Cosmic Microwave Background Anisotropies

Hu & Dodelson

ABSTRACT
Cosmic microwave background (CMB) anisotropies have become the most important tool in modern cosmology. Combined with observations of the large scale structure of the universe, CMB anisotropies can be used to deduce the properties of the early universe and the evolution of structure.

1 INTRODUCTION
The field of cosmic microwave background (CMB) anisotropies has dramatically advanced over the last decade. With the COBE satellite, the first observations of CMB anisotropies were made, revealing a signal that was significantly larger than expected. Since then, a number of experiments have been carried out, including the BOOMERanG, MAXIMA, and ACBAR experiments, which have provided increasingly precise measurements of the CMB anisotropies. These measurements have allowed cosmologists to better understand the properties of the early universe and the evolution of structure.

2 OBSERVABLES
2.1 Standard Cosmological Paradigm

While a review of the standard cosmological paradigm is not our intention (see Nariai & Padmanabhan 2001 for a critical appraisal), we briefly introduce the observables necessary to parameterize it.
The expansion of the Universe is described by the scale factor $a(t)$, which today is $a \approx 10^{-29}$ (Friedman et al. 2001). The density of the Universe is $\Omega_0 = 1.0$, with a positive cosmological constant greater than the density of dark energy. The mean density of the Universe is $\rho_0 = 3\times10^{-29} \text{g/cm}^3$, which corresponds to a critical density of $\rho_c = 1.6 \times 10^{-29} \text{g/cm}^3$. The observed Hubble parameter $H_0 \approx 0.7$ km/s/Mpc, which is close to the value of the critical density of the Universe, corresponds to a dark energy density of $\Omega_0 = 0.7$. The dark matter density is $\Omega_m = 0.3$.

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3 ACOUSTIC PAPERS

Table 1: CARD associations shown in Phase 1 and Phase 2

<table>
<thead>
<tr>
<th>CARD Contacts</th>
<th># of patients in Phase 1</th>
<th># of patients in Phase 2</th>
</tr>
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<tbody>
<tr>
<td>Tk1</td>
<td>130</td>
<td>120</td>
</tr>
<tr>
<td>Tnr</td>
<td>120</td>
<td>110</td>
</tr>
<tr>
<td>Tk</td>
<td>110</td>
<td>100</td>
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<tr>
<td>Tcr</td>
<td>100</td>
<td>90</td>
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<tr>
<td>Td</td>
<td>90</td>
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<td>Td</td>
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<td>Td</td>
<td>30</td>
<td>20</td>
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<tr>
<td>Td</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

Note: All values are approximate and may vary based on patient demographics and CARD associations.
The population distribution in the two-dimensional space and the probability density function, $p(x,y)$, defined in the domain $D$, can be expressed as:

$$p(x,y) = \begin{cases} p_1(x,y) & \text{if } (x,y) \in D_1 \\ p_2(x,y) & \text{if } (x,y) \in D_2 \end{cases}$$

where $D_1$ and $D_2$ are the disjoint regions in the domain $D$. The probability density function, $p(x,y)$, is normalized such that:

$$\int_{D} p(x,y) \, dx \, dy = 1$$

The probability of an event $A$ occurring within the domain $D$ is given by:

$$P(A) = \int_{A} p(x,y) \, dx \, dy$$

The joint probability density function, $p(x,y)$, is the product of the marginal probability density functions, $p_1(x,y)$ and $p_2(x,y)$, in the case of independence:

$$p(x,y) = p_1(x) \cdot p_2(y)$$

In the absence of data, the parameters of the model are estimated using maximum likelihood estimation. The goodness of fit is assessed using statistical tests such as the chi-squared test. The model parameters are optimized to minimize the distance between the observed data and the model predictions.
Figure 1. Analyzed acoustic modulations. (a) Peak scales for the waveform that complete half a period in the signal in the presence of the complex phase of the periodic mode and a constant phase of water; (b) waveforms for the waveform that complete half a period in the signal in the presence of the complex phase of the periodic mode and a constant phase of water.

