GENERALISED SUPERSYMMETRIC FLUXBRANES

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ABSTRACT. We study generalised supersymmetric fluxbranes in type II string theory obtained as Kaluza–Klein reductions of the Minkowski space vacuum of eleven-dimensional supergravity. We obtain a seven-parameter family of smooth solutions which contains all the known solutions, new solutions called nullbranes, and solutions interpolating between them.

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1. Introduction

There has recently been a lot of interest in the embedding of the Melvin universe [18] in M/string theory, in particular in the type IIA flux 7-brane (F7-brane), whose M-theory description [8, 6, 7, 15] strongly suggests that type IIA string theory with magnetic field $B = M^2_s/g_s^2$ is dual to type 0A string theory [4, 1]. They also play an important role in the supergravity description of the expansion of a D$p$-brane into a spherical D$(p + 2)$-brane due to the dielectric effect [5] and in the possible stabilization of tubular branes [9]. The study of analogous magnetic

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backgrounds allowing a conformal field theory description \cite{21, 22, 27} provides a framework to study the decay of unstable backgrounds into stable supersymmetric ones since they are smoothly connected to the supersymmetric closed string vacuum. Related work can also be found in \cite{19, 23, 3, 24, 25, 26}.

It was pointed out in \cite{15, 28} that when several magnetic parameters are taken into account the corresponding supergravity background may preserve some amount of supersymmetry. One of the goals of this paper is to classify these possibilities, or equivalently, to classify supersymmetric fluxbranes in string theory. To do so, we shall reexamine the geometrical setting giving rise to the flux-fivebrane (or F5-brane, for short) \cite{15} in the context of Kaluza–Klein reductions. Starting with eleven-dimensional Minkowski space $E^{10,1}$, the F5-brane is obtained by considering a Kaluza–Klein reduction along the orbits generated by a Killing vector $\xi$ consisting on a translation and a rotation. The reduction will preserve some supersymmetry if the rotation belongs to the isotropy algebra of some nonzero Killing spinor of the Minkowski vacuum. This suggests considering the Kaluza–Klein reduction of the most general Killing vector in Minkowski space

$$\xi = a^\mu \partial_\mu + \frac{1}{2} \omega^{\mu\nu} (x_\mu \partial_\nu - x_\nu \partial_\mu),$$

subject to some natural conditions, chiefly that we should have a smooth quotient inheriting a metric. Preservation of supersymmetry imposes no restrictions on translations but again enforces the Lorentz transformation to belong to the isotropy algebra of $\epsilon$. Reductions involving pure rotations generate flux p-branes configurations in type IIA (F5, F3 and F1), but we do also have the possibility of reducing along null rotations. Whereas the first ones can be interpreted as intersections of F7-branes, the last ones, that we shall denote as nullbranes, can be thought of as some sort of boosted F7-branes\footnote{JS would like to thank M Berkooz for suggesting this interpretation.}, even though its physical interpretation is not clear to us, these being time dependent backgrounds in string theory preserving one half of the spacetime supersymmetry. Besides these main supersymmetric building block solutions, one can also consider reductions along linear combinations of the Killing vectors giving rise to them. The type IIA configurations obtained in this way, do continuously interpolate among the Fp-branes and nullbranes discussed before.

Thus, one can think of the space of Kaluza–Klein reductions of the eleven dimensional Minkowski space being parametrised by $\{a^\mu, \omega^{\mu\nu}\}$. Arbitrary points in this configuration space will break supersymmetry completely. What it is done in this paper is the determination of which subspaces in $\{a^\mu, \omega^{\mu\nu}\}$ do preserve supersymmetry, the interpolation between supersymmetric and non-supersymmetric domains being smooth. The formulation developed in this paper allows an
extension to any supersymmetric M-theory background having some isometry group. This will be discussed and applied for M-brane backgrounds and their intersections in [14] giving rise to supersymmetric composite configurations of type IIA/B branes (D-branes, fundamental strings, NS5-branes, waves ... ) and flux(null) branes, but also for the remaining maximally supersymmetric M-theory backgrounds [13].

There remain many interesting open questions regarding Fp-branes such as which D-brane configurations are possible in these backgrounds (or their U-duals) and which is the corresponding field theory on them in certain decoupling limits. Furthermore, while standard D-branes admit an open string description on flat space, the existence of such a description for Fp-branes remains an open question.

This paper is organised as follows. In Section 2 we discuss the geometric context of the present paper: namely the supersymmetric Kaluza–Klein reduction of the flat M-theory vacuum and outline a classification problem to whose solution the present paper is a contribution. Section 3 contains the construction of smooth generalised supersymmetric fluxbranes from the Kaluza–Klein reduction of Minkowski space by a general class of spacelike Killing vectors preserving spinors. This class contains seven free continuous parameters and a result it yields a seven-parameter family of smooth generalised supersymmetric fluxbranes. This family contains the known fluxbrane solutions as well as new solutions (called here nullbranes), as well as solutions which interpolate between them. This is illustrated by some examples. Finally, Section 4 contains a comprehensive analysis of the U-dual configurations which can be obtained from the solutions constructed in Section 3.

Note added. While this paper was being completed [20] appeared having some overlap with the content presented here.

2. Kaluza–Klein reduction and generalised fluxbranes

We start by considering Kaluza–Klein reductions of the maximally supersymmetric vacuum of M-theory described by eleven-dimensional Minkowski space $E^{10,1}$, and in this way obtain a large class of supersymmetric solutions of type IIA supergravity whose geometries will be described explicitly in the next section.

2.1. The geometric set-up. The general geometric set-up is the following. We will consider a one-parameter subgroup of the group of isometries of Minkowski space, in other words a one-dimensional subgroup $G$ of the Poincaré group acting on Minkowski space. Topologically, $G$ is either a circle or a line. In traditional treatments of Kaluza–Klein reduction, $G$ is always taken to be a circle subgroup; but circles do not act freely on Minkowski space. It is possible to follow tradition and demand that $G$ be a circle; but in order to have a smooth quotient, one is forced to introduce “identifications” in Minkowski space which
effectively “close” the integral curves of $\xi$. This practice is standard in the context of fluxbranes, and although it has its merits, we prefer not to follow it here. To restate, all our solutions will be smooth quotients of Minkowski space by the action of $G \cong \mathbb{R}$. If desired, these solutions could be viewed as Kaluza–Klein reductions on a circle. Indeed, quotienting by $\mathbb{R}$ can be done in two steps: quotienting by $\mathbb{Z}$ (i.e., making identifications in Minkowski space) and then quotienting by the circle $\mathbb{R}/\mathbb{Z}$. The action of $\mathbb{Z}$ introduces a length scale in the problem, which is the radius (divided by $2\pi$) of the circle.

