} \theta^{b j} = \frac{1}{4} \varepsilon^{a i} \theta^{b j} , \quad \bar{\theta}^{a i} \theta^{b j} = \frac{1}{4} \varepsilon^{a i} \bar{\theta}^{b j} , \quad \bar{\theta}^{a i} \theta^{b j} = \frac{1}{3} \varepsilon^{a j} \theta^{b i} , \quad \theta^{a i} \theta^{b j} = \frac{1}{3} \varepsilon^{a j} \bar{\theta}^{b i} ,
\]
\[
\bar{\theta}^{a i} \bar{\theta}^{b j} = -\frac{1}{4} \delta^{a i} \delta^{b j} , \quad \bar{\theta}^{a i} \bar{\theta}^{b j} = 0 , \quad \bar{\theta}^{a i} \bar{\theta}^{b j} = \frac{1}{3} \varepsilon^{a i} \bar{\theta}^{b j} , \quad \bar{\theta}^{a i} \bar{\theta}^{b j} = -\frac{1}{3} \varepsilon^{a j} \bar{\theta}^{b i} .
\]

Definitions of products of covariant derivatives:
\[
D^2 : \quad D^{ij} \equiv D^{a i} D^a_j , \quad D^{a \beta} \equiv D^{a i} D^b_j , \quad \bar{D}^{ij} \equiv \bar{D}_a^i \bar{D}_a^j , \quad \bar{D}^{\dot{a} \dot{b}} \equiv \bar{D}_{\dot{a}}^i \bar{D}_{\dot{b}}^j ,
\]
\[
D^3 : \quad D^{3 i} \equiv D^a_a D^a_i , \quad D^{3 \beta} \equiv D^a_i D^b_j , \quad \bar{D}^{3 i} \equiv \bar{D}_{\dot{a}}^i \bar{D}_{\dot{a}}^j , \quad \bar{D}^{3 \beta} \equiv \bar{D}_{\dot{a}}^i \bar{D}_{\dot{a}}^j ,
\]
\[
D^4 : \quad D^4 \equiv D^{ij} D_{ij} = -D^{a \beta} D_{a \beta} , \quad D^4 \equiv \bar{D}^{ij} D_{ij} = -\bar{D}^{\dot{a} \dot{b}} \bar{D}_{\dot{a} \dot{b}} .
\]
Properties of the products of covariant derivatives:

\[ (12) : \, D_i^a D_{jk} = \frac{2}{3} \varepsilon_{ijl} D_k^{bl} , \quad \bar{D}_i^\alpha \bar{D}_{jk} = -\frac{2}{3} \varepsilon_{ijl} \bar{D}_l^{\alpha l} , \]

\[ D_{ai} D_{\beta j} = -\frac{2}{3} \varepsilon_{a(\beta D_\gamma j)i} , \quad \bar{D}_{ai} \bar{D}_{\beta j} = -\frac{2}{3} \varepsilon_{a(\beta D_\gamma j)i} , \]

\[ (13) : \, D_{ai} D^\beta_{\beta j} = \frac{4}{3} \varepsilon_{ai\beta} \varepsilon_{ij} D^4 , \quad \bar{D}_{ai} \bar{D}^\beta_{\beta j} = \frac{4}{3} \varepsilon_{ai\beta} \varepsilon_{ij} \bar{D}^4 , \]

\[ (22) : \, D_{ij} D_{kl} = \frac{4}{3} \varepsilon_{ijkl} D^4 , \quad D_{ij} \bar{D}_k = \frac{4}{3} \varepsilon_{ijkl} \bar{D}_j^4 , \]

\[ D_{ij} D_{a\beta} = 0 , \quad \bar{D}_j \bar{D}_{a\beta} = 0 , \quad D_{ij} \bar{D}^4 D_{kl} = \bar{D}_j \bar{D}_k D^4 . \]