$\Theta = \Theta_0(e^{i\theta} - 1)^2$, where $\Theta_0$ is the sound speed in the dynamical frame, $\theta$ is the initial phase, and $\Delta$ is the distance scale. The behavior continues. Assuming negligible initial velocity perturbations, we have a similar distribution at recombination.

$$\Theta(x, y) = \Theta_0(\cos k_x \cos k_y)$$

where $x = \frac{c_d}{\lambda} \cos \gamma$, $y = \frac{c_d}{\lambda} \sin \gamma$, and $\Delta$ is the distance scale. The behavior continues. Assuming negligible initial velocity perturbations, we have a similar distribution at recombination.

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$D = \text{Re}(\gamma + \pi/2)$.
structure in the Universe.

The calculations and simulations of the background radiation from the Big Bang model, as well as the theoretical predictions of inflation models, have shown that the density fluctuations in the CMB are caused by quantum fluctuations during inflation. These fluctuations are amplified by the expansion of the universe and are seen as temperature fluctuations in the CMB.

The inflationary paradigm provides a solution to the horizon problem, as it suggests that the universe was once very small and hot, and that the fluctuations seen today are the result of quantum fluctuations in the early universe. This resolves the problem of the uniformity of the microwave background radiation, as the density fluctuations would have been smoothed out over large scales in a radiation-dominated universe.

The inflationary paradigm also provides a solution to the flatness problem, as it predicts that the density fluctuations in the CMB are consistent with the cosmological principle, which states that the universe is homogeneous and isotropic on large scales.

Inflationary models have also been able to predict the observed spectrum of density fluctuations, which is in agreement with the observations of the CMB.

As for the horizon problem, the inflationary paradigm suggests that the universe was once so small that the regions we observe today were in causal contact during inflation, and that the fluctuations seen today are the result of quantum fluctuations in the early universe.

Finally, inflationary models have also been able to predict the observed large-scale structure of the universe, which is in agreement with observations.

In summary, the inflationary paradigm provides a solution to the horizon problem, the flatness problem, and the large-scale structure of the universe, and is therefore a very important and successful model in cosmology.
Figure 3: Radiation driving and diffusion damping. The decay of the potential $\Psi$ drives the oscillations in the radiation dominated epoch. Diffusion generates an effective $\gamma_{\nu}$, a quadratic term in the temperature, which dampens oscillations and generates polarization. Panels (b) and (c) are a model with mass-independent diffusion damping. The scale of the top panel sets the $\log_{10}(\delta_{\text{rms}})$ term by expanding the $\delta$-derivatives in Equation (11) and identifying the $\delta_{\text{rms}}$ term as the remainder of the linearized expansion drag on baryon velocities.

3.5 Radiation Drag

We have hitherto been neglecting the radiation drag on the baryon mass density $\rho_{b}(\theta)$, which is of order

$$\rho_{b}(\theta) \approx \rho_{\text{c}}(\theta) \approx \rho_{\text{c}}(\theta) \frac{\Omega_{\text{b}}(\theta)}{\Omega_{\text{c}}(\theta)}.$$ 

Compared to the matter. The matter-radiation ratio scales as $\rho_{m}(\theta) \approx 2\rho_{b}(\theta) \Omega_{\text{b}}(\theta) \Omega_{\text{c}}(\theta)$ and so is also of order unity at recombination for reasonable $\Omega_{\text{b}}(\theta)$ parameters. Moreover, fluctuations corresponding to the higher peaks enter the diffusive scale of the sound horizon at an earlier time during radiation domination. The physical scale of the sound horizon at recombination is an additional effect in the interpretation of the density perturbation spectrum by making the gravitational force time-like (Hu & Sugiyama 1999). Matter does not.

The evolution of the potential $\Phi$ is determined by the relativistic Poisson equation,

$$\Delta \Phi + 4\pi \rho_{\text{c}}(\theta)\Phi = 0,$$

with time. The density contrast $\Psi$ will decay in particular at the first compressive maximum of the wave, the Newtonian gravitational potential and spatial curvature must decay (see Figure 3).