If the $G$-action is free and has spacelike orbits\(^2\), the ten-dimensional space obtained by Kaluza–Klein reduction will be a solution of the type IIA supergravity. In addition, this solution will be supersymmetric provided that $G$ leaves some spinor invariant. Let $\xi$ be the Killing vector generating the $G$ action infinitesimally. The action will be (locally) free if and only if $\xi$ is nowhere-vanishing. With our choice of (mostly plus) metric, demanding that the norm $\|\xi\|^2$ be everywhere positive guarantees that the orbits are spacelike. Under further mild restrictions (namely that every point in Minkowski space should have trivial stabiliser) the space of orbits will be a smooth manifold.

In adapted coordinates, where $\xi = \partial_z$, we can write the eleven-dimensional Minkowski metric as

$$ds^2(\mathbb{E}^{10,1}) = e^{-2\phi/3}g + e^{4\phi/3}(dz + A)^2,$$

where $g$ is the ten-dimensional metric in the string frame, $\phi$ is the dilaton and $A$ is the RR 1-form potential of type IIA supergravity. By construction, the triple $(g, \phi, A)$ will satisfy the equations of motion of IIA supergravity, provided that we set the other field strengths $F_3$ and $F_4$ in the theory to zero. This solution will be supersymmetric if and only if $\xi$ preserves some Killing spinors.

The Killing vector $\xi$ acts on a Killing spinor $\varepsilon$ via the spinorial Lie derivative (see, e.g., [17] and also [10]), defined by

$$L_{\xi}\varepsilon = \nabla_{\xi}\varepsilon + \frac{1}{4}\nabla_a\xi_b\Gamma^{ab}\varepsilon.$$

Since Killing spinors are parallel in the Minkowski vacuum, the condition that $\xi$ preserves supersymmetry becomes the algebraic condition that, for some nonzero parallel $\varepsilon$,

$$\frac{1}{4}\nabla_a\xi_b\Gamma^{ab}\varepsilon = 0.$$

Alternatively, relative to adapted coordinates, a spinor is invariant under $\xi = \partial_z$ if and only if it does not depend explicitly on $z$.

\(^2\)Since we want to induce a metric in the space of orbits, it is important that the Killing vector never be null; hence it has to be always spacelike or always timelike. Since we are interested in constructing solutions of type II supergravity in signature $(9,1)$ we restrict ourselves to spacelike orbits. Nothing prevents us from considering timelike orbits and in this way obtain fluxbrane solutions of euclidean supergravity theories of the type considered in [16], but this will be left for another time.
The most general Killing vector in Minkowski space is the sum of a translation and a Lorentz transformation:

$$\xi = a^\mu \partial_\mu + \frac{1}{2} \omega^{\mu\nu} (x_\mu \partial_\nu - x_\nu \partial_\mu) ,$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. The condition (2) for a parallel spinor (which is constant in the flat coordinates $x^\mu$) to be invariant under $\xi$ becomes

$$\frac{1}{4} \omega^{\mu\nu} \Sigma_{\mu\nu} \varepsilon = 0 .$$

In other words, we see that the translation can be arbitrary and that the Lorentz transformation must belong to the isotropy algebra of a spinor.

For the purposes of this paper, by a (generalised) fluxbrane we will mean a solution of IIA supergravity obtained as the Kaluza–Klein reduction of eleven-dimensional Minkowski space. In what follows we will construct a multiparameter family of generalised fluxbranes containing as special cases all the fluxbranes which have been hereto considered.

2.2. Towards a classification. Although we will not attempt to solve it in the present paper, let us outline the mathematical problem of classifying supersymmetric generalised fluxbranes. This problem reduces to finding free actions of a group $G \cong \mathbb{R}$ on Minkowski space preserving the metric, leaving some parallel spinors invariant and possessing spacelike orbits. In other words, we are interested in classifying one-parameter subgroups of the spinor isotropy groups of (the spin cover of) the Poincaré group, which act freely on Minkowski space with spacelike orbits.

Killing vectors in Minkowski space are in one-to-one correspondence with the Poincaré algebra, so let us think of $\xi$ as an element in the Poincaré algebra $\mathfrak{so}(10, 1) \ltimes \mathbb{R}^{10,1}$, whence $\xi$ has a translation component and a Lorentz component. As we saw above, translations act trivially on spinors, so the translation component of $\xi$ is not constrained. Since Lorentz transformations fix the origin and we want $\xi$ to act freely, the translation component must be present. Moreover this translation has to be spacelike at the origin, and hence everywhere. Since we have the freedom to perform a change of coordinates which preserves the metric (in other words, we have the freedom to conjugate $\xi$ in the Poincaré algebra by an element of the Poincaré group), we can assume without loss of generality (although perhaps by rescaling $\xi$) that the translation component is simply $\partial_z$, where $z$ is the eleventh coordinate; that is,

$$\xi = \partial_z + \lambda ,$$

where $\lambda$ is an infinitesimal Lorentz transformation preserving spinors.

The eleven-dimensional spinor isotropy groups are known (see, e.g., [11]). An infinitesimal Lorentz transformation $\lambda$ fixes a spinor if and only if it belongs to one of two subalgebras of the Lorentz algebra: $\mathfrak{su}(5)$
or $\text{spin}(7) \times \mathbb{R}^9$. The former preserves a timelike direction and hence is contained in some $\mathfrak{so}(10)$ subalgebra; whereas in the latter, $\text{spin}(7)$ acts as rotations in some 8-plane, $\mathbb{R}^9 = \mathbb{R}^8 \oplus \mathbb{R}$, where $\mathbb{R}^8$ acts as null rotations in that 8-plane and $\mathbb{R}$ acts as null rotations in a perpendicular direction. It is not hard to show that if $\lambda$ does not fix any spacelike directions—which can only happen in the $\mathfrak{su}(5)$ case—then there are points in space time where the norm of the Killing vector $\xi$ becomes non-positive. Therefore we need only consider $\lambda$ in some $\text{spin}(7) \wr \mathbb{R}^9$ subalgebra. If $\lambda$ does not fix $\partial_z$, it is a delicate issue to characterise those $\lambda$ for which $\partial_z + \lambda$ is everywhere spacelike. Since our aim in this paper is not to solve the classification problem, but rather to point out the method and exhibit a large class of solutions, we will make the simplifying assumption that $\lambda$ leaves $\partial_z$ fixed. This means that it belongs to a $\text{spin}(7) \wr \mathbb{R}^8$ subalgebra of the $\mathfrak{so}(9,1)$ which fixes $\partial_z$.