C.2.1 Complex Conjugation

The general complex conjugation rule for the \( SU(2) \) indices is the following: \((\varphi_i)^* = \bar{\varphi}^\dagger \). As a consequence of this, we have \((\varphi_i)^* = (\varepsilon_{ij} \varphi_j)^* = \varepsilon_{ij} \varphi_j = -\varepsilon_{ij} \varphi_j = -\varphi_i \). For the specific products of \( \theta \)'s defined above, this leads to:

\[ (\theta^{ij})^* = \bar{\theta}_{ij} , \quad (\theta^{a\beta})^* = -\bar{\theta}^{\alpha\beta} , \quad (\theta^{a\alpha})^* = \bar{\theta}^{a\alpha} , \quad (\theta^3_\alpha)^* = \bar{\theta}^{3\alpha} , \quad (\theta^4)^* = \bar{\theta}^4 . \]

For the spinorial and covariant derivatives, this gives:

\[ (\partial^{i})^* = -\bar{\partial}^{i} , \quad (D^{i})^* = -\bar{D}^{i} , \]

and for higher order products:

\[ (D^{ij})^* = \bar{D}_{ij} , \quad (D^{a\beta})^* = -\bar{D}^{a\beta} , \quad (D^{a\alpha})^* = \bar{D}^{a\alpha} , \quad (D^4)^* = \bar{D}^4 . \]

C.2.2 Algebra of covariant Derivatives

\[ (12) : \]

\[ \begin{align*}
[D_{ij}, \bar{D}_{ak}] &= 4i \varepsilon_{k(i} D_{j)l}^{a} \partial_{a\bar{\alpha}} , \\
[\bar{D}_{ij}, D_{ak}] &= 4i \varepsilon_{k(i} \partial_{a\bar{\alpha}} \bar{D}_{j)l}^{\bar{\alpha}} , \\
[D_{a\beta}, \bar{D}_{ai}] &= 4i \varepsilon_{(a} \partial_{i}) \bar{D}_{\beta)i}^{\bar{\alpha}} , \\
[\bar{D}_{a\beta}, D_{ai}] &= -4i \varepsilon_{(a} \partial_{i}) \bar{D}_{\beta)i}^{\bar{\alpha}} , \\
[D_{ij}, D^{i}] &= 6i D_{i}^{a} \partial_{a\bar{\alpha}} , \\
[\bar{D}_{ij}, D^{i}] &= 6i \partial_{a\bar{\alpha}} \bar{D}_{i}^{\bar{\alpha}} .
\end{align*} \]

\[ (22) : \]

\[ \begin{align*}
[D_{ij}, \bar{D}_{kl}] &= 16 \varepsilon_{k(i} \varepsilon_{j)l} \partial_{a}\partial_{a\bar{\alpha}} \bar{D}_{l}^{\bar{\alpha}} , \\
&= -16 \varepsilon_{k(i} \varepsilon_{j)l} \partial_{a}\partial_{a\bar{\alpha}} \bar{D}_{l}^{\bar{\alpha}} , \\
[D_{ij}, \bar{D}_{a\beta}] &= 8i \partial_{a\alpha} D^{a}_{i} \bar{D}_{j}^{\bar{\alpha}} ,
\end{align*} \]

41
(13):

\[
\begin{align*}
\{D_{a\dot{a}}, \bar{D}_{\dot{a} j}^\dot{a}\} &= -3i\varepsilon_{\dot{a} j} \partial_{\alpha\dot{\alpha}} \bar{D}_{\dot{a} j}^\dot{a} - 3i\partial_{\alpha\dot{a}} \bar{D}_{\dot{a} j}^\dot{a}, \\
\{\bar{D}_{\dot{a} j}, D_{a j}^a\} &= 3i\varepsilon_{\dot{a} j} D_{a \beta} \partial_{\beta\dot{\beta}} - 3i\partial_{a\dot{a}} D_{a j}.
\end{align*}
\]