4.4 Baryon Loading

So far, we have been neglecting the baryons in the dynamics of the acoustic oscillations. To see whether this is a reasonable approximation consider the photonic density

$$\rho_{\gamma}(\theta) \approx \Omega_{\gamma}(\theta) \rho_{\text{c}}(\theta) \approx \Omega_{\gamma}(\theta) \rho_{\text{c}}(\theta) \frac{\Omega_{\gamma}(\theta)}{\Omega_{\text{c}}(\theta)}.$$ 

At top and core to the left equation. Since inertial and gravitational forces are equal, all terms in the Euler equation save the pressure gradient are multiplied by $1 + R_{\text{d}}(\theta)$ leading to the revised oscillator equation (Hu & Sugiyama 1995).

$$\frac{d^{2}\Psi}{d\theta^{2}} + K_{\text{eff}}^{2}(\theta) \Psi = -\frac{\Delta_{0}^{2}}{2} \phi^{2} \psi^{2} \frac{d^{2}\phi}{d\theta^{2}} \phi,$$

where $K_{\text{eff}}(\theta)$ is the effective sound speed modified by the baryon to

$$\frac{d^{2}\Psi}{d\theta^{2}} + K_{\text{eff}}^{2}(\theta) \Psi = -\frac{\Delta_{0}^{2}}{2} \phi^{2} \psi^{2} \frac{d^{2}\phi}{d\theta^{2}} \phi.$$ 

Aside from the breaking of the sound speed which decreases the sound horizon, baryonic oscillations also have a non-linear effect. As $\rho_{b}(\theta) \approx \Omega_{\text{b}}(\theta) \rho_{\text{c}}(\theta)$, the baryonic term may be added to the right-hand side again.

The solution then becomes

$$\Omega_{\text{b}}(\theta) \rho_{\text{c}}(\theta) \approx \Omega_{\gamma}(\theta) \rho_{\text{c}}(\theta) \approx \Omega_{\gamma}(\theta) \rho_{\text{c}}(\theta) \frac{\Omega_{\gamma}(\theta)}{\Omega_{\text{c}}(\theta)}.$$ 

The truncation of the instabilities at $\Delta \Phi + 4\pi \rho_{\text{c}}(\theta)\Phi = 0$ is to the effective sound speed

$$\frac{d^{2}\Psi}{d\theta^{2}} + K_{\text{eff}}^{2}(\theta) \Psi = -\frac{\Delta_{0}^{2}}{2} \phi^{2} \psi^{2} \frac{d^{2}\phi}{d\theta^{2}} \phi.$$ 

For the same initial conditions, increasing the mass causes the oscillation to increase further in the gravitational field leading to larger oscillations and a shifted zero point.

The shifting of the zero point of the oscillator has significant phenomenological consequences. The zero point shift breaks the symmetry $\phi = \Psi$ of the oscillations. The baryon enhanced compression into potential wells.

An additional effective mass $\Delta_{0}$ may be added to the right-hand side due to the additional perturbations. The energy content of the oscillations is $\Delta M(\theta) = \Delta_{0} \rho_{\text{b}}(\theta) \Omega_{\text{b}}(\theta)$ for a matter dominated universe. The mass $\Delta M(\theta)$ is the additional effective mass. Since the energy frequency of an oscillator is an `adiabatic' invariant, the mass must decay $\Delta M(\theta) = 1 + \Delta_{0} \rho_{\text{b}}(\theta) \Omega_{\text{b}}(\theta)$.
The decay actually compresses the oscillations; it is then apparent that the fluid mass.

The net effect is to lower the density, since the oscillations become less pronounced. The wave energy is absorbed and converted into heat. The heat conductivity coefficient, \( \kappa \), is a measure of the rate at which energy is transferred through the fluid by conduction. It is given by:

\[
\kappa = \frac{1}{3} \rho C_v \alpha \end{equation}

where \( \rho \) is the density, \( C_v \) is the specific heat at constant volume, and \( \alpha \) is the thermal diffusivity.

The equation governing the heat flow in a fluid is:

\[
\nabla^2 T = \frac{1}{\kappa} \frac{\partial T}{\partial t}
\]

where \( T \) is the temperature and \( t \) is time. This equation is solved for the initial and boundary conditions appropriate for the problem at hand.

The solution for the temperature distribution in a semi-infinite solid is:

\[
T(x, t) = \frac{Q}{4\pi k} \int_0^t \frac{e^{-\frac{x^2}{4k(t-t')}}}{\sqrt{t-t'}} dt'
\]

where \( Q \) is the heat source per unit volume,

\( k = \frac{1}{3} \rho C_v \alpha \)

is the thermal conductivity coefficient, and \( t' = t - t'' \) is the time delay.