In summary, there exists a coordinate system $(z, y^i, y^\pm)$, with $i = 1, \ldots, 8$, where the metric takes the form

$$ds^2(\mathbb{E}^{10,1}) = 2dy^+dy^- + \sum_{i=1}^8 dy^i dy^i + dz^2$$  \hspace{1cm} (3)

and the Killing vector takes the form

$$\xi = \partial_z + \frac{1}{2} \sum_{1 \leq i, j \leq 8} \omega^{ij}(y^i \partial_j - y^j \partial_i) + \sum_{i=1}^8 v^i(y^- \partial_i - y^i \partial_+)$$ \hspace{1cm} (4)

for some coefficients $v^i$ and some $\omega^{ij} = -\omega^{ji}$ such that the resulting rotation is in $\text{spin}(7) \subset \mathfrak{so}(8)$. It is clear from these two equations that $\xi$ is everywhere spacelike; indeed,

$$\|\xi\|^2 \geq 1 \quad \text{everywhere}.$$

The integrated action of $t \in \mathbb{R} \cong G$ on Minkowski space takes the form

$$(z, y^\mu) \mapsto (z + t, \Lambda(t)^\mu_\nu y^\nu)$$

whence the action is manifestly free: no point is left fixed and every point has trivial stabiliser. This means that the space of orbits $\mathbb{E}^{10,1}/G$ is smooth.

We are still free to conjugate $\lambda$ by $\mathfrak{so}(9,1)$ to bring it to a convenient normal form. In particular, we can rotate our coordinates in the 8-plane spanned by the $y^i$ so that $\omega^{ij}$ becomes

$$\omega = \begin{pmatrix} \beta_1 J & \beta_2 J & \beta_3 J & \beta_4 J \end{pmatrix}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and where $\sum_i \beta_i = 0$, since $\omega$ is in $\text{spin}(7)$. This normal form is still preserved by rotations in each of the 2-planes spanned by $(y^{2i-1}, y^{2i})$ for $i = 1, 2, 3, 4$. Using this freedom we can rotate the vector $(v^{2i-1}, v^{2i})$
to \((u^i, 0)\), say. In total, the Killing vector \(\xi\) depends on 7 independent parameters: 3 \(\beta\)'s and 4 \(u\)'s. The Kaluza–Klein reduction of Minkowski space along the orbits of \(\xi\) will result in a 7-parameter family of supersymmetric (generalised) fluxbranes which we now describe.

3. A family of supersymmetric fluxbranes

To best describe the explicit geometry of the generalised supersymmetric fluxbranes obtained by reducing Minkowski space along the orbits of the Killing vector (4), we will work in coordinates adapted to the Killing vector. This turns out to be very easy, once we observe that \(\xi\) is simply a dressed version of its translation component:

\[
\xi = U \partial_z U^{-1} \quad \text{where} \quad U = \exp(-z\lambda) .
\]

Let us introduce coordinates \((x^i, x^\pm)\) related to the coordinates \((y^i, y^\pm)\) by

\[
x = U y .
\]

It follows easily that \(\xi x = 0\), so that \(x\) are good coordinates for the space of orbits. Since \(\lambda\) is an infinitesimal Lorentz transformation, the \(x\) are linearly related to the \(y\) with \(z\)-dependent coefficients. Indeed, let us denote by \(B\) the constant \(10 \times 10\) real matrix such that

\[
\lambda y = By .
\]

Therefore the new coordinates are given by

\[
x = \exp(-zB) y ,
\]

where \(\exp\) here means the matrix exponential. For \(\lambda\) in the normal form described at the end of the last section, it is very easy to write an explicit expression for this matrix exponential; but we will refrain from doing so here, as it is not needed in order to write down the IIA background.

Indeed, it is a simple matter to rewrite the metric (3) in terms of the new variables, obtaining

\[
ds^2(\mathbb{E}^{10,1}) = \Lambda (dz + A)^2 + dx^\dagger \left(1 - \Lambda^{-1}(Bx)(Bx)^\dagger\right) dx ,
\]

where \(v^\dagger = v^t \eta\) is the adjoint relative to the Minkowski metric and

\[
\Lambda = 1 + (Bx)^\dagger (Bx) \quad \text{and} \quad A = \Lambda^{-1}(Bx)^\dagger dx .
\]

Using the Kaluza–Klein Ansatz (1) we can read off the IIA background which has \(F_3 = F_4 = 0\) and in the string frame the nontrivial fields are given by

\[
\phi = \frac{3}{4} \log \Lambda \quad \text{and} \quad g = \Lambda^{1/2} dx^\dagger \left(1 - \Lambda^{-1}(Bx)(Bx)^\dagger\right) dx ,
\]

with \(A\) in the previous equation, which also contains the definition of \(\Lambda\). The parameters in this equation are hidden in \(B\), which is the matrix of an infinitesimal Lorentz transformation in \(\text{spin}(7) \ltimes \mathbb{R}^8 \subset \text{so}(9, 1)\). As discussed in the previous section, we can always choose the
coordinates in such a way that $B$ only depends on 7 real parameters which are unconstrained. We thus obtain a seven-dimensional connected component of the moduli space of generalised supersymmetric fluxbranes. Some subvarieties in this moduli space are already known, as we now discuss.

To help comparison with the literature, let us write down the matrix $B$ explicitly relative to the basis $\{x^i, x^+, x^-\}$:

$$
B = \begin{pmatrix}
0 & -\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & u^1 \\
\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & u^2 \\
0 & 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta_3 & 0 & 0 & 0 & u^3 \\
0 & 0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\beta_4 & 0 & u^4 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta_4 & 0 & 0 \\
-u^1 & 0 & -u^2 & 0 & -u^3 & 0 & -u^4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

(7)

where $\sum_i \beta_i = 0$, with corresponding Killing vector (4), given by

$$\xi = \partial_z + \sum_{i=1}^4 \beta_i \left( y^{2i} \partial_{2i-1} - y^{2i-1} \partial_{2i} \right) + \sum_{i=1}^4 u^i \left( x^- \partial_{2i-1} - x^{2i-1} \partial_+ \right).$$

We will now consider several special cases.

3.1. **Supersymmetric fluxbranes.** If all the $u^i = 0$, so that the Lorentz transformation is purely a rotation, we obtain the standard supersymmetric fluxbranes considered previously in [15, 28, 20]. In this case we have several possibilities depending on how many of the $\beta$’s are zero. If three $\beta$’s vanish, then supersymmetry implies that all $\beta$’s must vanish and the resulting configuration is the type IIA vacuum. Therefore at most two $\beta$’s can be zero, if we are to obtain a nontrivial supersymmetric fluxbrane.