(14):

\[
\begin{align*}
\begin{bmatrix} D^a, \bar{D}^4 \end{bmatrix} &= 8i\partial^a\dot{a} \bar{D}_{\dot{a} j}^\dot{a} , \\
\begin{bmatrix} \bar{D}^i, D^4 \end{bmatrix} &= 8iD^{3a} \partial_{a\dot{a}}.
\end{align*}
\]

(23):

\[
\begin{align*}
\begin{bmatrix} D_{k j}^3, D_{i j} \end{bmatrix} &= \varepsilon_{k(i} \left( -24\bar{D}^a_{j)} \bar{D}_{k}^\dot{a} D_{\alpha\beta}^a \partial_{\beta\dot{\beta}} + 6i\partial^a\dot{a} D_{a(i} D_{k) j} \right), \\
\begin{bmatrix} \bar{D}_{\dot{a} j}^\dot{a}, D_{i j} \end{bmatrix} &= \varepsilon_{\dot{a} i} \left( 24\bar{D}_{j) \dot{a}}^\dot{a} \bar{D}_{k}^\dot{a} D_{a\beta}^\dot{a} \partial_{\beta\dot{\beta}} + 6i\partial_{\alpha\dot{a}} D_{\dot{a} i(} D_{j k)} \right), \\
\begin{bmatrix} D_{k j}^3, \bar{D}_{\dot{a} j}^\dot{a} \end{bmatrix} &= -24\partial_{(a\dot{a}} D_{\beta) j} \bar{D}_{k}^\dot{a} + 6iD_{(\dot{a} k} D^{3a} \partial_{\beta\dot{\beta})} + 6i\partial_{a\dot{a}} D_{a j} D_{k j} \bar{D}_{\dot{a} i(} D_{j) k} \right), \\
\begin{bmatrix} \bar{D}_{\dot{a} j}^\dot{a}, D_{a j} \end{bmatrix} &= -24\partial_{(a\dot{a}} D_{\beta) \dot{a}} \bar{D}_{k}^\dot{a} + 6iD_{(\dot{a} k} D^{3a} \partial_{\beta\dot{\beta})\dot{a}} + 6i\partial_{a\dot{a}} D_{a j} D_{k j} \bar{D}_{\dot{a} i(} D_{j) k} \right), \\
\begin{bmatrix} D_{k j}^3, \bar{D}_{j}^\dot{a} \end{bmatrix} &= 36\bar{D}_{i\dot{a}}^\dot{a} 9i\partial_{\dot{a} j} D_{\dot{a} i j} D_{a\beta}^\dot{a} \partial_{\beta\dot{\beta}} - 3i\partial_{a\dot{a}} D_{a j} D_{j\dot{a} i} \right), \\
\begin{bmatrix} \bar{D}_{j}^\dot{a}, D_{j}^\dot{a} \end{bmatrix} &= -36\bar{D}_{i\dot{a}}^\dot{a} 9i\partial_{\dot{a} j} D_{\dot{a} i j} D_{a\beta}^\dot{a} \partial_{\beta\dot{\beta}} + 3i\partial_{a\dot{a}} D_{a j} D_{j\dot{a} i} \right).
\end{align*}
\]

(24):