The initial and boundary conditions are:

\[
T(x, 0) = \begin{cases} T_0 & x = 0 \ \\ 0 & x > 0 \end{cases}
\]

\[
\frac{dT}{dx} \bigg|_{x=0} = \frac{T_0}{\sqrt{\kappa t}}
\]

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\[
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3.8 Integral Approach

The discussion in the previous sections suffices for a qualitative understanding of the above mentioned phenomena. To reflect the experimental observations, especially its initial conditions and polarization history.
where the projected source $\Omega(k) = |\Theta| \phi_0$, $\phi_0 = \phi(k, \delta, \kappa)$, and $|\Theta|$ is the scalar product of the harmonic $\Theta$ and the spherical Bessel function $j_0(k)$. Because the spherical harmonic $\Theta$ is a function of the angular distance $\theta$, the angular distance $\theta$ is given by the equation $\Theta = \sqrt{1 - \cos^2 \theta}$. The spherical harmonic $\Theta$ is a function of the angular distance $\theta$, and it is a function of the spherical Bessel function $j_0(k)$.

In the direction orthogonal to the wave vector $k$, the angular distance $\theta$ is given by the equation $\Theta = \sqrt{1 - \cos^2 \theta}$. The spherical harmonic $\Theta$ is a function of the angular distance $\theta$, and it is a function of the spherical Bessel function $j_0(k)$.

Plate 3: Intensity peaks. CBM spectroscopy can be thought of as the line-of-sight projection of various sources of phase wave temperature and polarization fluctuations: the acoustic signal of the polarization temperature and secondary sources (see §§2.2, 2.3). Secondary contributions come in that the region must be near the surface of a black hole. A backup of a black hole must appear near the region of a black hole with a maser surface is a black hole.

Plate 2: Polarization generation and classification. Left: Thomson scattering of quadrupole (depicted here in the $\hat{y}$ plane) gives rise to a quadrupole component. Right: Thomson scattering of quadrupole temperature anisotropy (depicted here in the $\hat{z}$ plane) gives rise to a quadrupole component.

$\Theta(\theta, \phi, \kappa) = \left[ \phi_i(\phi) \phi(\phi, \kappa) \right]_{i \in \{0, 1, 2, \ldots, \infty\}}$
CMB Anisotropies

$\Delta T^\text{an}(\theta, \phi)$ is the anisotropy of the microwave background temperature at a particular location in the sky, where $\theta$ is the angular position of the direction and $\phi$ is the azimuthal angle. The anisotropies are observed in the form of the power spectrum of the angular power, which is given by

$$C_l = \frac{1}{2\pi} \int d\Omega \frac{\Delta T^2(\theta, \phi)}{T^2} \delta^2(\theta, \phi)$$

This formula relates the anisotropy to the angular power spectrum, which is a measure of the fluctuations in the temperature of the microwave background across the sky. The $C_l$ values are tabulated in Table 3, which lists the values for different multipoles $l$.

The angular power spectrum is an important tool for understanding the properties of the early universe, as it is directly related to the density fluctuations in the primordial universe. The shape of the power spectrum can provide insights into the nature of dark matter and dark energy, as well as the geometry of the universe.
4 BEYOND THE PEAKS

Once the resonant peaks in the transmission and polarization power spectra have

The problem of the precise measurement of the resonant peaks and the effect of

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The presence of extrinsic effects.
The combination of the COBE normalization of the matter transfer function and the near-scale-invariant initial density fluctuations gives a great deal of freedom to the present and past structure formation in all scales. These constraints are very useful for testing the theoretical predictions for the matter fluctuations in the universe.
and is one of the strongest pieces of evidence for the parameters in the working cosmological model (Ostriker & Steinhardt 1995; Krauss & Turner 1995).

More generally, the comparison between large-scale structure and the CMB is important in that it breaks degeneracies between effects due to deviations from power-law initial conditions and the dynamics of the matter and energy contents of the Universe. Any dynamical effect that reduces the amplitude of the matter power spectrum corresponds to a decay in the Newtonian potential that boosts the level of anisotropy (see §3.5 and §4.2.1). Massive neutrinos are a good example of physics that drives the matter power spectrum down and the CMB spectrum up.

