We investigate the generation of primordial magnetic fields from stochastic currents created by the cosmological transition from inflation to reheating. We consider $N$ charged scalar fields coupled to the electromagnetic field in a curved background and derive self-consistent equations for the evolution of the two-point functions of the fields, which in the large-$N$ limit give a decoupled set for the scalar and the electromagnetic functions. The main contribution to the electric current comes from the infrared portion of the spectrum of created particles, and in this limit the damping of the magnetic field is not due to normal conductivity but to London currents in the scalar field. We found that the evolution equation for the magnetic field resembles a Mathieu equation. For a given set of the physical parameters of the problem, we solved this equation numerically and found that the field grows exponentially during the time interval in which our large-$N$ limit equations are valid. Although for the chosen parameters the induced field is weak, the present uncertainties on their actual values leave open the possibility for higher intensities.

I. INTRODUCTION

One of the most intriguing observational aspects of astrophysics is the detection of large scale magnetic fields in all the cosmological structures of the Universe [1–5]. The remarkable aspect of the observations of galactic magnetic fields, is that their intensity and structure are more or less the same, no matter if the galaxy is one of our neighbors or a highly redshifted one. Clusters of galaxies are also permeated with intense and quite coherent magnetic fields.

The main puzzle is the dynamical origin of these large scale fields. For galactic fields, two main lines of research are pursued: one proposes an origin based on local astrophysical processes, while the other advocates a primordial origin. The seducing aspect of a primordial origin of these fields is that it accounts for the observations in all the mentioned structures, no matter how high their redshift is.

The main difficulty for generating magnetic fields in the Early Universe is the need to break the conformal invariance of the Maxwell field. This can be done in many ways, which explains the large variety of proposed mechanisms in the literature [6–20]. After this first obstacle is overcome, the success of any given proposal hinges on whether the energy from the primordial source may be successfully stored in the magnetic field, or else it is dissipated away (over and above the strong reduction brought by cosmic redshift). This point was highlighted in a recent paper by Giovannini and Shaposhnikov [21]. Using kinetic theory, these authors showed that the amount of the magnetic field which may be generated during reheating is highly sensitive to the conductivity of the cosmic medium.

In this paper, we extend this analysis to fields generated in the earliest stages of reheating. At this time the fields and their sources span superhorizon scales and so they may hardly be described in terms of more or less localized excitations, counted by a distribution function obeying a transport equation. It becomes necessary to recast the analysis of [21] in terms of a more fundamental description of the process. We shall attempt to obtain an upper bound for the magnetic field which may be generated during reheating, directly from quantum field theory (for a discussion of the circumstances in which quantum field theory reduces to kinetic theory see [22]).

As a representative mechanism for field generation, we shall adopt the one proposed earlier in ref. [15,23]. In this proposal, conformal symmetry is broken by coupling the Maxwell field to charged scalars fields. The transition from inflation to reheating produces a strong amplification of the scalar fields (particle creation). This results in stochastic currents which eventually decay into the magnetic field. The main advantages of this mechanism over others proposed in the literature is that it is naturally superhorizoned and relatively simple, involving no new physics. The bulk of the field is generated early in the reheating era (the so-called preheating period) when on the other hand a quantum field theoretic treatment may be attempted. It is realistic in the sense that there are actual candidates for the scalar fields in supersymmetric versions of the standard model. We shall ignore the difficulties associated with the fact that the electromagnetic field does not exist as such during this period (we should rather follow the evolution of the several vector fields in some unified theory, a combination of which becomes the photon after electroweak symmetry breaking) as we believe our order of magnitude estimates shall be validated by this more complete treatment.

Given the present state of knowledge, a quantum field treatment can only be done within some perturbative scheme. We shall assume the model contains $N$ identical scalar fields, and thus avails ourselves of the $1/N$ expansion [24,25].
Working at leading order, we shall obtain the self-consistent evolution equations for mean fields and propagators using the closed time path, two particle irreducible effective action [24,26]. These tools have been applied to study a broad spectrum of non equilibrium problems, from heavy-ion collisions and pair production in strong electric fields [25] to the study of non equilibrium dynamics of the inflaton field during reheating [27] and to the evolution of quantum fields and production of density perturbation in inflationary dynamics [28].

We shall study the evolution of a system of $N$ charged, massive scalar fields coupled to one electromagnetic field, in an expanding universe which realizes a transition from an exponential expansion (inflationary period) to a less rapid expanding epoch (reheating period). For minimal coupling of the scalar field to gravity, we can assume that during inflation the charged field is in its invariant vacuum state [29], there being no particle creation and no electric currents. When the transition from inflation to reheating takes place, the vacuum state changes and the state of the field turns into a multi-particle state. From then on, we have an out of equilibrium system of charged and electromagnetic fields, that evolves in an expanding background, and it is this evolution that we shall study.

The reheating epoch of the universe is very difficult to address, both from the conceptual and from the technical point of view. Very little is understood about the process of decay of the inflaton into the other fields to which it is coupled, and the consequent establishment of the primordial plasma [30]. A complete description of the evolution of our system of scalar particles and electromagnetic fields should take into account their possible couplings to the other fields present. To attempt to take into account all these effects in a consistent way, makes the problem conceptually and technically unsolvable. However we can try and incorporate them in a phenomenological way by considering a “thermal mass” for the scalar particles, besides their bare one. On the other hand, any other coupling to the forming plasma would amount to consider an external electric conductivity, and this case was already considered in the literature [15,21,23].

We must observe that to preserve the very weak coupling of the inflaton during inflation [31], it may be presumed that the inflaton is neutral and uncoupled to other charged species. Therefore the plasma of charged particles would not result from the decay of the inflaton, but from some indirect mechanism (e. g., from gravitational particle creation [32]). As a result, the primordial charged plasma contributing to the thermal mass of our scalar fields, need not be in equilibrium with the neutral particles, including the inflaton, which contribute most to the energy density of the Universe during reheating, and it will take longer to form. As we shall assume that the primordial plasma is not established immediately after the end of inflation, we shall consider no external electrical conductivity in the period of time during which we follow the evolution of our system of charges and magnetic field: the damping of the fields and currents would be due to their own interaction.

The paper is organized as follows: In the next section we present the general tools that we shall use in the magnetic field evaluation: we evaluate in conformal time the scale factors of the Universe during inflation and reheating; we write down and solve the Klein Gordon equation for the scalar field modes in each epoch, and calculate the Bogolyubov coefficients. As it is not possible find a closed analytical solution to the scalar field equation during the reheating epoch, we split the momentum interval in two parts: an infrared and an ultraviolet one and find the solutions for each of them. The resulting Bogolyubov coefficients show that the bulk of the particles are created in the infrared portion. In section III we write down the Schwinger Dyson equation for the Hadamard two point function of the electromagnetic field four potential, $D_{\mu \nu}(x, x') = \langle \{ A_\mu (x), A_\nu (x) \} \rangle$ which will give us the information we seek about the induced magnetic field. This is in fact a set of coupled equations for the different components of the two point function. They posses two kernels: a local one, and a non local one that can be associated with dissipation. The main contribution to these kernels comes from the infrared sector. In this situation, the equations for the transverse part of the pure spatial components of the Hadamard function, i.e. the ones whose curl gives the magnetic field, decouple. When computing the kernels, the dissipative one turns out to be several orders of magnitude smaller than the local one and hence can be ignored. The remaining local kernel will be responsible for screening the field like in superconducting media, i.e. Meissner effect.

In section IV we translate our equations to a stochastic formulation, which allows a better understanding of the field evolution. We write down an equation for the magnetic field itself which resembles a Mathieu equation. We find a numerical solution for a given set of physical parameters of the problem. Finally in section V we shortly review and discuss our results.

In Appendix 1 we write down the main ingredients for our study, the Lagrangian for scalar electrodynamics in curved space time, we build the 2PI closed time path effective action and from it derive the set of Schwinger-Dyson equations for the scalar and electromagnetic two point functions. In taking the large $N$ limit we find that the equation for the scalar field propagator decouples from the one for the electromagnetic field, and hence there is no backreaction on the electric current. All the information about the dissipative properties of the system is encoded in the kernels of the equations for the electromagnetic two point functions. These equations are a set of coupled equations for the different components of the electromagnetic two point function.

In Appendix 2 we compute the kernels that appear in the Schwinger - Dyson equations for the ultraviolet sector.
of the mode spectrum and for the infrared one, and conclude that the latter dominates over the former. We work in natural units, in which \( c = \hbar = 1 \), and with signature \((-++,+,+)\).

II. SCENARIO FOR MAGNETOGENESIS

Assume that there exists a charged massive scalar field in the very early universe. Consider also de Sitter inflation, and that during that period of the universe the scalar field is in its invariant vacuum state and hence there is no particle creation [29]. When the transition from inflation to reheating takes place, due to the change in the geometry of the universe the vacuum state becomes a multiparticle state [33], which means that a stochastic electric current unfolds which induces a magnetic field.

As stated in the introduction, the reheating period is very difficult to address. It is widely accepted that the general picture is the decaying of the inflaton field into other matter fields through non linear oscillations. This is an out of equilibrium process, during which the plasma that will determine the evolution of the universe at the subsequent epochs is being formed. It is usually assumed that it ends when (conformal) thermal equilibrium is achieved, which in turn determines the beginning of the radiation dominated epoch. Very little is understood about the formation of the plasma, when the charged matter arises and how the temperature evolves [30].

Our created particles will dive through a sea of forming matter fields and it would be very naive to neglect a possible interaction between these two systems, no matter how little we know about them. We shall consider that this possible interaction produces a shift in the value of the scalar mass, that can be much larger than its bare value. Due to the uncertainties about the very process of particle creation (e.g. the change in the geometry is not instantaneous and hence neither is particle creation) and about the onset of reheating, we shall take into account this shift when calculating particle creation. When studying the propagation of the induced magnetic field, we shall not consider a possible conductivity of the forming plasma. On one side this is a reasonable assumption for the early stages of reheating, according to the comments made above. On other side this assumption will allow us investigate the damping in the magnetic field due to its own sources.

The scenario we are considering is then as follows: we evaluate particle creation by matching the modes of the scalar field and their first time derivatives at the instant of transition between inflation and reheating. We assume that this instant is \( \tau = 0 \) and that during inflation the universe expands exponentially. In the early stages of reheating, the dominant form of matter is still the oscillating inflaton field, and thus the universe expands like under matter dominance [34,31]. There being no compelling argument to the contrary, we consider minimal coupling of the matter field to the geometry. To model the generation of magnetic fields during this period, we shall follow the interaction between these two systems, no matter how little we know about them. We shall consider that this possible interaction produces a shift in the value of the scalar mass, that can be much larger than its bare value. Due to the uncertainties about the very process of particle creation (e.g. the change in the geometry is not instantaneous and hence neither is particle creation) and about the onset of reheating, we shall take into account this shift when calculating particle creation. When studying the propagation of the induced magnetic field, we shall not consider a possible conductivity of the forming plasma. On one side this is a reasonable assumption for the early stages of reheating, according to the comments made above. On other side this assumption will allow us investigate the damping in the magnetic field due to its own sources. The scenario we are considering is then as follows: we evaluate particle creation by matching the modes of the scalar field and their first time derivatives at the instant of transition between inflation and reheating. We assume that this instant is \( \tau = 0 \) and that during inflation the universe expands exponentially. In the early stages of reheating, the dominant form of matter is still the oscillating inflaton field, and thus the universe expands like under matter dominance [34,31]. There being no compelling argument to the contrary, we consider minimal coupling of the matter field to the geometry. To model the generation of magnetic fields during this period, we shall follow the interaction of the Maxwell field with \( N \) identical charged scalar fields, minimally coupled to the Maxwell field. To leading order in the \( 1/N \) expansion, these fields obey the free Klein-Gordon equation (see Appendix 1), and may be decomposed into modes in the usual way. We shall assume geometry is described by a spatially flat Friedmann-Robertson-Walker model.