3.1.1. **The supersymmetric F5-brane revisited.** If $\beta_1 = \beta_2 = 0$, and hence $\beta_3 = -\beta_4 = \beta$, we recover the supersymmetric flux-fivebrane of [15]. The matrix $B$ corresponds to an element in the Cartan subalgebra
of $\mathfrak{sp}(1) \subset \mathfrak{so}(4) \subset \mathfrak{so}(9,1)$:

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$  \hspace{1cm} (8)

Here $\beta$ is the parameter related to the magnetic field in [15]. The solution is given explicitly by (6) with

$$\Lambda = 1 + \beta^2 r^2 \quad \text{and} \quad A = \frac{\beta}{1 + \beta^2 r^2} \left( x^5 dx^6 - x^6 dx^5 - x^7 dx^8 + x^8 dx^7 \right),$$

where $r$ is the radial coordinate in the 4-plane spanned by the coordinates $x^5, \ldots, x^8$ transverse to the fluxbrane. We can rewrite this solution in a more familiar form, by introducing coordinates:

$$x^5 + ix^6 = r \cos \theta e^{i(\psi + \varphi)} \quad \text{and} \quad x^7 + ix^8 = r \sin \theta e^{i(\psi - \varphi)},$$  \hspace{1cm} (9)

where $\theta \in [0, \frac{\pi}{2}]$, $\psi \in [0, \pi]$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. In terms of the new coordinates,

$$A = \frac{\beta r^2}{1 + \beta^2 r^2} (d\varphi + \cos(2\theta)d\psi),$$  \hspace{1cm} (10)

and the metric becomes

$$g = \Lambda^{1/2} \left[ ds^2(\mathbb{E}^{5,1}) + dr^2 + r^2 d\theta^2 + r^2 \sin^2(2\theta) d\psi^2 \right]$$

$$+ \Lambda^{-1/2} r^2 (d\varphi + \cos 2\theta d\psi)^2.$$  \hspace{1cm} (11)

This solution represents a flux-fivebrane or $F5$-brane for short. Indeed, the IIA RR 2-form field $F = dA$ in this solution has a nontrivial “charge”, as can be seen from the (normalised) integral of $F \wedge F$ on the transverse $\mathbb{E}^4$:

$$\frac{1}{8\pi^2} \int_{\mathbb{E}^4} F \wedge F = \lim_{\rho \to \infty} \frac{1}{8\pi^2} \int_{r \leq \rho} F \wedge F = \lim_{\rho \to \infty} \frac{1}{8\pi^2} \int_{r = \rho} A \wedge F = \frac{1}{\beta^2},$$

where we have used that the orientation on the 3-sphere $r = \rho$ induced by the natural orientation $dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8$ on $\mathbb{R}^4$ agrees with $d\theta \wedge d\varphi \wedge d\psi$. The flux-fivebrane preserves one half of the supersymmetry of the eleven-dimensional vacuum.
The near-horizon geometry \((r \to 0)\) of the flux-fivebrane is flat. Indeed the metric is asymptotic to \(\mathbb{E}^{7,1}\) times a cone metric:
\[
dr^2 + r^2 \left[ d\theta^2 + \sin^2(2\theta) d\psi^2 + (d\varphi + \cos(2\theta) d\psi)^2 \right].
\]
The metric of the base of the cone (that is, the quantity in square brackets) is that of the round 3-sphere, whence the cone metric is that of \(\mathbb{E}^4\). In the other limit \((r \to \infty)\) we obtain a conformally cylindrical geometry. Indeed, \(h\) is asymptotically conformal to a metric which is a product of \(\mathbb{E}^{7,1}\) with
\[
dr^2 + r^2 \left[ d\theta^2 + \sin^2(2\theta) d\psi^2 \right] + \frac{1}{\beta^2} \left( d\varphi + \cos(2\theta) d\psi \right)^2.
\]
This is the metric on the total space of a circle bundle over \(\mathbb{R}^3\) where the radius of the circle is constant and equal to \(1/\beta\).

3.1.2. **The supersymmetric F3-brane.** If \(\beta_1 = 0\), but the rest of the \(\beta\)'s are different from zero (but add up to zero) we obtain the supersymmetric flux-threebrane discussed in [28, 20]. In this case the matrix \(B\) is a Cartan subalgebra of \(\text{su}(3) \subset \text{so}(6) \subset \text{so}(9, 1)\):

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\beta_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The solution is again given explicitly by (6) with
\[
\Lambda = 1 + \sum_{i=2}^{4} \beta_i^2 |z_i|^2 \quad \text{and} \quad A = \frac{-1}{2i} \sum_{i=2}^{4} \beta_i (\bar{z}_i dz_i - z_i d\bar{z}_i),
\]
where \(z_i, \ i = 2, 3, 4\), correspond to complex coordinates in the 6-plane spanned by the real coordinates \(x^3, \ldots, x^8\) transverse to the fluxbrane. We can rewrite this in terms of the radial distance to the fluxbrane by introducing coordinates:

\[
\begin{align*}
x^3 + ix^4 &= z_1 = r \cos \theta_1 e^{i\lambda_1} \\
x^5 + ix^6 &= z_2 = r \sin \theta_1 \cos \theta_2 e^{i\lambda_2} \\
x^7 + ix^8 &= z_3 = r \sin \theta_1 \sin \theta_2 e^{i\lambda_3},
\end{align*}
\]

where \(\theta_1 \in [0, \pi/2], \theta_2 \in [0, \pi/2]\) and \(\lambda_i \in \mathbb{R}/2\pi \mathbb{Z}\) for all \(i\). In terms of the new coordinates, the scalar function becomes
\[
\Lambda = 1 + r^2 \left( \beta_2^2 \cos^2 \theta_1 + \beta_3^2 \sin^2 \theta_1 \cos^2 \theta_2 + \beta_4^2 \sin^2 \theta_1 \sin^2 \theta_2 \right).
\]
whereas the RR 1-form potential is given by
\[ A = \Lambda^{-1} r^2 \left( \beta_2 \cos^2 \theta_1 d\lambda_1 \beta_3 \sin^2 \theta_1 \cos^2 \theta_2 d\lambda_2 + \beta_4 \sin^2 \theta_1 \sin^2 \theta_2 d\lambda_3 \right), \]
and the metric becomes
\[
g = \Lambda^{1/2} \left[ ds^2(E^{3,1}) + dr^2 + r^2 \left( d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 - r^2 \left( \cos^2 \theta_1 d\lambda_1^2 + \sin^2 \theta_1 \cos^2 \theta_2 d\lambda_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\lambda_3^2 \right) \right) \right] - \Lambda^{-1/2} r^4 \left( \beta_2 \cos^2 \theta_1 d\lambda_1 + \beta_3 \sin^2 \theta_1 \cos^2 \theta_2 d\lambda_2 + \beta_4 \sin^2 \theta_1 \sin^2 \theta_2 d\lambda_3 \right)^2.
\]
The solution preserves \(1/4\) of the supersymmetry.

3.1.3. Supersymmetric flux-strings. If none of the \(\beta\)'s vanish, there are two possibilities. Either the \(\beta\)'s add up to zero pairwise or they don't. In the former case, \(B\) belongs to a Cartan subalgebra of \(\text{sp}(1) \oplus \text{sp}(1) \subset \text{so}(4) \oplus \text{so}(4) \subset \text{so}(8) \subset \text{so}(9,1)\), whereas in the latter, it belongs to a Cartan subalgebra of \(\text{su}(4) \subset \text{so}(8) \subset \text{so}(9,1)\). In either case we have a supersymmetric fluxstring. In the former case the fluxstring preserves \(1/4\) of the supersymmetry, whereas in the latter case it preserves \(1/8\). The latter case was discussed in [28, 20], the former case seems new.