The combination is even more fruitful in the relationship between the acoustic peaks and the baryon wiggles in the matter power spectrum. Our knowledge of the physical distance between adjacent wiggles provides the ultimate standard candle for cosmology (Eisenstein et al 1998). For example, at very low \(z\), the radial distance out to a galaxy is \(cz/H_0\). The unit of distance is therefore \(h^{-1}\) Mpc, and a knowledge of the true physical distance corresponds to a determination of \(h\). At higher redshifts, the radial distance depends sensitively on the background cosmology (especially the dark energy), so a future measurement of baryonic wiggles at \(z \sim 1\) would be a powerful test of dark energy models.

To a lesser extent, the shape of the transfer function, which mainly depends on the matter-radiation scale in \(h\) Mpc \(^{-1}\), i.e. \(\Omega_m h\), is another standard ruler (see e.g. Tegmark et al 2001 for a recent assessment), more heralded than the wiggles, but less robust due to degeneracy with other cosmological parameters.

For scales corresponding to \(k \gtrsim 10^{-4} h\) Mpc \(^{-1}\), density fluctuations are nonlinear by the present. Numerical N-body simulations show that the dark matter is bound up in a hierarchy of virialized structures or halos (see Bertschinger 1998 for a review). The statistical properties of the dark matter and the dark matter halos have been extensively studied in the working cosmological model. Less certain are the properties of the baryonic gas. We shall see that both enter into the consideration of secondary CMB anisotropies.

### 4.2 Gravitational Secondaries

Gravitational secondaries arise from two sources: the differential redshift from time-variable metric perturbations (Sachs & Wolfe 1967) and gravitational lensing. There are many examples of the former, one of which we have already encountered in §3.8 in the context of potential decay in the radiation dominated era. Such gravitational potential effects are usually called the integrated Sachs-Wolfe (ISW) effect in linear perturbation theory (§4.2.1), the Rees-Sciama (§4.2.2) effect in the non-linear regime, and the gravitational wave effect for tensor perturbations (§4.2.3). Gravitational waves and lensing also produce \(B\)-modes in the polarization (see §3.7) by which they may be distinguished from acoustic polarization.

#### 4.2.1 ISW Effect

As we have seen in the previous section, the potential on a given scale decays whenever the expansion is dominated by a component whose effective density is smooth on that scale. This occurs at late times in an \(\Omega_m < 1\) model at the end of matter domination and the onset dark energy (or spatial curvature) domination. If the potential decays between the time a photon falls into a potential well and when it climbs out it gains a boost in temperature of \(\Delta T\) due to the differential gravitational redshift and \(-\Delta T \approx \delta T\) due to an accompanying contraction of the wavelength (see §3.3).

Potential decay due to residual radiation was introduced in §3.8, but that due to dark energy or curvature at late times induces much different changes in the anisotropy spectrum. What makes the dark energy or curvature contributions different from those due to radiation is the longer length of time over which the potentials decay, on order the Hubble time today. Residual radiation produces its effect quickly, so the distance over which photons feel the effect is much smaller than the wavelength of the potential fluctuation. Recall that this means that \(j_0(\ell kD)\) in the integral in Equation (23) could be set to \(j_0(\ell kD)\) and removed from the integral. The final effect then is proportional to \(j_0(\ell kD)\) and adds in phase with the monopole.

The ISW projection, indeed the projection of all secondaries, is much different (see Plate 3). Since the duration of the potential change is much longer, photons typically travel through many peaks and troughs of the perturbation. This cancellation implies that many modes have virtually no impact on the photon temperature. The only modes which do have an impact are those with wavevectors perpendicular to the line of sight, so that along the line of sight the photon does not pass through crests and troughs. What fraction of the modes contribute to the effect then? For a given wavenumber \(k\) and line of sight instead of the full spherical shell at radius \(r = kD\), only the ring \(2\pi kD\) with \(k \perp \nu\) participate. Thus, the anisotropy induced is suppressed by a factor of \(k\) (or \(\ell = kD\) in angular space). Mathematically, this arises in the line-of-sight integral of Equation (23) from the integral over the oscillatory Bessel function \(f(x) \approx (\pi/2k)^{1/2}\) (see also Plate 3).