A. Scale Factors of the Universe

To give a definite form to our model, we must find the scale factors of the Universe for the inflationary and the reheating periods in conformal time, which is defined as \( d\eta = dt/a(t) \), \( t \) being the physical or cosmological time. For the inflationary epoch, the scale factor of the Universe reads \( a_I(t) = a_0 \exp (Ht) \), so the conformal time is \( \eta - \eta_0 = - \exp (-Ht)/Ha_0 \) and in this variable the scale factor reads \( a_I(\eta) = H^{-1} (\eta_0 - \eta)^{-1} \).

Assuming that during reheating the universe expands as if it were matter dominated [31], in cosmological time the scale factor is \( a_R(t) = a_I(t + t_1)^{2/3} \). Therefore the conformal time reads \( \eta - \eta_1 = 3 (t + t_1)^{1/3}/a_1 \) and the corresponding scale factor \( a_R(\eta) = a_1^3 (\eta - \eta_1)^2/9 \).

In order to avoid overproduction of created particles, when matching the scale factors we must demand that at the time of transition, the scale factor changes smoothly. The minimum conditions to be fulfilled are that the functions and their first derivatives be continuous at the transition time, then obtaining \( a_I(\eta) = (1 - H\eta)^{-1} \) and \( a_R(\eta) = (1 + H\eta/2)^2 \). Defining the dimensionless time \( \tau = H\eta \) the corresponding functions read \( a_I(\tau) = (1 - \tau)^{-1} \) and \( a_R(\tau) = (1 + \tau/2)^2 \).
In this section we solve the Klein Gordon equation for the scalar field, in the two eras of the Universe involved in our study and calculate the Bogolyubov coefficients. From now on we will work with dimensionless variables \( \kappa = H k \), \( m \rightarrow m / H \), together with the dimensionless time variable \( \tau \) defined above.

Assuming minimal coupling to curvature the Klein Gordon equation for the modes of the scalar field reads

\[
\frac{d^2}{d\tau^2} + k^2 + \frac{m^2(\tau)}{H^2} a^2(\tau) - \frac{\ddot{a}(\tau)}{a(\tau)} f_k(\tau) = 0
\] (1)

In spite of our ignorance about the reheating epoch and the actual interactions of our scalar field during it, it would be very naive to neglect them, so as stated in the Introduction, we shall take them into account by considering a thermal mass given by the phenomenological expression

\[
m^2(\tau) = g T^2(\tau)
\] (2)

with \( g \) being a coupling constant which we consider of order one. We assume a generic form for the evolution of the thermal mass during reheating, i.e. one of the form

\[
T(\tau) = \frac{a^b(0)}{a^b(\tau)} T_M
\] (3)

where \( b \) is a parameter that satisfies \( 0 \leq b < 1 \), and \( T_M \) is the maximum value of the shift, attained at \( \tau \approx 0 \). This form of the mass shift is inspired in the process of preheating, where a peak temperature is attained which subsequently decreases as reheating unfolds. Setting \( b = 0, g = 1 \) implies \( T_M = m_0 \), where \( m_0 \) is the bare mass of the scalar field.

For any other values of \( b > 0 \) it must be \( T_M > m_0 \).

The Bogolyubov coefficients are given by the decomposition

\[
f_k(\tau) = \alpha_k f_R^k(\tau) + \beta_k f_R^* k(\tau)
\] (4)

with \( f_R^k(\tau) \) the modes of the scalar field during reheating, and can be obtained by demanding continuity of the mode functions and their derivatives at the time of transition.

1. Inflation

During this period, as the temperature of the Universe is practically zero, the modes satisfy the bare mass equation, i.e.

\[
\left[ \frac{d^2}{d\tau^2} + k^2 - \frac{2 - m_0^2/H^2}{(1 - \tau)^2} \right] f_k(\tau) = 0
\] (5)

Writing \( f_k(\tau) = (1 - \tau)^{1/2} h_k(\tau) \) we obtain a Bessel equation for \( h_k(\tau) \):

\[
\frac{d^2}{d\tau^2} \left( \frac{1}{(1 - \tau)} \right) \frac{d}{d\tau} + k^2 - \frac{9/4 - m_0^2/H^2}{(1 - \tau)^2} h_k(\tau) = 0
\] (6)

whose positive frequency solutions are the Hankel functions of the first kind, \( H^{(1)}{\nu}[k(1 - \tau)] \) with \( \nu = (3/2) \sqrt{1 - 4m_0^2/9H^2} \sim 3/2 \); the last relation stems from the fact that during inflation \( m_0^2/H^2 \ll 1 \). We then have that the normalized, positive frequency modes during inflation read

\[
f_k^I(\tau) \approx \frac{\sqrt{\pi}}{2} (1 - \tau)^{1/2} H^{(1)}{3/2}[k(1 - \tau)] = -\frac{e^{ik(1-\tau)}}{\sqrt{2k}} \left[ 1 + \frac{i}{k(1 - \tau)} \right]
\] (7)

The last equality follows from the exact polynomial expression for \( H^{(1)}{3/2}[k(1 - \tau)] \) [35].
For this epoch the Klein Gordon equation reads
\[ \left[ \frac{d^2}{d\tau^2} + k^2 + \frac{T_M^2}{H^2} \left( 1 + \frac{\tau}{2} \right)^4 e^{-4b} - \frac{1}{2 (1 + \tau/2)^2} \right] f_k(\tau) = 0 \tag{8} \]

In this case it has no closed analytic solution, so we shall solve it in two limits, namely the large wavenumber limit, for which \( k \gg \Delta \equiv g^{1/2} T_M / H \), and the small wavenumber limit, for which \( k \ll \Delta \). Care must be taken when doing this, because in eq. (8) the factor \( g^{1/2} T_M / H \) is multiplied by a growing functions of time. This means that in principle this splitting is time dependent. However this will prove to be unimportant at the end of the calculations.

\textbf{a. Large wavenumber limit} The Klein Gordon equation in this limit reads
\[ \left[ \frac{\partial^2}{\partial \tau^2} + k^2 - \frac{1}{2 (1 + \tau/2)^2} \right] f_{R(l)k}(\tau) = 0 \tag{9} \]
proposing again \( f_{R(l)k}(\tau) = (1 + \tau/2)^{1/2} h_{R(l)k}(\tau) \) we obtain a Bessel equation for \( h_k(\tau) \):
\[ h_{R(l)k}(\tau) + \frac{1}{(1 + \tau/2)} h_{R(l)k}(\tau) + \left[ 4k^2 - \frac{9/4}{(1 + \tau/2)^2} \right] h_{R(l)k}(\tau) = 0 \tag{10} \]
whose positive frequency solutions are the Hankel functions of the second kind \( H_{\frac{3}{2}}^{(2)} [2k (1 + \tau/2)] \). The normalized mode functions read
\[ f_{R(l)k}(\tau) = \sqrt{\frac{\pi}{2} \left( 1 + \frac{\tau}{2} \right)^{1/2}} H_{\frac{3}{2}}^{(2)} \left[ 2k \left( 1 + \frac{\tau}{2} \right) \right] \]
\[ = \frac{\sqrt{\pi}}{\sqrt{2k}} \left( 1 + \frac{\tau}{2} \right)^{1/2} e^{-2k (1 + \tau/2)} \left( 1 - \frac{i}{2k (1 + \tau/2)} \right) \tag{11} \]
where in the last line we have used the polynomial expression of the Hankel function.

Replacing the corresponding modes and their derivatives in the general expression for the Bogolyubov coefficients we get
\[ \alpha_{(l)k} = e^{ik} \left[ 1 + \frac{3i}{2k} - \frac{9}{8k^2} - \frac{3i}{8k^3} \right] \tag{12} \]
\[ \beta_{(l)k} = e^{-ik} \left[ \frac{3}{8k^2} - \frac{3i}{8k^3} \right] \tag{13} \]

\textbf{b. Small wavenumber limit} For small wavenumbers, i.e. those for which \( k \ll g^{1/2} T_M / H \), the equation reads
\[ \left[ \frac{d^2}{d\tau^2} + g \frac{T_M^2}{H^2} \left( 1 + \frac{\tau}{2} \right)^{4-4b} - \frac{1/2}{(1 + \tau/2)^2} \right] f_{(s)k}(\tau) = 0 \tag{14} \]
Assuming as time variable \( u = (1 + \tau/2) \) and replacing \( f_{(s)k}^{R}(z) = u^{1/2} h(u) \) we get
\[ \ddot{h}_{(s)k}(u) + \frac{1}{2} \dot{h}_{(s)k}(u) + \left[ 4g \frac{T_M^2}{H^2} u e^{-2/\gamma} - \frac{9/4}{u^2} \right] h_{(s)k}(u) = 0 \tag{15} \]
with \( c = 4 - 4b \). Defining \( x = u^\gamma \) the equation now reads
\[ \frac{d^2}{dx^2} h_{(s)k}(x) + \frac{1}{x} \frac{d}{dx} h_{(s)k}(x) + \left[ 4g \frac{T_M^2}{H^2} x^{c/\gamma - 2 + 2/\gamma} - \frac{9}{4x^2} \right] h_{(s)k}(x) = 0 \tag{16} \]
which can be cast in the form of a Bessel equation by demanding \( c/\gamma + 2/\gamma - 2 = 0 \). We get \( \gamma = (3 - 2b) \) and the equation reads
\[
\frac{d^2}{dx^2} h_{(s)k}(x) + \frac{1}{x} \frac{d}{dx} h_{(s)k}(x) + \left[4g \frac{T^2_m}{H^2} \frac{1}{\gamma^2} - \frac{9}{4} \frac{1}{\gamma^2 x^2} \right] h_{(s)k}(x) = 0
\]  

(17)

whose positive frequency solutions are again Hankel functions of the second kind, \(H_{3/2\gamma}^{(2)} \left[ (2g^{1/2}T_m/H) u \right] \). The normalized solutions to the field equation are then

\[
f(\tau) = \sqrt{\frac{\pi}{2\gamma z_0^{1/2}}} H_{3/2\gamma}^{(2)} [z(\tau)]
\]

(18)

with

\[
z(\tau) = z_0 \left(1 + \frac{\tau}{2}\right)^{\gamma}, \quad z_0 = \frac{2g^{1/2} T_m}{\gamma H}
\]

(19)

Replacing in the general expressions we get by a straightforward calculation

\[
\alpha_k = -i \sqrt{\frac{\pi}{2\gamma}} \frac{e^{ik}}{\sqrt{2k}} \left\{ \left[ 1 + \frac{i}{k} \right] g^{1/2} \frac{T_m}{H} H_{3/2\gamma+1}^{(1)} [z_0] - ik H_{3/2\gamma}^{(1)} [z_0] \right\}
\]

(20)

and in the limit \(z_0 \ll 1\), which is always valid, they reduce to

\[
\alpha_k = \beta_k \sim -i \sqrt{\frac{\pi}{2\gamma}} g^{3/2\gamma-1} \Gamma \left(\frac{2\gamma + 3}{2\gamma}\right) z_0^{-3/2\gamma} \frac{e^{ik}}{k^{3/2}}
\]

(22)

Observe that the apparent breakdown of these formulae at \(\gamma = 0\) (due to the inadequacy of the Bessel function representation of the solution) lies outside the physical range.