\[
\begin{align*}
\begin{bmatrix} D_{\dot{a} j}, \bar{D}^4 \end{bmatrix} &= 48\bar{D}_{\dot{a} j} + 16i\partial_{a\dot{a}} \bar{D}_{\dot{a} j}^\dot{a} \bar{D}^a_{(\dot{a} i)} \bar{D}^3_{(j)}^a \right), \\
\begin{bmatrix} D_{a j}, \bar{D}^4 \end{bmatrix} &= -48\bar{D}_{\dot{a} j} - 16i\partial_{a\dot{a}} \bar{D}_{\dot{a} j}^\dot{a} \bar{D}^3_{(\dot{a} j)} \right), \\
\begin{bmatrix} D_{\dot{a} j}, \bar{D}^4 \end{bmatrix} &= -48\partial_{(a\dot{a}} D_{\beta) j} \bar{D}^\dot{a} \dot{a} + 16i\partial_{a\dot{a}} \bar{D}_{\dot{a} j}^\dot{a} D_{(\dot{a} \dot{a})} D_{j\dot{a} i} \right), \\
\begin{bmatrix} D_{a j}, \bar{D}^4 \end{bmatrix} &= 48\partial_{(a\dot{a}} D_{\beta) j} \bar{D}^\dot{a} \dot{a} + 16i\partial_{a\dot{a}} \bar{D}_{\dot{a} j}^\dot{a} D_{(\dot{a} \dot{a})} D_{j\dot{a} i} \right), \\
\begin{bmatrix} D_{\dot{a} j}, D^4 \end{bmatrix} &= -48\partial_{(a\dot{a}} D_{\beta) j} D_{a\beta} - 16i\partial_{a\dot{a}} D_{a j} D_{j\dot{a} i} \right), \\
\begin{bmatrix} D_{a j}, D^4 \end{bmatrix} &= 48\partial_{(a\dot{a}} D_{\beta) j} D_{a\beta} + 16i\partial_{a\dot{a}} D_{a j} D_{j\dot{a} i} \right).
\end{align*}
\]
\{D^3_{\alpha i}, \bar{D}^3_{\alpha i}\} = -36\theta^{\alpha i}, \quad D^3_{\alpha i} \theta^{\alpha j} = 9\delta^{\alpha j}_{\alpha i}, \quad D^4 \theta^{\alpha i} = 144, \\
D^3_{\alpha i} \bar{\theta}^{\beta j} = -12\bar{\theta}^{\beta j}, \quad D^3_{\alpha i} \bar{\theta}^{\beta j} = 6\delta^{\beta j}_{(\alpha i)}, \quad \bar{D}^3 \bar{\theta}^{\alpha i} = -9\bar{\theta}^{\alpha i}, \\
\bar{D}^3 \bar{\theta}^{\beta j} = -9\delta^{\beta j}_{\alpha i}, \quad \bar{D}^3 \bar{\theta}^{\alpha i}, \quad \bar{D}^3 \bar{\theta}^{\beta j} = 144, \\
D^4(\theta^{\alpha i}X)\} = 36D_{\alpha i}X^{\alpha i}, \quad \bar{D}^4(\bar{\theta}^{\alpha i}X)\} = -36\bar{D}_{\alpha i}X^{\alpha i}, \\
D^4(\theta^{\alpha i}X^{\alpha j})\} = -12D_{\alpha i}X^{\alpha j}, \quad \bar{D}^4(\bar{\theta}^{\alpha i}X^{\alpha j})\} = -12\bar{D}_{\alpha i}X^{\alpha j}, \\
D^4(\theta^{\alpha i}X^{\alpha j})\} = 4D_{\alpha i}X^{\alpha i}, \quad \bar{D}^4(\bar{\theta}^{\alpha i}X^{\alpha j})\} = 4\bar{D}_{\alpha i}X^{\alpha i}. \\
\begin{align*}
\{D_{\alpha i}, \bar{D}^3_{\alpha i}\} & = -72i\partial_{\alpha a} \bar{\theta}^{\beta j} \{D^4_{\alpha j}, \bar{D}^3_{\alpha j}\} - 9i \frac{9i}{4} \partial_{\alpha a} \{D_{k(i}, \bar{D}^4_{j)}\} - 9i \frac{9i}{4} \partial_{\alpha a} \{D^{3}_{\alpha \beta}, \bar{D}_{\alpha \beta}\}, \\
\{D^3_{\alpha i}, \bar{D}^3_{\alpha j}\} & = -9i \frac{9i}{4} \left( \partial_{\alpha a} \{D_{k(i}, \bar{D}^4_{j)}\} + \partial_{\beta a} \{D^{3}_{\alpha \beta}, \bar{D}_{\alpha \beta}\} + \partial_{\alpha a} \{D_{\alpha i}, \bar{D}^3_{\alpha j}\} \right). 
\end{align*}