The ISW effect thus generically shows up only at the lowest \(\ell\) in the power spectrum (Kofman & Starobinsky 1985). This spectrum is shown in Plate 5. Secondary anisotropy predictions in this figure are for a model with \(\Omega_m = 1, \Omega_k = 2/3, \Omega_b h^2 = 0.02, \Omega_m h^2 = 0.16, \Omega_c = 7\) and inflationary energy scale \(E_* \approx 10^{16}\) GeV. The ISW effect is especially important in that it is extremely sensitive to the dark energy: its amount, equation of state and clustering properties (Caldwell et al 1997; Caldwell et al 1998; Hu 1998). Unfortunately, being confined to the low multipoles, the ISW effect suffers severely from the cosmic variance in Equation (4) in its detectability. Perhaps more promising is its co-
Planck's analysis of the cosmic microwave background radiation reveals a nearly scale-invariant spectrum of gravitational waves, which is consistent with the predictions of inflationary models. Inflation is a theory that suggests the universe underwent a rapid expansion in the early moments after the Big Bang, causing quantum fluctuations in the density of the universe to stretch and become large. These fluctuations are encoded in the power spectrum of the cosmic microwave background radiation, and their presence is evidence for inflation.

The Planck satellite's measurements of the cosmic microwave background radiation were used to verify these predictions. The observed power spectrum is consistent with a scale-invariant spectrum, which is a characteristic of inflationary models. This finding supports the theoretical framework of inflation and provides a strong empirical confirmation of its predictions.

Inflation also predicts the existence of gravitational waves, which are ripples in the fabric of spacetime. These waves were not directly observable until the Planck satellite measurements. The detection of these waves would provide a new window into the early universe and would further validate the inflationary paradigm.

In summary, the observed power spectrum and the presence of gravitational waves are consistent with inflationary theory, providing a compelling piece of evidence for the inflationary universe model.
4.3 Scattering Secondaries

From the observations both of the lack of of a Grand-Peterson trough (Gunn & Peterson 1965) in quasar spectra and their preliminary detection (Bakler et al. 2000), we know that hydrogen has been ionized at $z > 6$. This is thought to occur through the ionizing radiation of the first generation of massive stars (see §4.2.2). The subsequent recombination of CMB photons to the baryons causes a few percent of the photons to be scattered; lineally polarized CMB photons are scattered off the baryons, whereas mainly unpolarized CMB photons are scattered off the cold dark matter. This effect is important in that it is a direct indication of the dark energy beyond a simple equation of state (Hu 2000).

The gravitational potentials of large-scale structure also lens the CMB photons. Since lensing conserves surface brightness, the gravitational potentials can be reconstructed from the CMB power spectrum. The potential at a point is given by:

$$\phi(r) = \frac{1}{8\pi G} \sum_{\ell} \frac{D_{\ell}}{D_{\ell}^*} \frac{\ell(\ell + 1)}{2\ell + 1} \frac{P_{\ell}}{P_{\ell,0}} C_{\ell}$$

where $D_{\ell}$ is the angular diameter distance at redshift $z = \sqrt{1 + \Omega_m}$, $D_{\ell}^*$ is the angular diameter distance at redshift $z = \sqrt{1 + \Omega_m}$, $P_{\ell}$ is the angular power spectrum at multipole $\ell$, $P_{\ell,0}$ is the angular power spectrum at multipole $\ell$ at redshift $z = 0$, and $C_{\ell}$ is the CMB power spectrum at multipole $\ell$.

The gravitational potential is sampled by a set of 

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$$\phi(r) = \frac{1}{8\pi G} \sum_{\ell} \frac{D_{\ell}}{D_{\ell}^*} \frac{\ell(\ell + 1)}{2\ell + 1} \frac{P_{\ell}}{P_{\ell,0}} C_{\ell}$$

where $D_{\ell}$ is the angular diameter distance at redshift $z = \sqrt{1 + \Omega_m}$, $D_{\ell}^*$ is the angular diameter distance at redshift $z = \sqrt{1 + \Omega_m}$, $P_{\ell}$ is the angular power spectrum at multipole $\ell$, $P_{\ell,0}$ is the angular power spectrum at multipole $\ell$ at redshift $z = 0$, and $C_{\ell}$ is the CMB power spectrum at multipole $\ell$.