### III. DYNAMICS OF THE ELECTROMAGNETIC TWO POINT FUNCTIONS

Cosmological particle creation is a stochastic process. If the particles are charged, electromagnetic fields are induced by the created currents. But their mean values being zero, they manifest through their variances or two point functions. The functions we need to obtain information about the evolution and the state of the field are respectively the retard \(D_{\nu\gamma}^{\text{ret}}(x, x')\) and the Hadamard \(D_{1\nu\gamma}(x, x')\) two point functions, defined as

\[
D_{\nu\gamma}^{\text{ret}}(x, x') = i \langle [A_\nu(x), A_\gamma(x')] \rangle \Theta(\tau - \tau')
\]

(23)

\[
D_{1\nu\gamma}(x, x') = \langle [A_\nu(x), A_\gamma(x')] \rangle
\]

(24)

The evolution equations for this propagators, known as Schwinger-Dyson equations (see Appendix 1), are

\[
\left[ \eta^{\mu\nu} \square_x + \left(1 - \frac{1}{\zeta} \right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu}_{11}(x, x) \right] D_{\nu\gamma}^{\text{ret}}(x, x')
\]

(25)

\[+ie^2 \int dx'' \Sigma_{\nu\gamma}^{\mu\nu}(x, x'') D_{\nu\gamma}^{\text{ret}}(x'', x') = -\delta_\gamma^a \delta(x - x')
\]

\[
\left[ \eta^{\mu\nu} \square_x + \left(1 - \frac{1}{\zeta} \right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu}_{11}(x, x) \right] D_{1\nu\gamma}(x, x')
\]

(26)

\[+ie^2 \int dx'' \Sigma_{\nu\gamma}^{\mu\nu}(x, x'') D_{1\nu\gamma}(x'', x') = -\frac{e^2}{2} \int dx'' \Sigma_1^{\mu\nu}(x, x'') D_{\nu\gamma}^{\text{adv}}(x'', x')
\]

where \(\zeta\) is a gauge fixing constant, \(\square_x = -\partial_x^2 + \nabla^2\), and
\[
\Gamma^{\mu\nu}_{cd}(x,x') = \eta^{\mu\nu} \left[ G^{1}_{cd}(x,x) + G^{2}_{cd}(x,x) \right]
\]  

(27)

\[
\Sigma^{cc',dd'}_{x,x''} = \eta^{\mu\alpha}(x)\eta^{\nu\beta}(x'') \left[ G^{1}_{cc'}(x,x'') \partial_\alpha \partial_\beta G^{2}_{dd'}(x,x') \right]
\]  

(28)

c and \(d\) are closed time path indices whose values are \(1\) for the forward directed time path and \(2\) for the backward directed time path and the supraindex \(i = 1, 2\) denotes the real and imaginary parts of the complex scalar field. According to the values of the closed time path indices we have that \(G^{1}_{21}(x,x') = \langle \phi(x) \phi(x') \rangle\) is the positive frequency two point function for the scalar field and \(G^{2}_{12}(x,x') = \langle \phi(x') \phi(x) \rangle\) the negative frequency one. With these basic propagators we can build the antisymmetric two point function, also known as Jordan propagator, \(G(x,x') = G^{1}_{21}(x,x') - G^{2}_{12}(x,x')\) and the symmetric, or Hadamard one, \(G_{11}(x,x') = G^{1}_{21}(x,x') + G^{2}_{12}(x,x')\). \(G_{11}(x,x') \equiv \langle T(\phi(x) \phi(x')) \rangle\) is the Feynman propagator and \(G_{22}(x,x') \equiv \langle T(\phi(x') \phi(x)) \rangle\) the Dyson one. The retarded and advanced two point functions are defined as \(G_{ret}(x,x') = iG(x,x') = G_{adv}(x',x)\), or else \(G_{ret}(x,x') = i[G_{11}(x,x') - G_{12}(x,x')]\), \(G_{adv}(x,x') = i[G_{22}(x,x') - G_{12}(x,x')]\). The same definitions apply for the two point functions for the electromagnetic field, i.e. \(D^{\mu\nu}_{ret}(x,x') \equiv \langle A^\mu(x) A^\nu(x') \rangle\) is the positive frequency two point function, \(D^{\mu\nu}_{adv}(x,x') \equiv \langle A^\mu(x') A^\nu(x) \rangle\) the negative frequency two point function, and so on. Based on these definitions, we have built the non local kernels in eqs. (25) and (26), as \(\Sigma^{\mu
u}_{ret}(x,x'') = \Sigma^{\mu
u}_{11,11}(x,x'') - \Sigma^{\mu
u}_{12,12}(x,x'') \equiv \Sigma^{\mu
u}_{21,21}(x,x'') - \Sigma^{\mu
u}_{22,22}(x,x'')\) and \(\Sigma^{\mu
u}_{1}(x,x'') \equiv \Sigma^{\mu
u}_{12,12}(x,x'') + \Sigma^{\mu
u}_{21,21}(x,x'')\). Each of equations (25) and (26) is in fact a set of coupled differential equations for the different components of the retarded and Hadamard two point functions of the electromagnetic four potential. Equation (26) is the one that we shall use to evaluate the magnetic field.

A. Equation for the spatial components of the electromagnetic two point functions.

The equations for the electromagnetic two point functions form a set, where the equations for the pure spatial and pure temporal propagators are coupled by the mixed functions, i.e. by two point functions with one temporal and one spatial component. This coupling is realized through the non local kernels, specifically by its \(0-i\) components, which in general do not vanish.

When we take into account particle creation, we see that local and non local kernels split into a vacuum and a particle contribution, i.e. \(\Gamma^{\mu\nu}_{11}(x,x) = \Gamma^{\mu\nu(V)}_{11}(x,x) + \Gamma^{\mu\nu(P)}_{11}(x,x)\) and \(\Sigma^{\mu
u}_{ret(1)}(x,x'') = \Sigma^{\mu
u(V)}_{ret(1)}(x,x'') + \Sigma^{\mu
u(P)}_{ret(1)}(x,x'')\). We are interested in the contribution of the created particles and hence shall solve

\[
\left[ \eta^{\mu\nu} \Box_x + \left( 1 - \frac{1}{\zeta} \right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu(P)}_{11}(x,x) \right] D_{1\nu\gamma}(x,x') = \frac{e^2}{2} \int \! dx'' \Sigma^{\mu\nu(P)}_{ret(1)}(x,x'') D_{1\nu\gamma}(x'',x')
\]  

(29)

For our purposes of magnetic field evaluation, we shall need only the spatial components of the propagator \(D_{1ij}(x,x')\), i.e. the solution to the \(i-j\) equations, and in particular its transverse parts. Since the bulk of particle creation occurs for long wavelengths, the main contribution to the kernels comes from that sector of the spectrum (see Appendix 2). The \(0-i\) components of the non local kernels take the form of a gradient and in that limit the mode functions depend on the momenta only through their modulus. When transforming Fourier the spatial part of the two point functions, there remains an integral over the momenta that vanishes for the transverse part of the \(0-i\) component. This means that the equations for the transverse part of \(D_{1ij}(x,x')\) decouple from the set, a fact that facilitates enormously their resolution. The Fourier transformed equations that we shall work with then read (see eq. (26) and Appendix 1)

\[
\left[ \eta^{\mu} (\partial^2_\tau + k^2) + e^2 \int \! \frac{d^3p}{(2\pi)^{3/2}} \Gamma^{i(P)}_{11}(p,\tau,\tau) \right] D_{1ij}(k,\tau,\tau') = \frac{e^2}{2} \int \! d\tau'' \Sigma^{i(P)}_{ret}(p, k - p, \tau, \tau'') D_{1ij}^{adv}(k, \tau', \tau'')
\]  

\[
- \frac{e^2}{2} \int \! d\tau'' \Sigma^{i(P)}_{1}(p, k - p, \tau, \tau'') D_{1ij}^{adv}(k, \tau', \tau'')
\]  

(30)
where \( k \) is the comoving wavenumber of the spatially Fourier transformed two point function of the electromagnetic field, and \( p \) the corresponding one of the scalar field. For example when replacing the mode decomposition (4) the local kernel reads

\[
\Gamma_{11}^{il}(p, \tau, \tau) = \eta^{il} \left\{ 2 |\beta_p|^2 |f_p(\tau)|^2 + \alpha_p/\beta_p f_p^2(\tau) + \beta_p/\alpha_p f_p^{12}(\tau) \right\}
\]  

(31)

with \( f_p(\tau) \) given by eq. (18) or (11) and the Bogolyubov coefficients by (22), and similarly the non local kernels. We assume that the vacuum part may be absorbed into a renormalization of the classical action, the remainder being negligible.

**B. Computing the kernels**

In this section we shall compute the kernels found in the equations (30) for the Maxwell field propagators. It may be checked that the contribution from the large wavenumber sector is negligible (see Appendix 2) in comparison to the infrared one, a fact that can be understood by looking at the expressions for the Bogolyubov coefficients, eqs. (12), (13) and (22). The mode functions are given by eq. (18) and we see that they do not depend on the wavenumbers \( p \). The Bogolyubov coefficient on the other side retain their full momentum dependence.

When we replace for the modes and coefficients, eqs. (18) and (22), we find a logarithmic divergence in the momentum integral of the local kernel eq. (31). We introduce an infrared cut-off, which we can choose as the comoving, dimensionless mode \( \bar{\Upsilon} \) corresponding to the original inflationary patch that gave rise to our universe. For the upper limit we take \( \Delta \sim T_M/H \). We then have

\[
\Gamma_{11}^{il}(\tau, \tau) = \int_{\bar{\Upsilon}}^{\Delta} \frac{d^3p}{(2\pi)^{3/2}} \Gamma_{11}^{il}(p, \tau, \tau) \simeq \eta^{il} \bar{\Gamma},
\]  

(32)

\[
\bar{\Gamma} = 2^{3/\gamma-1} \Gamma^2 \left( \frac{2\gamma + 3}{2\gamma} \right) \ln \left( \frac{\Delta}{\bar{\Upsilon}} \right) z_0^{-4/\gamma} F^2(z)
\]  

(33)

where

\[
F(z) = z^{1/2\gamma} J_{3/2\gamma}(z)
\]  

(34)

\( J_{3/2\gamma}(z) \) being a Bessel function of the first kind on \( z \) and \( z_0 \) being defined above in eq. (19).

The non local noise kernel has no infrared divergencies and is \( \Delta^2 \) times smaller than the local one, with \( \Delta \sim T_M/H \ll 1 \) (see Appendix 2). This difference in the orders of magnitude will never decrease, (even if we evaluate the remaining time integral in the non local part of the equation) and therefore for our purposes we can ignore the contribution of this kernel.

This is a very important approximation, not only from the technical point of view but also because it shows that dissipation due to the ordinary conductivity of the own charges is negligible.

We now turn to the Hadamard noise kernel \( \Sigma_{11}^{ij(N)}(p, k - p, \tau, \tau') \) that is the source in the equation (30) for \( D_{11}^{ij}(k, \tau, \tau') \). The same considerations made for the retarded kernel about the gradient structure of the \( 0 - i \) component of the two point function hold for this one. So we shall only need to evaluate the pure spatial components, given by

\[
\Sigma_{11}^{il(P)}(p, k - p, \tau, \tau) \sim \frac{p^l p^l}{p^3 |k - p|^3} S_1(\tau, \tau'),
\]  

(35)

\[
S_1(\tau, \tau') = 2^{6/\gamma-1} \Gamma^4 \left( \frac{2\gamma + 3}{2\gamma} \right) z_0^{-8/\gamma} F^2[z(\tau)] F^2[z(\tau')]
\]  

(36)

It is remarkable that the same function \( F \) appears in this kernel and in (33).