Let us examine, first of all, the case when \(B\) belongs to a Cartan subalgebra of \(\text{sp}(1) \oplus \text{sp}(1) \subset \text{so}(4) \oplus \text{so}(4) \subset \text{so}(8) \subset \text{so}(9,1)\):

\[
B = \begin{pmatrix}
0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\tilde{\beta} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{\beta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{\beta} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\tilde{\beta} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

It is natural to use the same coordinates introduced in (9), this time to parametrise the 4-plane spanned by \(x^1, \ldots, x^4\)
\[
x^1 + i x^2 = r \cos \theta e^{i(\psi + \varphi)} \quad \text{and} \quad x^3 + i x^4 = r \sin \theta e^{i(\psi - \varphi)}, \tag{12}
\]
where \(\theta \in [0, \pi/2], \psi \in [0, \pi]\) and \(\varphi \in \mathbb{R}/2\pi\mathbb{Z}\), and proceed analogously with the second 4-plane spanned by \(x^5, \ldots, x^8\)
\[
x^5 + i x^6 = \tilde{r} \cos \tilde{\theta} e^{i(\tilde{\psi} + \tilde{\varphi})} \quad \text{and} \quad x^7 + i x^8 = \tilde{r} \sin \tilde{\theta} e^{i(\tilde{\psi} - \tilde{\varphi})},
\]
where \(\tilde{\theta} \in [0, \pi/2], \tilde{\psi} \in [0, \pi]\) and \(\tilde{\varphi} \in \mathbb{R}/2\pi\mathbb{Z}\). The full distance to the flux-string is measured by the square root of \(r^2 + \tilde{r}^2\). In terms of the new coordinates,
\[
\Lambda = 1 + r^2 \beta^2 + \tilde{r}^2 \tilde{\beta}^2
\]
\[ A = \Lambda^{-1} \beta r^2 (d\varphi + \cos(2\theta)d\psi) + \Lambda^{-1} \tilde{\beta} \tilde{r}^2 \left( d\tilde{\varphi} + \cos(2\tilde{\theta})d\tilde{\psi} \right), \]

and the metric becomes
\[
g = \Lambda^{1/2} \left[ ds^2(E^{1,1}) + dr^2 + r^2 d\theta^2 + r^2 \sin^2(2\theta)d\psi^2 
+ d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2 \sin^2(2\tilde{\theta})d\tilde{\psi}^2 \right] 
+ \Lambda^{-1/2} \left[ r^2 (d\varphi + \cos 2\theta d\psi)^2 (1 + \tilde{\beta} \tilde{r}^2) + \tilde{r}^2 \left( d\tilde{\varphi} + \cos 2\tilde{\theta} d\tilde{\psi} \right)^2 (1 + \beta r^2) 
- 2\beta \tilde{\beta} r^2 \tilde{r}^2 (d\varphi + \cos 2\theta d\psi) \left( d\tilde{\varphi} + \cos 2\tilde{\theta} d\tilde{\psi} \right) \right]. \]

Finally, let us consider the case in which \( B \) belongs to a Cartan subalgebra of \( \mathfrak{su}(4) \subset \mathfrak{so}(8) \subset \mathfrak{so}(9,1) \):
\[
B = \begin{pmatrix}
0 & -\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_4 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \quad (13)
\]

The solution is again given explicitly by (6) with
\[
\Λ = 1 + \sum_{i=1}^{4} \beta_i^2 |z_i|^2 \quad \text{and} \quad A = \frac{\ Lambda^{-1} }{2i} \sum_{i=1}^{4} \beta_i (\bar{z}_i dz_i - z_i d\bar{z}_i),
\]
where \( z_i \) \( i = 1, 2, 3, 4 \) correspond to complex coordinates in the 8-plane spanned by the real coordinates \( x^1, \ldots, x^8 \) transverse to the fluxbrane. We can rewrite this in terms of the radial distance to the fluxbrane by introducing coordinates:
\[
x^1 + ix^2 = z_1 = r \cos \theta_1 e^{i\lambda_1} \\
x^3 + ix^4 = z_2 = r \sin \theta_1 \cos \theta_2 e^{i\lambda_2} \\
x^5 + ix^6 = z_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3 e^{i\lambda_3} \\
x^7 + ix^8 = z_4 = r \sin \theta_1 \sin \theta_2 \sin \theta_3 e^{i\lambda_4},
\]
where \( \theta_i \in [0, \frac{\pi}{2}] \) and \( \lambda_i \in \mathbb{R}/2\pi \mathbb{Z} \) for all \( i = 1, 2, 3, 4 \). In terms of the new coordinates, the scalar function becomes
\[
\Λ = 1 + r^2 \left( \beta_1^2 \cos^2 \theta_1 + \beta_2^2 \sin^2 \theta_1 \cos^2 \theta_2 
+ \beta_3^2 \sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 + \beta_4^2 \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 \right),
\]
whereas the RR 1-form potential is given by

\[ A = \Lambda^{-1} r^2 \left( \beta_1 \cos^2 \theta_1 d\lambda_1 + \beta_2 \sin^2 \theta_1 \cos^2 \theta_2 d\lambda_2 \\
+ \beta_3 \sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 d\lambda_3 + \beta_4 \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 \right), \]

and the metric becomes

\[ g = \Lambda^{1/2} \left[ ds^2(E^{1,1}) + dr^2 + r^2 \left( d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 \right) \\
+ r^2 \left( \cos^2 \theta_1 d\lambda_1^2 + \sin^2 \theta_1 \cos^2 \theta_2 d\lambda_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 d\lambda_3^2 \\
+ \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 d\lambda_4^2 \right) \right] \\
- \Lambda^{-1/2} r^4 \left[ \beta_1 \cos^2 \theta_1 d\lambda_1 + \beta_2 \sin^2 \theta_1 \cos^2 \theta_2 d\lambda_2 + \beta_3 \sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 d\lambda_3 \\
+ \beta_4 \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 d\lambda_4 \right]^2. \]

3.2. **Supersymmetric nullbranes.** On the other extreme, we have those reductions where the \( u \)'s are nonzero but the \( \beta \)'s are zero. These reductions give rise to solutions with null RR fieldstrengths which we tentatively call *nullbranes*. In this case we have the freedom to choose a coordinate system in which the null rotation is along one of the coordinates, say \( x^1 \). The corresponding matrix \( B \) has all \( \beta \)'s equal to zero and only \( u^1 = u \) is different from zero. Moreover, it is possible to reabsorb \( u \), provided it is nonzero, by rescaling \( x^\pm \mapsto u^1 x^\pm \). The resulting solution seems to be new.\(^3\) Explicitly, we have

\[ \Lambda = 1 + (x^-)^2 \quad \text{and} \quad A = \frac{1}{1 + (x^-)^2} \left( x^- dx^1 - x^1 dx^- \right), \quad (14) \]

with IIA metric in the string frame given by

\[ g = \Lambda^{1/2} \left[ 2dx^+ dx^- - (x^-)^2 dx^- + ds^2(E^7) \right] \\
+ \Lambda^{-1/2} \left( dx^1 + x^1 dx^- \right)^2. \quad (15) \]

The RR 2-form field strength is null

\[ F = \frac{2}{\Lambda^2} dx^- \wedge dx^1, \quad (16) \]

hence our name for these solutions. These solutions always preserve one half of the supersymmetry.