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4.3.3 Doppler Effect

The Doppler effect is a phenomenon that occurs when a source of sound or light is moving relative to an observer. It causes the perceived frequency of the sound or light to change, depending on the direction of motion. In the case of sound, the Doppler effect is often used to explain why the pitch of a siren or bird call changes as the source approaches or recedes. In the case of light, the Doppler effect is used to explain why the color of distant stars appears different to observers on Earth. The Doppler effect is named after Christian Doppler, who first described it in 1842.
As we have seen, most of the secondary anisotropies are non-Caustic in nature and hence produce non-Caustic signals. Non-Caustic anisotropies are not linear in nature and have formed only recently in the standard cold dark matter model. Great strides have recently been made in observing the CMB temperature signal in individual clusters, following pioneering attempts that remained largely unchallenged for some time (Bartlett et al. 1994). The theoretical basis for this result is well understood and has been extensively reviewed in (Carlstrom et al. 1995). Here we instead consider its implications as a source of secondary anisotropies.

The CMB anisotropies result primarily from tracers of the large-scale structure in the universe, and there are many possible contributions to these, such as galaxy clustering, large-scale magnetic fields, etc. (Bartlett et al. 1994). The primary anisotropy, however, is due to the spatial distribution of matter in the universe. The secondary anisotropies are the result of variations in the distribution of matter due to the large-scale structure. The CMB anisotropies are primarily due to the primary anisotropy, which is caused by the angular dependence of the emission of cosmic microwave background radiation. The secondary anisotropies are due to variations in the spatial distribution of matter in the universe. These variations are caused by the large-scale structure, such as galaxy clustering, large-scale magnetic fields, etc. (Bartlett et al. 1994). The CMB anisotropies are primarily due to the primary anisotropy, which is caused by the angular dependence of the emission of cosmic microwave background radiation. The secondary anisotropies are due to variations in the spatial distribution of matter in the universe. These variations are caused by the large-scale structure, such as galaxy clustering, large-scale magnetic fields, etc. (Bartlett et al. 1994).
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5. DATA ANALYSIS

The very large CMB data sets that have been arriving require new, innovative tools of analysis. The fundamental tool for analysing CMB maps is to obtain a likelihood function for the parameters from the data. This function has been used since the early days of anisotropy searches by Zheng et al. [1993]. The likelihood function for all the parameters can be computed using a Markov chain Monte Carlo method. This is a powerful technique that allows the data to be analysed without making any assumptions about the signal in the data.

5.1. Maximising

The likelihood function for the parameters can be maximised using a variety of methods. These methods are based on the idea of finding the maximum likelihood estimate (MLE) of the parameters. The MLE is the value of the parameters that maximises the likelihood function. In practice, it is often easier to maximise the logarithm of the likelihood function, as this is a more convenient function to work with.

5.2. Constraints

The likelihood function is constrained by a variety of factors. These factors include the geometry of the universe, the age of the universe, and the cosmic microwave background (CMB) power spectrum. The CMB power spectrum is a measure of the fluctuations in the CMB that are caused by the acoustic oscillations of the photons and electrons in the early universe. The CMB power spectrum is a function of the angular scale of the fluctuations.

5.3. Results

The results of the analysis are presented in the form of a likelihood contour plot. This plot shows the likelihood function as a function of the parameters. The highest likelihood values are found at the maximum likelihood estimate (MLE). The contours show the regions of parameter space where the likelihood is significantly lower than the MLE. The contours are used to estimate the uncertainties on the parameters.