(C.2.3) Derivatives acting on $\theta$

\[
\partial_{\alpha i} \theta^4 = -4\delta^3_{\alpha i}, \quad \bar{\partial}_{\alpha i} \bar{\theta}^4 = -4\delta^3_{\alpha i}, \\
\partial_{\alpha i} \theta_{3\beta j} = -\frac{3}{2} \left( \delta^3_{\alpha i} \theta_{3 j} - \delta^3_{\alpha j} \theta_{3 i} \right), \quad \bar{\partial}_{\alpha i} \bar{\theta}_{3 j} = -\frac{3}{2} \left( \delta^3_{\alpha i} \bar{\theta}_{3 j} - \delta^3_{\alpha j} \bar{\theta}_{3 i} \right), \\
D^\theta_{\alpha i} = -12\theta^i, \quad D_{\alpha i} \theta_{3\alpha k} = 6\delta^k_{(i} \theta^j_{j)}, \quad D^\bar{\theta}_{\alpha i} = -9\bar{\theta}^i, \\
D^\theta_{\alpha i} \theta_{\gamma d} = -12, \quad D_{\alpha i} \theta_{\gamma d} = -4\delta_{(\alpha i} \theta_{\gamma d)}, \\
D^\theta_{\alpha i} \theta_{3\alpha j} = -36\theta^{\alpha i}, \quad D^\theta_{\alpha i} \theta_{3\alpha j} = 9\delta^j_{(\alpha i}, \quad D^\theta_{\alpha i} = 144, \\
\bar{D}^\theta_{\alpha i} = -12\bar{\theta}^i, \quad \bar{D}_{\alpha i} \bar{\theta}_{3\alpha k} = 6\delta^k_{(i} \bar{\theta}^j_{j)}, \quad \bar{D}^\theta_{\alpha i} = -9\bar{\theta}^i, \\
\bar{D}^\theta_{\alpha i} \theta_{\gamma d} = -12, \quad \bar{D}_{\alpha i} \theta_{\gamma d} = -4\delta_{(\alpha i} \theta_{\gamma d)}, \\
\bar{D}^\theta_{\alpha i} \theta_{3\alpha j} = -36\bar{\theta}^{\alpha i}, \quad \bar{D}^\theta_{\alpha i} \theta_{3\alpha j} = 9\delta^j_{(\alpha i}, \quad \bar{D}^\theta_{\alpha i} = 144, \\
\bar{D}^\theta(\bar{\theta}^{\alpha i}X)\} = 144X\}, \quad \bar{D}^\theta(\bar{\theta}^{\alpha i}X)\} = 144X\}, \\
\bar{D}^\theta(\bar{\theta}_{\alpha i}X^{\alpha j})\} = 36D_{\alpha i}X^{\alpha j}, \quad \bar{D}^\theta(\bar{\theta}_{\alpha i}X^{\alpha j})\} = -36\bar{D}_{\alpha i}X^{\alpha j}, \\
\bar{D}^\theta(\bar{\theta}_{\alpha i}X^{\alpha j})\} = -12D_{\alpha i}X^{\alpha j}, \quad \bar{D}^\theta(\bar{\theta}_{\alpha i}X^{\alpha j})\} = -12\bar{D}_{\alpha i}X^{\alpha j}, \\
\bar{D}^\theta(\bar{\theta}_{\alpha i}X^{\alpha j})\} = 4D_{\alpha i}X^{\alpha i}, \quad \bar{D}^\theta(\bar{\theta}_{\alpha i}X^{\alpha j})\} = 4\bar{D}_{\alpha i}X^{\alpha i}.
\]
References


44