We see that after replacing the modes and Bogolyubov coefficients the resulting kernels given by eq. (33) and (36) are real functions. This happens because of exp. (22) for the Bogolyubov coefficients and the fact that the modes (18) do not depend on the wavenumbers. In other words, after replacing the Bogolyubov coefficients, the modes (18) combine to give real functions that oscillate coherently, an indication that they are indeed superhorizoned and hence frozen.
After all the considerations we have made, the equation for the transverse part of the pure spatial Hadamard two point function for the electromagnetic four potential reads

\[ \left[ \eta^l (\frac{d^2}{d\tau^2} + k^2) + e^2 \Gamma_{11}^{il(P)} (\tau, \tau) \right] D_{ij} (k, \tau, \tau') = \Xi_{ij} (k, \tau, \tau') \]  

(37)

with

\[ \Xi_{ij} (k, \tau, \tau') = \frac{e^2}{2} \int_0^{\tau'} d\tau'' \int_\Delta \frac{d^3 p}{(2\pi)^{3/2}} \Sigma_{il} (p, k - p, \tau, \tau'') D_{ij}^{ret} (k, \tau', \tau'') \]  

(38)

Let us introduce the Hadamard two point function for the magnetic field:

\[ \langle \{ B_i (\tau, \vec{r}), B_j (\tau', \vec{r'}) \} \rangle = H^4 e^{ikl} e^{ik'l'} \int \frac{d^3 k}{(2\pi)^{3/2}} k_k k_{l'} D_{ij} (k, \tau, \tau') \exp [i \vec{k} \cdot (\vec{r} - \vec{r'})] \]  

(39)

where the prime refers to the prime coordinate and \( H^4 \) gives the dimensions.

We are interested in the field \( B_i (\tau, \lambda) \) coherent over a scale \( \lambda \), so we must filter the high frequency contribution with a window function of "size" \( \lambda \), i.e. we must evaluate

\[ \frac{1}{V_\lambda} \int_{V_\lambda} d^3 r \int_{V_\lambda} d^3 r' \langle \{ B_i (\tau, \vec{r}), B_j (\tau', \vec{r'}) \} \rangle \]  

(40)

where \( V_\lambda \sim \lambda^3 \) is a comoving volume in which we seek homogeneity. Eq. (40) amounts to calculate

\[ \frac{1}{V_\lambda} \int_{V_\lambda} d^3 r \exp [i \vec{k} \vec{r}] = \frac{1}{V_\lambda} \int_{V_\lambda} d^3 r' \exp [-i \vec{k} \vec{r}'] \equiv W_\lambda (k) \]  

(41)

This window function can be approximated by

\[ W_\lambda (k) \sim 1 \quad \text{if} \quad k \leq K = 1/\lambda \]  

\[ = 0 \quad \text{otherwise} \]  

(42)

which can be implemented as a cut-off in the \( k \)-momentum integral.

For our purposes it is enough to compute the self-correlation \( \langle \{ B^i (\tau, \lambda), B_i (\tau', \lambda) \} \rangle \). From the equation for the Hadamard propagator we get

\[ \left[ \frac{d^2}{d\tau^2} + e^2 \Gamma (\tau, \tau) \right] \langle \{ B^i (\tau, \lambda), B_i (\tau', \lambda) \} \rangle = 2e^2 H^4 \int_0^K \frac{d^3 k}{(2\pi)^{3/2}} \int_\Delta \frac{d^3 p}{(2\pi)^{3/2}} \int_0^\tau d\tau'' \frac{[\vec{k} \times \vec{p}]}{p^3 |k - p|^3} S_1 (\tau'', \tau') D^{ret} (\tau', \tau'') \]  

(43)

where we used the fact that, because \( K \ll 1 \), the spatial gradients are negligible, and \( D^{ret} \) is therefore independent of the wavenumber. The momentum integral can be easily evaluated and we obtain

\[ \left[ \frac{d^2}{d\tau^2} + e^2 \Gamma (\tau, \tau) \right] \langle \{ B^i (\tau, \lambda), B_i (\tau', \lambda) \} \rangle = 2e^2 H^4 K^4 \int_0^\tau d\tau'' S_1 (\tau, \tau'') D^{ret} (\tau', \tau'') \]  

(44)

### IV. EQUIVALENT STOCHASTIC PROBLEM

Rather than solving directly equation (44), we move to a physically more transparent language and translate our problem to a stochastic formulation. We introduce a stochastic field \( B (\tau) \) obeying the Langevin equation

\[ \left[ \frac{d^2}{d\tau^2} + e^2 \Gamma (\tau, \tau) \right] B (\tau) = \xi (\tau) \]  

(45)
where $\xi (\tau)$ is Gaussian noise with zero mean and self correlation

$$\langle \xi (\tau) \xi (\tau') \rangle_{\xi} = 4e^2 H^4 K^4 S_1 (\tau, \tau')$$ (46)

It is then easy to see that

$$\langle \{ B' (\tau, \lambda), B_i (\tau', \lambda) \} \rangle = \langle B (\tau) B_i (\tau') \rangle_{\xi}$$ (47)

Moreover, because of the structure of the self-correlation, we may realize the noise as

$$\xi (\tau) = e H^2 K^2 2^{3/\gamma} T^2 \left( \frac{2\gamma + 3}{2\gamma} \right) z_0^{-4/\gamma} F^2 [z (\tau)] X$$ (48)

where $X$ is a single gaussian variable with $\langle X \rangle = 0$ and $\langle X^2 \rangle = 1$.

Since the equation is linear, we may write

$$B (\tau) = e H^2 K^2 2^{3/\gamma} T^2 \left( \frac{2\gamma + 3}{2\gamma} \right) z_0^{-4/\gamma} X B_s (\tau)$$ (49)

The equation to be solved is then

$$\left[ \frac{d^2}{d\tau^2} + C^2 F^2 [z (\tau)] \right] B_s (\tau) = F^2 [z (\tau)]$$ (50)

where

$$C^2 = e^2 z_0^{-4/\gamma}$$ (51)

where $e^{2^3/\gamma - 4T^2 [(2\gamma + 3)/2\gamma]} \ln (\Delta / \Upsilon) = e^2$ with the boundary conditions $B_s (0) = \dot{B}_s (0) = 0$.

Eq. (50) resembles the London equation in a superconducting medium [36]. This means that the induced field will be weaker than the one obtained in the case of free propagation. We may write $B_s (\tau) = C^{-2} (1 + B_h (\tau))$ and the equation for $B_h (\tau)$ reads

$$\left[ \frac{d^2}{d\tau^2} + C^2 F^2 [z (\tau)] \right] B_h (\tau) = 0$$ (52)

with initial conditions $B_h (0) = -1, \dot{B}_h (0) = 0$. It is convenient to adopt the variable $z$ as independent variable.

Equation (52) now reads

$$\left[ \frac{d^2}{dz^2} + \frac{\gamma - 1}{\gamma z} \frac{d}{dz} + c^2 g^2 (z) \right] B_h (z) = 0$$ (53)

where $c^2 = 4e^2 z_0^{-6/\gamma} / \gamma^2$ and $g^2 (z) = z^{-2(\gamma - 1)/\gamma} F^2 [z] = z^{(3 - 2\gamma)/\gamma} J^2_{3/2\gamma} (z)$. The magnetic field $B (\tau)$ will then be given by

$$B (\tau) \sim H^2 K^2 (1 + B_h (\tau))$$ (54)

Because of the oscillatory behaviour of $g^2 (z)$, equation (52) resembles a Mathieu equation and hence we expect that the attenuation of the induced magnetic field due to the London currents will not be so strong. We shall integrate it numerically and evaluate the induced field from eq. (54). In view of the uncertainties in the actual values of $\gamma$ and $T_M$ we shall not attempt to estimate the induced field for a wide interval of their possible values, but shall illustrate the effect by considering one specific choice, namely $\gamma = 5/2$ and $T_M / H = 10^{-2}$. This value of $\gamma$ corresponds to $b = 1/4$ and describes a strong deviation of the thermal mass from conformal thermal evolution. We recall that the parameter $b$ does not describe the evolution of the temperature of the reheating plasma, i.e. the one formed by the inflaton decays, but the one of the thermal shift of the scalar charges.

Considering the maximum allowed value for the Hubble constant during inflation, namely $H = 10^{13} \text{ GeV}$, $T_M / H = 10^{-2}$ corresponds to $T_M = 10^{11} \text{ GeV}$.
Eq. (53) correctly describes the evolution of $B_t$ during a time interval in which the equations obtained in the leading order of the $1/N$ approximation, eqs. (25) and (26) are valid. To estimate this interval we can compare the growth of the electromagnetic energy density with the one of the scalar field: assuming efficient conversion of scalar energy into electromagnetic one, the moment at which the latter overtakes the former can be considered as the limiting time to integrate the equations. In fact, in the leading order of the $1/N$ approximation the equation for the scalar field propagator shows no backreaction from the magnetic field (see Appendix 1). This means that to this order the induced field is very weak and hence its energy density must be smaller than the one of the scalar field. When this condition is broken, the evolution equations for the two point functions cease to be valid. The energy density is given by the 00 component of the stress energy tensor of the system of fields, $T^{00}$. To evaluate this tensor component we consider the electromagnetic field as classical and the scalar one as quantum. We can split $T^{00}$ in a pure scalar contribution and an electromagnetic part as

$$T^{00}_\varphi = H^4 \left[ \dot{\varphi}^2 - \frac{\dot{\varphi}}{a^2} \varphi + \frac{1}{2} \eta^{ij} \partial_i \varphi \partial_j \varphi + \left( \frac{m^2}{H^2} - a^2 \right) \varphi^2 \right]$$

$$T^{00}_A = H^4 \left[ \frac{1}{4} \left( \ddot{F}^{ij} F_{ij} + 2 \dot{F}^{ij} \dot{F}_{ij} \right) + \frac{1}{2} \eta^{ij} e^2 A_i A_j \left( \dot{\varphi_1}^2 + \dot{\varphi_2}^2 \right) \right]$$

where it is understood that all quantities between brackets are dimensionless. We shall compare $\langle T^{00}_\varphi \rangle$ with $\langle T^{00}_A \rangle$: the instant at which the latter equals the former will be considered as the time at which the equations obtained in the large $N$ limit cease to be valid.

### 1. Stress energy tensor for the scalar field

Considering the electromagnetic field as classical and the scalar as quantum, and defining $T^{00} = H^4 \tilde{T}^{00}$ we can write

$$\langle \tilde{T}^{00}_\varphi \rangle = \int \frac{d^3p}{(2\pi)^{3/2}} \left\{ \dot{\varphi}_p^* (\tau) \dot{\varphi}_p (\tau) - \frac{\dot{a}}{a} \left[ \varphi_p^* (\tau) \dot{\varphi}_p (\tau) + \dot{\varphi}_p^* (\tau) \varphi_p (\tau) \right] + \eta^{ij} p_i p_j \varphi_p^* (\tau) \varphi_p (\tau) + \left( a^2 \frac{m^2 (\tau)}{H^2} + \frac{\dot{\varphi}_p^* (\tau)}{a^2} \right) \frac{\varphi_p^* (\tau)}{\varphi_p (\tau)} \right\}$$

Writing the modes as $\varphi_p (\tau) \sim \alpha_p f_p (\tau) + \beta_p f_p^* (\tau)$ with the modes $f_p (\tau)$ given by eq. (18) and the Bogolyubov coefficients by eq. (22), the contribution from the created particles to $\langle \tilde{T}^{00}_\varphi \rangle$ reads

$$\langle \tilde{T}^{00}_\varphi \rangle_\beta \simeq \int_T^\Delta \frac{d^3p}{(2\pi)^{3/2}} |\beta_p|^2 \left\{ \left[ \dot{f} (\tau) + \dot{f}^* (\tau) \right]^2 - 2 \frac{\dot{a}}{a} \left[ \dot{f} (\tau) + \dot{f}^* (\tau) \right] |f (\tau) + f^* (\tau)| + \eta^{ij} p_i p_j |f (\tau) + f^* (\tau)|^2 \right\}$$