Notice that the metric (15) has an \( E^7 \) subspace, so this suggests computing the Hodge dual of the null field strength (16). Indeed, \( \star F \) couples naturally to a seven dimensional extended object. Doing so one finds that \( \star F \) is actually constant in this coordinate system and is given by

\[ \star F = 2dx^- \wedge dvol(E^7). \]

\(^3\)The possibility of quotienting by boosts was mentioned briefly in [15]. The resulting fluxbranes are electrically charged but are not supersymmetric.
for our choice of orientation. This is reminiscent of the homogeneous branes introduced in [12].

### 3.3. Generalised supersymmetric fluxbranes.

When the matrix $B$ contains both rotations and null rotations, the resulting solution interpolates between the fluxbranes and the nullbranes discussed above. To illustrate the geometry of these solutions, let us consider a matrix $B$ which is the sum of the matrix (8) giving rise to the F5-brane and the matrix giving rise to the N7-brane:

$$
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & u \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 \\
0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\
-u & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

As in the case of the nullbrane, one can reabsorb $u$ by a rescaling $x^\pm \mapsto u^{1/2}x^\pm$. Applying the general formulae (5) and (6), it is easy to write down the resulting IIA background. In the coordinates (9), the dilaton is again given by $\phi = \frac{3}{4} \log \Lambda$, the RR 1-form potential by

$$
A = \Lambda^{-1} \left( x^- dx^1 - x^1 dx^- + \beta r^2 (d\varphi + \cos 2\theta d\psi) \right)
$$

and the metric by

$$
g = \Lambda^{1/2} \left[ 2dx^+ dx^- - (x^1)^2 (dx^-)^2 + dx^2 (E^3) + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2) \right] + \Lambda^{-1/2} \left[ (dx^1 + x^1 x^- dx^-)^2 + r^2 (d\varphi + \cos 2\theta d\psi)^2 \right] \left( 1 + (x^-)^2 \right) + (\beta r)^2 \left( (dx^1)^2 + (x^1)^2 (dx^-)^2 \right) - 2\beta r^2 (x^- dx^1 - x^1 dx^-) (d\varphi + \cos 2\theta d\psi),
$$

where the scalar function is $\Lambda = 1 + (x^-)^2 + \beta^2 r^2$.

This solution smoothly interpolates between the F5-brane—which is recovered in the region $x^- \ll 1$ keeping $\beta r$ fixed—and the N7-brane, in the region $\beta r \ll 1$ keeping $x^-$ fixed. It preserves 1/4 of the spacetime supersymmetry. There are similar solutions involving lower dimensional fluxbranes.

On the other hand, one can consider the case where the null rotation in the matrix $B$ takes place in the same 4-plane as the rotation; for
example, in the direction $x^5$. The matrix $B$ is now

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & u \\
0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 \\
0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. $$

In the same coordinates as above (which allows us to reabsorb $u$), the resulting IIA configuration can be written as follows. The dilaton is $\phi = \frac{1}{4} \log \Lambda$, the RR 1-form field is given by

$$A = \Lambda^{-1} \left( A_0 + \beta r^2 (d\varphi + \cos 2\theta d\psi) \right),$$

and the metric is given by

$$g = \Lambda^{1/2} \left[ 2dx^+ dx^- + ds^2(E^4) + dr^2 + r^2 (d\theta^2 + \sin^2(2\theta) d\psi^2) \right] + \Lambda^{-1/2} \left[ r^2 (d\varphi + \cos 2\theta d\psi)^2 (\Lambda - (\beta r)^2) - A_0 \left( A_0 + 2\beta r^2 (d\varphi + \cos 2\theta d\psi) \right) \right],$$

where the scalar function is now

$$\Lambda = 1 + (x^-)^2 + \beta^2 r^2 - 2x^- \beta r \cos \theta \sin(\psi + \varphi),$$

and we have introduced

$$A_0 = x^- \left[ \cos \theta \cos(\psi + \varphi) dr - r \sin \theta \cos(\psi + \varphi) d\theta \right. - \left. r \cos \theta \sin(\psi + \varphi) (d\psi + d\varphi) \right] - r \cos \theta \cos(\psi + \varphi) dx^-.$$

The latter solution also preserves $1/4$ of the spacetime supersymmetry. For $\beta \to 0$ one recovers the N7 branes, whereas if we reintroduce the parameter $u$ by rescaling $x^\pm \mapsto u^{\pm1} x^\pm$, and take the limit $u \to 0$ we recover the F5-brane. Hence this solution also interpolates between the F5- and the N7-branes.

We trust that the interested reader should have no difficulty in applying the formalism described in this section to construct more general solutions by taking as a starting point a more general matrix $B$ of the form (7). Let us make one final comment concerning the supersymmetry preserved by these reductions. For the most general matrix $B$ given by (7), the solution will preserve $1/16$ of the supersymmetry: the most general rotation (all $\beta$’s different from zero and no two adding to zero) will preserve $1/8$ as we saw above. Adding any null rotations further cuts this by $1/2$, yielding a minimum of $1/16$. This is to be expected from the group theory: indeed any element in SU(5) fixes two linearly independent spinors, whereas any one element of Spin(7) $\rtimes \mathbb{R}^9$, since it actually belongs to a SU(4) $\rtimes \mathbb{R}^9$ subgroup, also preserves two spinors.
4. Dualities and flux branes

Any supergravity background gives rise to a family of related solutions through the use of U-duality transformations and other solution-generating techniques. The purpose of this section is to discuss the family of solutions obtained by U-duality from some of the generalised flux branes found in the previous section, namely flux branes and null-branes. It is well known that any time a T-duality transformation is used to generate a new solution, the latter is delocalised in the T-dual direction. It is often possible to find localised versions of the new solutions, but it remains to be seen whether this is indeed true for the solutions we find here. We will make much use of the T-duality rules derived by Bergshoeff, Hull and Ortín in [2].