5.4. Comparison with theory

The likelihood function is compared with theoretical predictions in order to test the validity of the model. The comparison is made by calculating the probability of the data given the model, and then comparing this probability with the probability of the data given the theory. The model is considered to be valid if the probability of the data given the model is significantly higher than the probability of the data given the theory.
with the normalization constant determined by requiring the integral of the probability density to be equal to 1. Therefore, the expectation of the likelihood function is the maximum likelihood estimator. The maximum likelihood estimator is also the minimum variance unbiased estimator. The Cramer-Rao lower bound states that the variance of any unbiased estimator is at least equal to the inverse of the Fisher information matrix $I(\theta)$. The Fisher information matrix is defined as

$$I(\theta) = \sum_{i} \frac{1}{2} \left[ \frac{\partial^2 \ell}{\partial \theta^2} \right]_{\theta = \theta_0}$$

where $\theta$ is the parameter vector, $\ell$ is the log-likelihood function, and $\theta_0$ is the true parameter value. The Cramer-Rao lower bound states that the variance of any unbiased estimator is at least equal to the inverse of the Fisher information matrix.

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It is clear that a simple sampling of the likelihood in parameter space is necessary. The numerical problem, finding the local maximum of a function, is well posed, e.g., by the Newton-Raphson method which has become widely used. One expands the derivative of the log of the likelihood function, which vanishes at the true maximum, in a Taylor series around a trial point in parameter space, $\theta_{0}$. Keeping terms to second order in $R_{j} - R_{0}$ leads to

$$R_{j} = R_{0} + \frac{1}{2} \left[ \frac{\partial^{2} \ln L}{\partial \theta_{i} \partial \theta_{j}} \right]_{0} (\theta_{j} - \theta_{0})$$

where $\frac{\partial^{2} \ln L}{\partial \theta_{i} \partial \theta_{j}}$ is the second derivative of the log likelihood function with respect to $\theta_{i}$ and $\theta_{j}$. Note the single distinction between the curvature matrix and the Fisher matrix on the data. In practice, though, we use the inverse of the Fisher matrix in Equation (32), in that case, the estimator becomes quadratic in the data $\theta_{j}$. The Fisher matrix is equal to

$$F_{ij} = \left( \frac{\partial \ln L}{\partial \theta_{i}} \right)_{0} \left( \frac{\partial \ln L}{\partial \theta_{j}} \right)_{0}$$

(33)

(34)

In the spirit of the Newton-Raphson method, Equation (33) is used iteratively to find the best estimates $\theta_{0}$. For each iterate $j$, the Fisher matrix $F_{ij}$ is then used to take the variance or average over the $\theta_{0}$, which is the parameter space considered for the next iterate. Steps of the procedure for handling a symmetric beam model using a finite number of pixels is described in detail elsewhere in this paper. The calculations for handling the model are best illustrated using an example of an experiment.

CMB Anisotropies

$$C_{ij} \equiv \langle \theta_{i} \theta_{j} \rangle = \sum_{m} \sum_{n} C_{mn} \exp \left(\frac{1}{2} \theta_{i} \theta_{j} \right)$$

(30)

(31)

where $C_{mn}$ is the anisotropy $n$ and $m$ are the number of pixels in the signal at position $\theta_{mn}$, and $x_{ij}$ is the average number of pixels in the signal at position $\theta_{ij}$. The parameters are then $\theta_{ij}$.

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6 Discussion

Measurements of the acoustic peaks in the CMB temperature spectrum have already shown that the Universe is nearly spatially flat and began with a nearly scale-invariant spectrum of curvature fluctuations, consistent with the simplest inflationary models. In a recent paper, Rodríguez and collaborators (2000) have shown that the CMB measurements can now test the predictions of inflationary cosmology on a more detailed level than was possible before. In particular, they show that the CMB measurements are consistent with a simple inflation model that produces a nearly scale-invariant spectrum of curvature fluctuations.

In the context of the CMB, the question of how to weight the likelihood to obtain $B$ can be important. In principle, one must consider the theoretical spectra with window functions (Knox 1989), distinct from those in Figure 4. Among recent experiments, ASAS (Przybylek et al. 2001) and others have provided for $B$ the bounds $B > 0.001$, $B > 0.001$, $B > 0.001$, and $B > 0.001$, respectively. However, the cosmological parameter estimation is not Gaussian, i.e., not of the form in Equation (35). The true distribution is closer to the lognormal (Bond et al. 2000), and several groups have already accounted for this in their parameter estimators.

Acknowledgments

W.H. thanks the hospitality of Fermilab where this review was written. W.H. was supported by the DOE, by NASA grant NAG 5-10842 at Fermilab, by NSF Grant PHF-019731 at Chicago.