Neglecting the gradient term because it will give a negligible contribution in comparison to the other terms and performing the momentum integral, we get

$$\langle \tilde{T}^{00}_\varphi \rangle_\beta = \frac{4\pi}{(2\pi)^{3/2}} \frac{\gamma}{\pi} \frac{\gamma_0^{-3/\gamma}}{a^{\gamma-1}} \times \left\{ a^2 \frac{m^2 (\tau)}{H^2} |f (\tau) + f^* (\tau)|^2 + \left( \left[ \dot{f} (\tau) + \dot{f}^* (\tau) \right] - \frac{\dot{a}}{a} |f (\tau) + f^* (\tau)| \right)^2 \right\}$$

where we have used $a^2 m^2 (\tau)/H^2 = a^{\gamma-1} T_M H$.
Replacing the modes in terms of the Bessel functions through eq. (18), the scale factor and its derivative in terms of $z$ and $z_0$ and using $T_M/H = \gamma z_0/2$ we are left with

$$\langle \tilde{T}_{\varphi\varphi}^{00} \rangle = \frac{2\pi \gamma^2}{(2\pi)^{3/2}} \frac{z^{2-1/\gamma}}{z_0^{2/\gamma}} \left\{ J^2_{3/2\gamma} [z(\tau)] + J^2_{3/2\gamma+1} [z(\tau)] \right\}$$  \hfill (60)

For $z \gg 1$ we can replace the asymptotic expressions of the Bessel functions [35], i.e. $J_\nu [z(\tau)] \sim \sqrt{2/(\pi z)} \cos [z - \nu \pi/2 - \pi/4]$ and so obtain the final expression for $\langle \tilde{T}_{\varphi\varphi}^{00} \rangle$:

$$\langle \tilde{T}_{\varphi\varphi}^{00} \rangle \sim \frac{4\gamma^2}{(2\pi)^{3/2}} \frac{z^{1-1/\gamma}}{z_0^{2/\gamma}}$$  \hfill (61)

which is plotted in Fig. 2.

2. Stress-energy tensor of the electromagnetic field

As stated in eq. (56), the stress energy tensor for the electromagnetic field consists of the pure electromagnetic and the interaction energy density. Proceeding with eq. (56) similarly as for $T_{00}$ we are left with

$$\langle \tilde{T}_{AA}^{00} \rangle = \frac{1}{2} \left( \tilde{B}^2 + \tilde{E}^2 \right) + e^2 \frac{B^2}{K^2} \frac{z^{1/\gamma}}{z_0^{4/\gamma}} J^2_{3/2\gamma} (z)$$  \hfill (62)

Using that $\tilde{B} = B/H^2$ and

$$\tilde{E} = \frac{d}{d\tau} \frac{\tilde{B}}{K} = \frac{\gamma}{2K} \frac{z_0^{1/\gamma}}{z^{2-1/\gamma}} \frac{d\tilde{B}}{dz}$$  \hfill (63)

we get

$$\langle \tilde{T}_{AA}^{00} \rangle = \frac{1}{2} \left[ \tilde{B}^2 + \frac{\gamma^2}{4K^2} \frac{z_0^{2/\gamma}}{z^{2-1/\gamma}} \left( \frac{d\tilde{B}}{dz} \right)^2 \right] + e^2 \frac{B^2}{K^2} \frac{z^{1/\gamma}}{z_0^{4/\gamma}} J^2_{3/2\gamma} (z)$$  \hfill (64)

We solved numerically eq. (53) and with the outcomes reconstruct $B$ from eq. (54). Eq. (54) gives us the field coherent over a comoving scale $K^{-1}$ in which we might be interested. We see from that equation that the field intensity grows as $K^2$. In order to make a honest computation of $\langle \tilde{T}_{AA}^{00} \rangle$ we must take into account the highest possible magnetic field intensity produced by our mechanism, and this is produced at a scale $K_{\text{max}} \sim T_M/H$, i.e. the cut-off that we used to split the momentum spectrum. In Fig. 2 we have also plotted the outcomes of these calculations.

V. MAGNETIC FIELD EVALUATION

To estimate the intensity of the induced magnetic field, coherent over a physical scale $\kappa^{-1}$ at a given time $\tau$, we must replace

$$a(\tau) \kappa = HK$$  \hfill (65)

So, from eqs. (54) we get

$$B_{\text{phys}}(\tau) = \frac{B(\tau)}{a^2(\tau)} \sim \kappa^2 (1 + B_h(\tau))$$  \hfill (66)

With the parameters given above, i.e. $\gamma = 5/2$ and $T_M/H = 10^{-2}$, we have numerically solved eq. (53) and show the results in Fig. 1. We see that $B_h([z])$ oscillates with an amplitude that increases exponentially. This is in agreement with the assertion we made in the previous section, about the resemblance of eq. (52) with a Mathieu equation. We
have estimated numerically the Floquet exponent of the exponential envelope in the $z$ interval considered, finding $\mu = 1/8$.

![Plot of $B_h$ vs. $z$](image)

**FIG. 1.** Plot of $B_h$ vs. $z$: the magnetic field intensity grows exponentially, as is shown by the dashed line. The numerically estimated Floquet exponent is $\mu = 1/8$ and the amplitude $B_{0h} = 200$.

To calculate $\langle T^{00}_A \rangle$ we considered the exponential envelope and its derivative to estimate the magnetic and electric fields respectively. It can be seen from Fig. 2 that at $z \sim 70$ the magnetic energy density equals the one of the scalar field and hence our equations cease to be valid. At this time, $(1 + B_h(\tau)) \sim 10^9$.

![Plot of the energy density of the scalar field (solid line) and of the electromagnetic field (dashed, oscillatory curve) as a function of $z$.](image)

**FIG. 2.** Plot of the energy density of the scalar field (solid line) and of the electromagnetic field (dashed, oscillatory curve) as a function of $z$. The magnetic energy overtakes the scalar one at $z \sim 70$.

For example, if we consider the comoving galactic scale today, $\kappa \simeq 10^{-38}$ GeV and use the equivalence $1 \text{GeV}^2 \simeq 10^{20}$ Gauss, we obtain a magnetic field intensity $B_{\text{gal,phys}}(\tau_{\text{ tod}}) \simeq 10^{53}$ Gauss, a value too weak to seed the galactic dynamo.

**VI. DISCUSSION**

In this paper we have evaluated in a selfconsistent manner the induction of magnetic fields from electric currents created at the inflation-reheating transition. As the mean value of the current vanishes, this current arises due to the quantum and stochastic fluctuations around the mean.

Using techniques specially suited for studying quantum fields out of equilibrium such as the 2PI closed time path effective action together with the large $N$ approximation, we have obtained consistent evolution equations for the two point functions for the scalar and electromagnetic fields, known as Schwinger - Dyson equations.

To evaluate the field we took into account the possible couplings of the created particles to the other fields present in the reheating epoch, by using a phenomenological expression of a temperature dependent mass. Any other coupling to
the forming matter fields could be account as an external conductivity, which would give ordinary dissipation already discussed in the literature [15,21,23].

The first important result of our work is that at leading order in the 1/N expansion there is no backreaction of the induced magnetic field on its sources, and hence the charged field propagates as a free field.

When computing particle creation, we find that the largest number is created in the infrared portion of the spectrum, giving this sector the main contribution to the different terms that enter in the Schwinger Dyson equations.

The evolution equations for the magnetic field two point function show two kernels, a local and a non local one. When we evaluate these kernels with the infrared modes and Bogolyubov coefficients, we find that the local kernel dominates over the non local, dissipative one, by several orders of magnitude during all reheating. Physically this means that dissipation in our system of fields will not be due to an ordinary electric conductivity. This fact also allows the equations for the pure spatial components of the electromagnetic two point functions to decouple, a fact that facilitates enormously their resolution.

In order to estimate the induced field we translate our equations to a more familiar and physically clearer Langevin equation, which due to the presence of the local kernel, resembles the London equation for a superconducting medium. This means that there exists attenuation produced by the same created particles, which is not due to ordinary conductivity but to screening due to the fact that the created particles behave as a superconducting medium. This is the second important result of our paper and could be obtained because we work with the full quantum field theoretical features of our system.

To visualize the process of field induction, we have solved numerically eq. (53) for $\gamma = 5/2$ and $T_M/H = 10^{-2}$ and found that, due to the fact that the London current oscillates, the damping is not perfect. The magnetic field grows exponentially, a fact that can be understood in view of the similarity of eq. (52) with a Mathieu equation. The growth rate is given by the Floquet index, which in the case analyzed is $\mu \simeq 1/8$. In view of our ignorance about the actual couplings of our scalar field and specially about the process of reheating, we did not pursue a thorough analysis of the possible parameter space in order to give bounds to the final field intensity.

In general we can estimate the upper limit of integration of eq. (53) as the one at which the energy density in the electromagnetic field overtakes the one in the scalar field. This estimation is based on the fact that to leading order in the 1/N expansion, there is no backreaction of the electromagnetic fields on their sources. When this condition is violated, i.e. when the electromagnetic field becomes so intense that its energy density equals the one of its sources, the approximation breaks down and the equations cease to be valid.

For the set of parameters used in this paper to illustrate the field induction, this time interval is rather short and the resulting field intensity too weak to have an astrophysical effect.

Due to the different powers of the momenta, the kinetic term in eq. (64) for the 00 component of the electromagnetic stress-energy tensor is negligible with respect to the interaction term, i.e. $c^2 A^2 \langle \varphi^2 \rangle$. The growth of this term is given by the growth in the magnetic field, which in turn is determined by the Floquet index of the exponential. Therefore the smaller this index the later the magnetic energy will catch up with the one of the scalar field, which means that the final field intensity will be higher. Given our present ignorance about the actual values of the physical parameters that determine that index, a very small Floquet index and hence a stronger field cannot be ruled out.

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VII. APPENDIX 1: SELF CONSISTENT SCALAR ELECTRODYNAMICS: CLOSED TIME PATH TWO PARTICLE IRREDUCIBLE EFFECTIVE ACTION AND 1/N APPROXIMATION

In this appendix we derive the selfconsistent, causal equations for the scalar electrodynamics, in the large N approximation, using the closed time path formulation of quantum field theory in a Friedmann Robertson Walker universe [26,27]. This formulation helps to correctly identify the Feynmann graphs that contribute to each order in the 1/N expansion of the equations.

The starting point for its derivation is the action for a set of N charged scalar field coupled to the electromagnetic field, which in curved spacetime reads

$$S_F \left( \phi_i, \phi_i^\dagger, A_\mu \right) = - \int d^4 x \sqrt{-g} \left\{ g^{\mu\nu} \left( \partial_\mu - ie A_\mu \right) \phi_i \left( \partial_\nu + ie A_\nu \right) \phi_i^\dagger + \left( m^2 + \chi R^2 \right) \phi_i \phi_i^\dagger + \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} - \frac{1}{2\xi} g^{\mu\nu} g^{\alpha\beta} \partial_\mu A_\alpha \partial_\nu A_\beta \right\} \quad (67)$$
To implement the large $N$ limit, where the plus (minus) sign corresponds to forward (backward) time direction, reads

\[ \phi_n^i \rightarrow \sqrt{N} \phi_n^i, \quad A_\mu \rightarrow \sqrt{N} A_\mu, \quad e \rightarrow \frac{e}{\sqrt{N}} \]

obtaining

\[ S_F^N = -N \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_1^i \partial_\nu \phi_2^j + \frac{1}{2} \left( m^2 + \chi R^2 \right) \left( \phi_1^i + \phi_2^j \right) \right\} + g^{\mu\nu} e A_\mu \left( \partial_\nu \phi_1^i \phi_2^j - \phi_1^i \partial_\nu \phi_2^j \right) + \frac{e^2}{2} g^{\mu\nu} A_\mu A_\nu \left( \phi_1^i + \phi_2^j \right) + \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \partial_\mu A_\alpha \partial_\nu A_\beta \right\} \]

which means that each vertex appears $N$ times in the interaction.