4.1. A word on the notation. Applying the T-duality rules to the solutions found above, we will obtain many solutions for which no name yet exists. In this section we have introduced a notation in order to be able to identify them. Since the notation is not standard, let us take a moment to explain it. By an $F(p,q)$-brane, we shall denote a solution with full Poincaré invariance in $p+1$ dimensions, but which is nevertheless delocalised in $q$ of them. Whenever an S-duality transformation is applied, an $s$ is added to the notation together with one of the letters $a$ or $b$ to emphasise that this is a configuration of type IIA or IIB, respectively. The is done to distinguish solutions which, although formally equivalent, belong to different theories. The rest of the notation is standard.

4.2. F5-brane dualities. The starting solution is the F5-brane, with metric (11) and RR 1-form potential (10), whereas the dilaton is given by $\phi = \frac{3}{4} \log(1 + \beta^2 r^2)$.

Performing a T-duality along $5-p$ worldspace directions of the F5-brane, one obtains a family of $F(p,5-p)$-brane configurations, for $p = 4, 3, 2, 1, 0$ belonging to type IIA for odd $p$ and to type IIB for even $p$. They can be jointly described by

$$e^\phi = \Lambda^{(p-2)/4}, \quad C_{(6-p)} = A_{(1)} \wedge dy_{p+1} \wedge \cdots \wedge dy_5,$$
$$ds_{10}^2 = \Lambda^{1/2} ds^2(\mathbb{R}^{p,1}) + \Lambda^{-1/2} [dy_{p+1}^2 + \cdots + dy_5^2] + \Lambda^{1/2} ds^2_{\text{SU}(2)},$$

where $ds^2_{\text{SU}(2)}$ is given by

$$ds^2_{\text{SU}(2)} = dr^2 + r^2 (d\theta^2 + \sin^2 2\theta d\psi^2) + \frac{r^2}{\Lambda} (d\varphi + \cos 2\theta d\psi)^2.$$

Notice that the F(2,3)-brane is self-dual under S-duality transformations, whereas the F(4,1)-brane gives rise to the F(4,1)bs-string

$$e^\phi = \Lambda^{-1/2}, \quad b_{(2)} = -A_{(1)} \wedge dx_{11},$$
$$ds_{10}^2 = ds^2(\mathbb{R}^{4,1}) + \Lambda^{-1} dx_{11}^2 + ds^2_{\text{SU}(2)}, \quad (17)$$
and the F(0, 5)-brane to the F(0, 5)bs-brane

\[
e^\phi = \Lambda^{1/2}, \quad b_{(6)} = A_{(1)} \wedge dy_1 \wedge \cdots \wedge dy_5
\]
\[
ds_{10}^2 = \Lambda [-dt^2 + ds_{SU(2)}^2] + ds^2(\mathbb{E}^5). \tag{18}
\]

By T-duality along \( \mathbb{R}^4 \) in the F(4, 1)bs-string, the F(4, 1)as-string is obtained, which is formally equivalent to its T-dual. Both are T-duals along the twisted directions to the corresponding vacuum solutions in type IIA/B with non-trivial identifications. Analogously, the new F(0, 5)bs-brane configuration also generates through T-duality a new type IIA solution, the F(0, 5)as-brane, which is again equivalent to its type IIB T-dual.

Having generated new type IIA configurations, we can understand all of them as coming from the Kaluza–Klein reduction of certain new M-theory configurations. In particular, the F(4, 1)as-brane and the F(3, 2)-brane come from the F(4, 2)-brane

\[
A_{(3)} = A_{(1)} \wedge dx_{11} \wedge dy
\]
\[
ds_{11}^2 = \Lambda^{1/3}[ds^2(\mathbb{E}^{4,1}) + ds_{SU(2)}^2] + \Lambda^{-2/3}[dx_{11}^2 + dy^2]. \tag{19}
\]

whereas the F(1, 4)-brane and the F(0, 5)as-brane from the F(1, 5)-brane

\[
*_{10}dA_{(3)} = dA_{(1)} \wedge dvol(\mathbb{E}^4)
\]
\[
ds_{11}^2 = \Lambda^{2/3}[-ds^2(\mathbb{E}^{1,1}) + ds_{SU(2)}^2] + \Lambda^{-1/3}[dx_{11}^2 + ds^2(\mathbb{E}^4)]. \tag{20}
\]

We must check the three form up there !!.

4.3. **F3-brane dualities.** The starting point is again the flat eleven dimensional geometry reduced on a circle whose action involves an \( SU(3) \subset SO(6) \) rotation acting in \( \mathbb{R}^6 \). By reducing and using T-duality in flat transverse directions we get the same vacuum structures in type IIA and IIB theories in the corresponding decompactifying
Figure 1. Web of dualities associated with an F5-brane. A bold line represents an S-duality transformation, while unidirectional lines indicate Kaluza–Klein reductions from M-theory, and bidirectional lines a T-duality transformation.
limits. Reducing along the twisted direction, we get the F3-brane

\[ e^\phi = \Lambda^{3/4} \]

\[ A_{(1)} = \frac{r^2}{R^2} \sum_{i=1}^{3} \frac{\Lambda_i}{\beta_i} d\varphi_i \]

\[ ds_{(10)}^2 = \Lambda^{1/2} \left\{ ds^2(\mathbb{E}^{3,1}) + ds_{SU(3)}^2 \right\} \]

\[ ds_{SU(3)}^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r^2}{\Lambda} \sum_{i=1}^{3} \frac{\Lambda_i}{\beta_i^2} d\varphi_i^2 \left[ \Lambda - \frac{r^2}{R^2} \Lambda_i \right] \]

\[ - \frac{r^4}{\sqrt{R^2}} \sum_{i \neq j} \frac{\Lambda_i \Lambda_j}{\beta_i \beta_j} d\varphi_i d\varphi_j. \]

Using T-duality iteratively along the worldspace directions of the F3-brane, we can generate a family of solutions F(p, 3 − p)-branes for \( p = 0, 1, 2 \) \(^5\) belonging to type IIA for \( p = 1 \) and to type IIB for \( p = 0, 2 \). They are jointly described by

\[ e^\phi = \Lambda^{p/4}, \quad A_{(4-p)} = A_{(1)} \wedge dy_{p+1} \wedge \cdots \wedge dy_3 \]

\[ ds_{(10)}^2 = \Lambda^{1/2} \left[ ds^2(\mathbb{E}^{p,1}) + ds_{SU(3)}^2 \right] + \Lambda^{-1/2} \left[ dy_{p+1}^2 + \cdots + dy_3^2 \right] \]

Notice that the F(0, 3)-brane is self-dual under S-duality, in particular it has a constant dilaton, whereas the F(2, 1)-brane gives rise to the F(2, 1)bs-brane described by

\[ ds_{(10)}^2 = ds^2(\mathbb{E}^{2,1}) + ds_{SU(3)}^2 + \Lambda^{-1} R^{-2} d\chi^2 \]

\[ e^{\phi'} = \Lambda^{-1/2} R^{-1} \]

\[ b_{(2)}' = -r^2 (\Lambda R^{-2})^{-1} \sum_{i=1}^{3} \frac{\Lambda_i}{\beta_i} d\varphi_i \wedge d\chi \]  \( (21) \)

Using T-duality along the longitudinal \( \mathbb{E}^2 \) subspace, the same solution in type IIA is generated, the F(2, 1)as-brane. Both are the T-duals of the corresponding IIA/B vacuum when applying a T-duality transformation along the twisted direction.