### A. Closed time path generating functional for $N$ scalar fields in curved spacetime

The closed time path generating function for the n-point functions is obtained by coupling the scalar and electromagnetic fields to sources $J_a(\mathbf{x})$ in each of the time paths

\[ Z[J, K] = Z_G \prod_{n=1}^{N} DA_\pm^a D\phi_1^a D\phi_2^a \exp \left\{ iS_F^N \left[ \phi_1^a, \phi_2^a, A_\pm^a \right] \right\} + i \int d^4x \sqrt{-g} C^{ab} \left[ J_1^a(x) \phi_1^a(x) + J_2^a(x) A_\mu^a(x) \right] + i \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g} C^{ab} C^{cd} \times \left[ K^{ijnm}(x, x') \phi_1^i(x) \phi_1^j(x') + K^{ijmu}(x, x') A_\mu(x) A_\nu(x') \right] + i \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g} C^{ab} C^{cd} \times \left[ K^{iin\mu}(x, x') \left[ \phi_1^i(x) A_\mu(x') + \phi_2^i(x) A_\mu(x') \right] \right] \equiv \exp \left\{ iW(J, K) \right\} \]

with

\[ S_F^N \left[ \phi_1^a, \phi_2^a, A_\pm^a \right] = S_F^N \left[ \phi_1^a, \phi_2^a, A_+^a \right] - S_F^N \left[ \phi_1^a, \phi_2^a, A_-^a \right] \]

where the plus (minus) sign corresponds to forward (backward) time direction, $n, m = 1 \ldots N$. and $i, j = 1$ or $2$ and $Z_G$ is an overall constant factor which arisen from the integration of the ghost fields included in the action to ensure that the result is gauge invariant. As we are dealing with Abelian gauge fields, that integral decouples from the others. From now on we omit $Z_G$ in the forthcoming calculations. $C^{ab}$ is the closed time path metric which reads

\[ C_{11} = 1 \]
\[ C_{22} = -1 \]
\[ C_{12} = C_{21} = 0 \]

The generating function for connected n-point functions is given by

\[ W = -i \ln Z[J, K] \]

From it we define the mean values
and two point functions

\[ \hat{\phi}_{ia} (x) \hat{\phi}^m_{jb} (x') + G_{ijab} (x, x') = 2C^a r_C^{b s} \frac{\delta W}{\delta K_{ir}^{j m} (x, x')} \] (76)

\[ \hat{A}_{\mu a} (x) \hat{A}_{\nu b} (x') + D_{\mu \nu a b} (x, x') = 2C^a r_C^{b s} \frac{\delta W}{\delta K_{ir}^{j m} (x, x')} \] (77)

\[ \frac{1}{2} \left[ \hat{\phi}_{ia} (x) \hat{A}_{\mu b} (x') + \hat{\phi}_{ia} (x') \hat{A}_{\mu b} (x) \right] + H_{i a b} (x, x') = 2C^a r_C^{b s} \frac{\delta W}{\delta K_{ir}^{j m} (x, x')} \] (78)

B. 2PI Effective Action

The closed time path 2PI effective action is defined as

\[ \Gamma = W - C^{a b} \int d^4 x \sqrt{-g} \hat{\phi}_{i a}^{m n} (x) J_{b}^{i n} (x) - C^{a b} \int d^4 x \sqrt{-g} \hat{A}_{\mu a} (x) J_{b}^{i n} (x) \]

\[ - \frac{1}{2} C^a r_C^{b s} \int d^4 x \sqrt{-g} \int d^4 x' \sqrt{-g'} \times K_{i a b}^{j m n} (x, x') \left[ \hat{\phi}_{i r}^{m n} (x) \hat{\phi}_{i j}^{m n} (x') + G_{i j a b}^{m n} (x, x') \right] \]

\[ - \frac{1}{2} C^a r_C^{b s} \int d^4 x \sqrt{-g} \int d^4 x' \sqrt{-g'} \times K_{i a b}^{j m n} (x, x') \left[ \hat{A}_{\mu r} (x) \hat{A}_{\nu s} (x') + D_{\mu \nu a b} (x, x') \right] \] (79)

where we have neglected the terms with mixed mean values, because as we shall work with a system for which the mean values of the fields vanish, these terms will vanish too for they couple through the mean values.

From this action we define the sources

\[ \frac{\delta \Gamma}{\delta \hat{\phi}_{i a} (x)} = -C^{a b} \sqrt{-g} J_{b}^{i n} (x) \] (80)

\[ - \frac{1}{2} C^a r_C^{b s} \int d^4 x \sqrt{-g} \sqrt{-g'} \left[ K_{i j a b}^{m n} (x, x') + K_{i j a b}^{m n} (x, x') \right] \hat{\phi}_{i j}^{m n} (x') \]

\[ - \frac{1}{2} C^a r_C^{b s} \int d^4 x' \sqrt{-g'} \left[ K_{i a b}^{j m n} (x, x') + K_{i a b}^{j m n} (x, x') \right] \hat{A}_{\mu a} (x') \]

\[ \frac{\delta \Gamma}{\delta \hat{A}_{\mu a} (x)} = -C^{a b} \sqrt{-g} J_{b}^{i n} (x) \] (81)

\[ - \frac{1}{2} C^a r_C^{b s} \int d^4 x \sqrt{-g} \sqrt{-g'} \left[ K_{i j a b}^{j m n} (x, x') + K_{i j a b}^{j m n} (x, x') \right] \hat{A}_{\mu a} (x') \]

\[ - \frac{1}{2} C^a r_C^{b s} \int d^4 x' \sqrt{-g'} \left[ K_{i a b}^{j m n} (x, x') + K_{i a b}^{j m n} (x, x') \right] \hat{\phi}_{i a}^{m n} (x') \]

\[ \frac{\delta \Gamma}{\delta K_{i r}^{j m} (x, x')} = - \frac{1}{2} C^a r_C^{b s} \sqrt{-g} \sqrt{-g'} K_{i j a b}^{m n} (x, x') \] (82)

\[ \frac{\delta \Gamma}{\delta D_{\mu \nu a b} (x, x')} = - \frac{1}{2} C^a r_C^{b s} \sqrt{-g} \sqrt{-g'} K_{i a b}^{j m n} (x, x') \] (83)
Inverting these expressions and replacing them in the Effective Action and shifting the fields by

\[ \phi_a^{in} (x) = \phi_a^{in} (x) + \varphi_a^{in} (x) \]  \hspace{1cm} (84)
\[ A_{\mu r} (x) = A_{\mu r} (x) + a_{\mu r} (x) \]  \hspace{1cm} (85)

we obtain

\[
\begin{align*}
\Gamma &= C^{ap} C^{aq} \int d^4x \int d^4x' \left[ \frac{\delta \Gamma}{\delta G_{pq}^{nm} (x, x')} \hat{G}_{rs}^{nm} (x, x') + \frac{\delta \Gamma}{\delta D_{\mu pq} (x, x')} D_{\mu vs} (x, x') \right] \\
&- i \ln \int_{CTP} D \varphi D a_{\mu} \exp \left\{ i S_F^N - i C^{ap} \int d^4x \left[ \frac{\delta \Gamma}{\delta \phi_a^{in} (x)} \bar{\varphi}_a^{in} (x) + \frac{\delta \Gamma}{\delta A_{\mu p} (x)} \varphi_a^{in} (x) \right] \right. \\
&\left. - i C^{ap} C^{aq} \int d^4x \int d^4x' \left[ \frac{\delta \Gamma}{\delta G_{pq}^{nm} (x, x')} \varphi_a^{in} (x') \varphi_a^{in} (x) + \frac{\delta \Gamma}{\delta D_{\mu pq} (x, x')} \varphi_a^{in} (x) a_{\mu r}^{in} (x') \right] \right\} \\
&\text{This expression can be interpreted as an implicit equation whose solution is the Effective Action.} \hspace{1cm} (86)
\end{align*}
\]

Formaly this solution can be written as [24]

\[
\Gamma = S_F^N - i \ln \det \left[ G_1^{-1} \right] - i \ln \det \left[ G_2^{-1} \right] - i \ln \det \left[ D^{-1} \right] + \Gamma_2 \]  \hspace{1cm} (87)
\]

where \( \Gamma_2 \) is \(-i\hbar \) times all the two particle irreducible vacuum graphs with lines given by \( G_{ab} \) and vertices given by a shifted action \( S_{int} \) given by

\[
S_{int} [\varphi] = S \left[ \phi + \varphi \right] - S \left[ \phi \right] - C^{ab} \int d^4x \frac{\delta S}{\delta \phi_a} \varphi_b \\
- C^{ab} C^{a'b'} \int d^4x \int d^4x' \frac{\delta^2 S}{\delta \phi_a (x) \delta \phi_b (x')} \varphi_b (x) \varphi_b (x') \hspace{1cm} (88)
\]

Writing

\[
G_{nab}^{ij} (x, x') = \frac{\delta^2 S_{F}}{\delta \phi_a^{i} (x) \delta \phi_b^{j} (x')} \hspace{1cm} (89)
\]

\[
D^{\mu \nu}_{ab} (x, x') = \frac{\delta^2 S_{F}}{\delta A_{\mu a} (x) \delta A_{\nu b} (x')} \hspace{1cm} (90)
\]

performing the scalings

\[
G_{1ab}^{ij} \rightarrow \frac{1}{N} G_{1ab}^{ij} \\
G_{2ab}^{ij} \rightarrow \frac{1}{N} G_{2ab}^{ij} \\
D_{\mu ab}^{\nu} (x, x') \rightarrow \frac{1}{N} D_{\mu ab}^{\nu} (x, x') \\
G_{1ab}^{ij} (x, x') \rightarrow N G_{1ab}^{ij} (x, x') \delta^{ij} \\
G_{2ab}^{ij} (x, x') \rightarrow N G_{2ab}^{ij} (x, x') \delta^{ij} \\
D_{\mu ab}^{\nu} (x, x') \rightarrow N D_{\mu ab}^{\nu} (x, x') \hspace{1cm} (91)
\]

and taking into account that each loop of scalar field counts \( N \), that each trace over scalar field indices counts \( N \) and that there is an overall factor of \( N \) in the action, the effective action reads

\[
\begin{align*}
\Gamma &= \frac{i}{2} N \ln \det \left[ (G_1)^{-1} \right] + \frac{i}{2} N \ln \det \left[ (G_2)^{-1} \right] + \frac{i}{2} \ln \det \left[ (D)\right] \\
&+ \frac{N^2}{2} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g} \hat{G}_{1ab} (x, x') G_{1ab} (x, x') \\
&+ \frac{N^2}{2} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g} \hat{G}_{2ab} (x, x') G_{2ab} (x, x') \\
\end{align*}
\]
positive frequency propagator and 

\[ G \]

electromagnetic potential decouple and therefore there is no backreaction of the electromagnetic field over its sources.

where the delta function is a coordinate function. We see that to this order in \( 1/N \),

where

\( u \)

Replacing the expression for \( G \),

\( \mathcal{G} \rightarrow \mathcal{G} \)

with \( G^{ab} \) the propagators for the scalar field, and \( D_{\mu \nu}^{ab} \) the propagators for the electromagnetic field. The overbar denotes \( \rightarrow \).

It is easy to check that no other loops contribute to the 2PI effective action to this order in \( 1/N \). Indeed consider a graph that is order \( N^0 \), adding a new scalar line that maintains the 2PI condition amounts to add two vertices (and hence to multiply by \( N^{-2} \)) one propagator (another \( N^{-1} \)) and one momentum integration (a factor of \( N \)), which gives a graph that is order \( N^{-2} \). Adding a new photon line also amounts to add two vertices and one propagator but this time there is no loop integration, so the new graph is order \( N^{-3} \). As described in Section III, \( G_{21} (x, x') \) is the positive frequency propagator and \( G_{12} (x, x') \) the negative frequency one. Their spatial Fourier transforms read

\[
G_{21} (x, x') = \int \frac{d^3 \kappa}{(2\pi)^{3/2}} u_\kappa (t) u^*_\kappa (t') \exp [i\kappa. (\vec{r} - \vec{r}')]
\]

\[
G_{12} (x, x') = \int \frac{d^3 \kappa}{(2\pi)^{3/2}} u^*_\kappa (t) u_\kappa (t') \exp [i\kappa. (\vec{r} - \vec{r}')]
\]

where \( u_\kappa (t) \) are positive frequency modes, that are solutions to the Klein Gordon equation for the scalar field.

In the next subsections we shall find the evolution equations for the different two point functions involved in our calculations.

C. Equation for the scalar field propagator

The equation for the scalar field propagator is obtained by taking the functional derivative of the effective action with respect to the desired two point function, i.e.

\[
\frac{\delta \Gamma}{\delta G_{ab}} = -\frac{i}{2} NC_{ab}^{-1} + 2N G_{1ab} (x, x') - i e^2 g^{\mu \nu} (x) C^{abcd} D_{\mu \nu}^{ab} (x, x') \delta (x - x') \]

from where, to leading order in \( 1/N \) we have

\[
iG^{-1}_{ab} (x, x') = G_{1ab} (x, x')
\]

Replacing the expression for \( G_{1ab} (x, x') \) we obtain

\[
\left\{ \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \right) \partial_{\nu} + g^{\mu \nu} \partial_{\mu} \partial_{\nu} - (m^2 + \chi R) \right\} C_{ab} G^{bc} (x, x') = \frac{i}{\sqrt{-g}} \delta^3 (x - x')
\]

where the delta function is a coordinate function. We see that to this order in \( 1/N \), the equations for the scalar and electromagnetic potential decouple and therefore there is no backreaction of the electromagnetic field over its sources.

D. Equation for the electromagnetic field propagator

It is given by
\[
\frac{\delta \Gamma}{\delta D_{\mu \nu}} = -i N D_{\mu \nu}^{ab} + \frac{1}{2} N D_{\alpha \beta} (x, x') - N \frac{e^2}{2} (G_{1ab} (x, x) + G_{2ab} (x, x)) \\
+ i N \frac{e^2}{2} C^{abcd} C^{bc'd'} g^{\mu \alpha} (x) g^{\nu \beta} (x') \left[ G_{1cc'} (x, x') \overline{\partial}_a \overline{\partial}_b G_{2dd'} (x, x') \right]
\]
\[
= 0
\]

where the factor \( N \) in the term \( N D_{\mu \nu}^{ab} \) comes from the scaling of the inverse of the propagator. This is already the equation to leading order in the \( 1/N \) expansion. Replacing \( D_{\mu \nu}^{ab} (x, x') \):

\[
D_{\mu \nu}^{ab} (x, x') = \sqrt{-g (x)} \left\{ g^{\mu \alpha} (x) \Box_x + \left( 1 - \frac{1}{\zeta} \right) g^{\mu \alpha} (x) g^{\nu \beta} (x) \partial_{\alpha} \partial_{\beta} \right.
\]
\[
- \frac{1}{\sqrt{-g}} \partial_{\alpha} \left[ \sqrt{-g} g^{\mu \alpha} g^{\nu \beta} \right] A_{\beta} F_{\mu \nu} - \frac{1}{\zeta} \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} g^{\mu \alpha} g^{\nu \beta} \right] A_{\alpha} F_{\mu \nu} \right\} C_{ab}
\]
\[
= \Delta_{\mu \nu}^{ab} (x, x')
\]

The equation for the propagators for the electromagnetic field then reads

\[
\Delta_{\mu \nu}^{ab} (x, x') D_{\nu \rho}^{bf} (x, x') - e^2 g^{\mu \nu} (x) [G_{1ab} (x, x) + G_{2ab} (x, x)] D_{\nu \rho}^{bf} (x, x') \\
+ i e^2 C^{abcd} C^{bc'd'} \int dx'' \sqrt{-g (x'')} g^{\mu \alpha} (x) g^{\nu \beta} (x'')
\]
\[
\times [G_{1cc'} (x, x'') \overline{\partial}_a \overline{\partial}_b G_{2dd'} (x, x'')] \overline{D}_{\nu \rho}^{bf} (x'', x')
\]
\[
= i \delta_{\mu \rho} \delta_{ab} \frac{1}{\sqrt{-g (x)}} \delta (x - x')
\]

### E. Changing to conformal time

By changing to conformal time \( d\eta = dt/a (t) \), using \( \tau = H \eta \) and performing the scalings

\[
G_{1ab} (x, x') \rightarrow \frac{G_{1ab} (x, x')}{a (\tau) a (\tau')}
\]

it can be checked by a straightforward calculation that the equation for the propagators read

\[
\left\{ \Box_x - \frac{m^2}{H^2} a^2 (\tau) \right\} C_{ab} G^{bc} (x, x') = i \delta_{\alpha \beta} \delta (x - x')
\]
\[
(102)
\]

and

\[
\left[ \tau^{\mu \nu} \Box_x + \left( 1 - \frac{1}{\zeta} \right) \tau^{\mu \alpha} \tau^{\nu \beta} \partial_{\alpha} \partial_{\beta} \right] C_{ab} D_{\nu \rho}^{bf} (x, x')
\]
\[
- e^2 \Gamma^{\mu \nu} \Box_x \left( x, x' \right) D_{\nu \rho}^{bf} (x, x') + i e^2 C^{abcd} C^{bc'd'} \int dx'' \Gamma^{\mu \nu}_{cc', dd'} (x, x'') \overline{D}_{\nu \rho}^{bf} (x'', x')
\]
\[
= i \delta_{\mu \rho} \delta_{ab} \delta (x - x')
\]
\[
(103)
\]

where

\[
\Box_x = \eta^{\alpha \beta} \partial_{\alpha} \partial_{\beta} = - \partial_{\alpha}^2 + \nabla^2
\]
\[
(104)
\]

and

\[
\Gamma^{\mu \nu}_{cd} (x, x) \equiv \eta^{\mu \nu} \left[ G_{1cd} (x, x) + G_{2cd} (x, x) \right]
\]
\[
(105)
\]

\[
\Sigma^{\mu \nu}_{cc', dd'} \equiv \eta^{\mu \nu} (x) \eta^{\rho \beta} (x'') \left[ G_{1cc'} (x, x'') \overline{\partial}_a \overline{\partial}_b G_{2dd'} (x, x'') \right]
\]
\[
(106)
\]

To avoid overnotation, we have also used \( x \) as the dimensionless variables.
In this section we shall find the equations for the retarded and Hadamard propagators. The latter encodes all the information about the state of the electromagnetic field while the former evolves the initial conditions. For the electromagnetic field we have

\[
\begin{align*}
\left[\eta^{\mu\nu}\square_x + \left(1 - \frac{1}{\zeta}\right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu\nu}_{11} (x, x') \right] D^{11}_{\nu\gamma} (x, x') \\
+ ie^2 \int dx'' \Sigma^{\mu\nu}_{11,11} (x, x'') D^{11}_{\nu\gamma} (x'', x') \\
- ie^2 \int dx'' \Sigma^{\mu\nu}_{12,12} (x, x'') D^{21}_{\nu\gamma} (x'', x') = i\delta^\mu_\nu \delta (x - x') \\
\end{align*}
\]

(107)

\[
\begin{align*}
\left[\eta^{\mu\nu}\square_x + \left(1 - \frac{1}{\zeta}\right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu\nu}_{11} (x, x) \right] D^{12}_{\nu\gamma} (x, x') \\
+ ie^2 \int dx'' \Sigma^{\mu\nu}_{11,21} (x, x'') D^{11}_{\nu\gamma} (x'', x') \\
- ie^2 \int dx'' \Sigma^{\mu\nu}_{12,22} (x, x'') D^{21}_{\nu\gamma} (x'', x') = 0 \\
\end{align*}
\]

(108)

\[
\begin{align*}
\left[\eta^{\mu\nu}\square_x + \left(1 - \frac{1}{\zeta}\right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu\nu}_{22} (x, x) \right] D^{21}_{\nu\gamma} (x, x') \\
+ ie^2 \int dx'' \Sigma^{\mu\nu}_{21,21} (x, x'') D^{11}_{\nu\gamma} (x'', x') \\
- ie^2 \int dx'' \Sigma^{\mu\nu}_{22,22} (x, x'') D^{21}_{\nu\gamma} (x'', x') = 0 \\
\end{align*}
\]

(109)

\[
\begin{align*}
\left[\eta^{\mu\nu}\square_x + \left(1 - \frac{1}{\zeta}\right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu\nu}_{22} (x, x) \right] D^{22}_{\nu\gamma} (x, x') \\
+ ie^2 \int dx'' \Sigma^{\mu\nu}_{21,21} (x, x'') D^{12}_{\nu\gamma} (x'', x') \\
- ie^2 \int dx'' \Sigma^{\mu\nu}_{22,22} (x, x'') D^{22}_{\nu\gamma} (x'', x') = - i\delta^\mu_\nu \delta (x - x') \\
\end{align*}
\]

(110)

By substracting the first two (or the last two) equation and defining \(\Sigma^\mu_\nu_{ret} = \Sigma^\mu_\nu_{11} - \Sigma^\mu_\nu_{12,12} = \Sigma^\mu_\nu_{21,21} - \Sigma^\mu_\nu_{22,22}\), we get

\[
\begin{align*}
\left[\eta^{\mu\nu}\square_x + \left(1 - \frac{1}{\zeta}\right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu\nu}_{11} (x, x) \right] D^{ret}_{\nu\gamma} (x, x') \\
+ ie^2 \int dx'' \Sigma^{\mu\nu}_{ret} (x, x'') D^{ret}_{\nu\gamma} (x'', x') = - i\delta^\mu_\nu \delta (x - x') \\
\end{align*}
\]

(111)

and by adding the homogenous equations

\[
\begin{align*}
\left[\eta^{\mu\nu}\square_x + \left(1 - \frac{1}{\zeta}\right) \partial^\mu \partial^\nu - e^2 \Gamma^{\mu\nu\nu}_{11} (x, x) \right] D_{1\nu\gamma} (x, x') \\
+ ie^2 \int dx'' \Sigma^{\mu\nu}_{ret} (x, x'') D_{1\nu\gamma} (x'', x') = - \frac{e^2}{2} \int dx'' \Sigma^{\mu\nu}_1 (x, x'') D_{\nu\gamma}^{adv} (x'', x') \\
\end{align*}
\]

(112)

where \(\Sigma^\mu_1 = \Sigma^\mu_{12,12} + \Sigma^\mu_{21,21} = \Sigma^\mu_{11,11} + \Sigma^\mu_{22,22}\).
For the electromagnetic two point functions we write
\[ G_{cd}(x, x') = \int \frac{d^3p}{(2\pi)^{3/2}} G_{cd}(p, \tau, \tau') \exp [-i\vec{p} \cdot (\vec{y} - \vec{y}')] \] (113)
where in terms of the modes the functions \( G_{cd}(p, \tau, \tau') \) read
\[ G_{21}(p, \tau, \tau') = f_p(\tau) f_p^*(\tau') \] (114)
\[ G_{12}(p, \tau, \tau') = f_p(\tau') f_p^*(\tau) \] (115)
For the non local kernel we have
\[ \Sigma_{\text{ret}}(x, x'') = \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3q}{(2\pi)^{3/2}} \Sigma_{\text{ret}}(p, q, \tau, \tau'') \exp [i(\vec{p} + \vec{q}) \cdot (\vec{y} - \vec{y}'')] \] (116)
with
\[ \Sigma_{\text{ret}}^{00}(p, q, \tau, \tau'') = \eta^{00} \eta^{00} \left\{ G_{11}^1(p, \tau, \tau'') \tilde{\partial}_0 \tilde{\partial}_0 G_1^2(q, \tau, \tau'') - G_{12}^1(p, \tau, \tau'') \tilde{\partial}_0 \tilde{\partial}_0 G_{12}^2(p, \tau, \tau'') \right\} \] (117)
\[ \Sigma_{\text{ret}}^{0i}(p, q, \tau, \tau'') = -i (q_j - p_j) \eta^{00} \eta^{ij} \left\{ G_{11}^1(p, \tau, \tau'') \tilde{\partial}_0 G_1^2(q, \tau, \tau'') - G_{12}^1(p, \tau, \tau'') \tilde{\partial}_0 G_{12}^2(q, \tau, \tau'') \right\} \] (118)
\[ \Sigma_{\text{ret}}^{ij}(p, q, \tau, \tau'') = i (q_j - p_j) \eta^{00} \eta^{ij} \left\{ G_{11}^1(p, \tau, \tau'') \tilde{\partial}_0 G_1^2(q, \tau, \tau'') - G_{12}^1(p, \tau, \tau'') \tilde{\partial}_0 G_{12}^2(q, \tau, \tau'') \right\} \] (119)
\[ \Sigma_{\text{ret}}^{ij}(p, q, \tau, \tau'') = \left\{ q^i q^j + p^i p^j - \eta^{ij} p^i p^j \right\} \times \left\{ G_{11}^1(p, \tau, \tau'') G_1^2(q, \tau, \tau'') - G_{12}^1(p, \tau, \tau'') G_{12}^2(q, \tau, \tau'') \right\} \] (120)
For the electromagnetic two point functions we write
\[ D_{\nu\gamma}(x, x') = \int \frac{d^3k}{(2\pi)^{3/2}} D_{\nu\gamma}(k, \tau, \tau') \exp [i\vec{k} \cdot (\vec{y} - \vec{y}')] \] (121)
so the spatially Fourier transformed equations read
\[ \left[ \eta^{\mu\nu} \left( \partial_\tau^2 - \nabla^2 \right) - \left( 1 - \frac{1}{\zeta} \right) \partial_\mu \partial_\nu + e^2 \int \frac{d^3p}{(2\pi)^{3/2}} \Gamma_{\nu\gamma}^{\mu\nu}(p, \tau, \tau') \right] D_{\nu\gamma}^{\text{ret}}(k, \tau, \tau') \] (122)
\[ -ie^2 \int d\tau'' \Sigma_{\text{ret}}^{\mu\nu}(p, k - p, \tau, \tau'') D_{\nu\gamma}^{\text{ret}}(k, \tau'', \tau') = \delta^\alpha_\mu \delta^\delta_\gamma \left( \tau - \tau'' \right) \]
and by adding the homogenous equations
\[ \left[ \eta^{\mu\nu} \left( \partial_\tau^2 - \nabla^2 \right) - \left( 1 - \frac{1}{\zeta} \right) \partial_\mu \partial_\nu + e^2 \int \frac{d^3p}{(2\pi)^{3/2}} \Gamma_{\nu\gamma}^{\mu\nu}(p, \tau, \tau') \right] D_{\nu\gamma}(k, \tau, \tau') \] (123)
\[ -ie^2 \int d\tau'' \Sigma_{\text{ret}}^{\mu\nu}(p, k - p, \tau, \tau'') D_{\nu\gamma}(k, \tau'', \tau') = \frac{e^2}{2} \int d\tau'' \Sigma_{\text{ret}}^{\mu\nu}(p, k - p, \tau, \tau'') D_{\nu\gamma}^{\text{adv}}(k, \tau', \tau'') \]
The contribution from short wavelength modes to the local term is

$$
\Gamma_{11}^{il(P)}(\tau, \tau) = \eta \int_{\Lambda} \frac{d^3p}{(2\pi)^{3/2}} \left\{ 2 |\beta_p|^2 |f_p(\tau)|^2 + \alpha_p^2 \beta_p^2 f_p^2(\tau) + \beta_p \alpha_p^2 f_p(\tau) \right\}
$$

(124)

where $\Lambda$ is the minimum wavenumber for which the approximation of short wavelengths becomes valid. Replacing eqs. (11) and (13) and performing the momentum integrals we get.

$$
\Gamma_{11}^{il(P)}(\tau, \tau) \simeq \frac{9\pi}{32 (2\pi)^{3/2} \Lambda^2} - \frac{3}{2 (2\pi)^{3/2}} \text{Ci} [2\Lambda \tau]
$$

(125)

For the mixed non local:

$$
\Sigma_{ret}^{il(P)}(p, k - p, \tau, \tau'') = 0
$$

(126)

and for the pure spatial non local

$$
\Sigma_{ret}^{il(P)}(p, k - p, \tau, \tau'') \simeq \Theta (\tau - \tau'') \ 4p^2 p' I
$$

(127)

Replacing the modes and Bogolyubov coefficients we are left with

$$
\Sigma_{ret}^{il(P)}(p, k - p, \tau, \tau'') \simeq 4\Theta (\tau - \tau'') p^2 p' \left\{ -\frac{9i}{64 p^2} \sin [2p (\tau - \tau'')] + \frac{3}{8 p^4} \cos [2p \tau] - \frac{3}{8 p^2} \cos [2p \tau'''] \right\}
$$

(128)

To evaluate the momentum integral, we must take into account that we shall be interested in the transverse component of the electromagnetic two point function, which is obtained by taking the curl of the corresponding equations. If we assume that the magnetic field propagates along the $z$-direction, then the curl picks out the components $x - x$, $x - y$ and $y - y$ of the equation, and of these only the $x - x$ and $y - y$ are non vanishing. For either of them we get

$$
\int_{\Lambda} \frac{d^3p}{(2\pi)^{3/2}} \Sigma_{rel}^{xx(P)}(p, k - p, \tau, \tau') =
$$

$$
\eta^{xx} \int_{\Lambda} \frac{d^3p}{(2\pi)^{3/2}} \Sigma_{rel}^{xx(P)}(p, k - p, \tau, \tau') =
$$

$$
\eta^{xx} \left\{ \frac{9i}{16 \pi} \sin [2\Lambda (\tau - \tau'')] + \frac{9i}{8 \Lambda} (\tau - \tau'') \text{Ci} [2\Lambda (\tau - \tau'')] \right\}
$$

(129)

$$
- \frac{3}{4 \pi} \cos [2\Lambda \tau] + \frac{3}{4 \pi} \cos [2\Lambda \tau']
$$

We see that for both kernels the only momentum contribution is due to the lower limit of integration and these terms must cancel against the ones that come from the upper limit of the integral in the intermediate momenta.

For the pure spatial Hadamard kernel we keep only the terms quadratic in the Bogolyubov coefficients, as they will give the main contribution in this momentum interval, i.e. we have

$$
\Sigma_{1}^{il(P)}(p, k - p, \tau, \tau'') \simeq 4p^2 p' I
$$

(130)

$$
\left\{ 2 |\beta_p|^2 \left[ 2 |f_p(\tau)|^2 |f_p(\tau')|^2 + f_p^2(\tau) f_p^2(\tau') + f_p^2(\tau') f_p^2(\tau) \right] + 2\beta_p \alpha_p^2 \left[ f_p^2(\tau) f_p^2(\tau') + |f_p(\tau)|^2 f_p^2(\tau') \right] + 2\beta_p^2 \alpha_p \left[ f_p^2(\tau')^2 f_p^2(\tau) + |f_p(\tau')|^2 f_p^2(\tau) \right] \right\}
$$

and for which the same reasoning as for the retarded kernel applies. We can conclude that the ultraviolet sector of the created particles spectrum makes no contribution to the noise kernels. A way to understand it is to observe that the mode functions for inflation and reheating are given by the same Bessel function, $H_{3/2}(z)$, but with different arguments. This fact makes very small the amount of created particles, as can be seen from the expression for $\beta_p$. 
The local kernel for this portion of the momenta spectrum was computed in the body of the paper. In this part of the appendix we shall evaluate the retarded non local kernel and show that it is negligible with respect to the local one. The computation of the non local Hadamard kernel $\Sigma^0_{\text{nl}}$ is straightforward.

For the $i = 0$ components of the retarded kernel we have:

$$
\Sigma^0_{\text{ret}}(p, k - p, \tau, \tau') = -i\Theta(\tau - \tau'')(k^i - 2p^i)
$$

$$
\left\{ (\beta_p^x a_p - \beta_k^x a_k)^f(\tau) \hat{f}(\tau') f(\tau) - f^*(\tau') \hat{f}(\tau') \right\}
+ (\alpha_k^x - \alpha_p^x \beta_p)^{\hat{f}}(\tau) f^*(\tau') f(\tau)
+ 2 \left| \beta_k - \beta_p \right|^2 f^*(\tau) f(\tau) \hat{f}(\tau') f(\tau') \right\}
$$

while for the pure spatial components of the same kernel we get

$$
\Sigma^0_{\text{ret}}(p, k - p, \tau, \tau') = \Theta(\tau - \tau'') \left( 4p^i p^i + k^i k^i - 2p^i k^i - 2k^i p^i \right)
\left\{ \left( |\beta_p|^2 + |\beta_k|^2 \right) f^2(\tau) f^2(\tau') - f^2(\tau) f^2(\tau') \right\}
+ (\alpha_k^x - \alpha_p^x \beta_p)^{\hat{f}}(\tau) f^*(\tau') f(\tau) f(\tau') \right\}
$$

Recalling again that we shall take the curl of the equations and considering for example, that the magnetic field propagates along the $z$ direction, we need only the $0 - x$ and $0 - y$ components of the mixed kernel and the $x - x$, $x - y$ and $y - y$ of the pure spatial one. But except for the $x - x$ and $y - y$ of the pure spatial noise kernel, the rest of the components, by virtue of the momentum independence of the mode function (18), are odd functions of $p$ and hence their momentum integral vanishes. We then find the desired decoupling of the equation for the transverse part of the pure spatial propagator. Of course the equations for longitudinal part of the propagators are still coupled and carry the information about charge conservation.

Performing the momentum integral of the component $\Sigma^{xx(N)}_{\text{ret}}$ for example, we get

$$
\int_\tau^\Delta \frac{d^3p}{(2\pi)^{3/2}} \Sigma^{xx(P)}_{\text{ret}}(p, k - p, \tau, \tau') \sim \eta^{\text{zz}} \Theta(\tau - \tau'') \frac{8}{3} \Delta^2
$$

$$
2^{3/\gamma - 41^2 \left( \frac{3 + 2\gamma}{2\gamma} \right) \left( \frac{\pi}{\gamma} \right) \frac{z^{1/\gamma} z^{1/\gamma}}{z_0^{\gamma/\gamma}}}
\left[ H^{(1)}_{3/2\gamma}(z) H^{(1)}_{3/2\gamma}(z) \right]
+ \left[ H^{(1)}_{3/2\gamma}(z) H^{(1)}_{3/2\gamma}(z) \right]
+ \left[ H^{(1)}_{3/2\gamma}(z) H^{(1)}_{3/2\gamma}(z) \right]
$$

Comparing the prefactor in this equation with the corresponding one in eq. (33) we see that by virtue of the logarithmic dependence in the latter, the non local kernel turns out to be several orders of magnitude smaller than the local one.

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