Finally, both the F(1, 2)-brane and the F(2, 1)as-brane can be understood as the Kaluza–Klein reduction of the F(2, 2)-brane in eleven dimensions

\[ C_{(3)} = A_{(1)} \wedge dy_2 \wedge dy_3 \]

\[ ds_{(11)}^2 = \Lambda^{1/3} \left[ ds^2(\mathbb{E}^{2,1}) + ds_{SU(3)}^2 \right] + \Lambda^{-2/3} \left[ dy_2^2 + dy_3^2 \right] \]  \( (22) \)

This finishes the web of dualities for the F3-brane, which is summarized in Figure 2.

\(^4\)In the following formulae, we have reintroduced a length scale \( R \) corresponding to the radius of the M-theory circle. The coordinate \( z \) along the Killing vector can be thought of as \( z = R\chi \). It is then also convenient to rescale the RR 1-form potential by \( R \), so that in the eleven-dimensional metric one has the combination \( d\chi + A \).

\(^5\)p=3 would correspond to the above F3-brane solution.
Figure 2. Web of dualities associated with an F3-brane. A bold line represents an S-duality transformation, while unidirectional lines indicate Kaluza–Klein reductions from M-theory, and bidirectional lines a T-duality transformation.
4.4. **F1-string dualities.** The starting eleven-dimensional geometry is given by

\[
    ds_{11}^2 = ds^2(\mathbb{E}^{1,1}) + \Lambda R^2 (d\chi + A)^2 + ds^2_{SO(8)}
\]

\[
    A_{(1)} = \frac{r^2}{\Lambda R^2} \sum_{i=1}^{4} \frac{\Lambda_i}{\beta_i} d\varphi_i
\]

\[
    ds_{SO(8)}^2 = dr^2 + r^2 (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2)
\]

\[
    + \frac{r^2}{\Lambda} \sum_{i=1}^{4} \frac{\Lambda_i}{\beta_i^2} d\varphi_i^2 \left[ \Lambda - \frac{r^2}{R^2} \Lambda_i \right] - \frac{r^4}{\Lambda R^2} \sum_{i \neq j} \frac{\Lambda_i \Lambda_j}{\beta_i \beta_j} d\varphi_i d\varphi_j.
\]

Given such a solution, we can either reduce it along the spacelike direction \((R_{\parallel})\) in the flat \(\mathbb{E}^{1,1}\) subspace, giving rise to the same construction applied to the flat ten dimensional vacuum in type IIA or along the twisted direction \((R_{\perp})\), to generate the F1-string

\[
    e^{\phi} = \Lambda^{3/4}, \quad C_{(1)} = A_{(1)}
\]

\[
    ds_{10}^2 = \Lambda^{1/2} \left[ ds^2(\mathbb{E}^{1,1}) + ds^2_{SO(8)} \right].
\]

By applying T-duality along its worldspace direction, the type IIB F(0,1)-string is generated

\[
    e^{\phi} = \Lambda^{1/2}, \quad C_{(2)} = A_{(1)} \wedge dy
\]

\[
    ds_{10}^2 = \Lambda^{1/2} \left[ -dt^2 + ds^2_{SO(8)} \right] + \Lambda^{-1/2} dy^2.
\]

(23)

Close to the core \((r^2 R^2 \ll 1 \forall i)\) where the supergravity description is reliable since the string coupling constant is weak, (23) is flat, whereas far away from the core \((r^2 R^2 \gg 1 \forall i)\), it is no longer conformal to a cylindrical metric, but to a singular metric whose \(dy^2\) coefficient vanishes. It is in any case not appropriate to use the supergravity description in that region since the string coupling constant blows up there and one should take into account higher order corrections in \(g_s\) into the classical supergravity equations of motion or use the S-dual description, the F(0,1)bs-string

\[
    e^{\phi} = \Lambda^{-1/2}, \quad b_{(2)} = -A_{(1)} \wedge dy
\]

\[
    ds_{10}^2 = -dt^2 + ds^2_{SO(8)} + \Lambda^{-1} dy^2.
\]

(24)

where the string coupling became weak again. Notice that the above solution is equivalent to the type IIB configuration obtained by applying a T-duality transformation along the twisted direction on the type IIA vacuum with topological non-trivial identifications.

Of course, one could have started from the flat background in type IIB with non-trivial identifications and generate the F(0,1)as-string, which is formally equivalent to (24). This new type IIA solution can
be seen as the Kaluza–Klein reduction of the $F(0, 2)$-brane

$$C_{(3)} = -A_{(1)} \wedge dy_1 \wedge dy_2$$

$$ds_{11}^2 = \Lambda^{1/3} \left[ -dt^2 + ds^2_{SO(8)} \right] + \Lambda^{-2/3}(dy_1^2 + dy_2^2).$$

This finishes the web of dualities obtained from the F1-string and which is summarized in Figure 3.

4.5. **N7-brane dualities.** The purpose of this subsection is to give a preliminary analysis of the family of solutions related to the N7-brane (16). It will be convenient to introduce the following notation

$$h = \Lambda^{1/2} \left( 2dx^+dx^- - (x^1)^2(dx^-)^2 \right) + \Lambda^{1/2} \left( dx^1 + x^1x^-dx^- \right)^2.$$

where $\Lambda$ is the one appearing in (14).

By T-duality along $E^7$ in (16), one generates a family of $N(p, 7-p)$-branes $p = 0, \ldots, 7$ whose dilaton is

$$\phi = \frac{p - 4}{4} \log \Lambda,$$

RR field strength

$$F_{(9-p)} = F \wedge d Vol E^{7-p},$$

and metric

$$g = \Lambda^{1/2}ds^2(E^p) + \Lambda^{-1/2}ds^2(E^{7-p}) + h,$$  \hspace{1cm} (25)
where $F$ is the one defined in (16).

One can then act with S-duality on the new type IIB configurations ($p = 6, 4, 2, 0$). The $N(4,3)$-brane is $S$ self-dual (so it has constant dilaton) whereas the $N(6,1)$-brane gives rise to the $N(6,1)bs$-brane

$$\phi = -\frac{1}{2} \log \Lambda \quad \text{and} \quad H = - F \wedge dx^7,$$

with metric

$$g = ds^2(\mathbb{E}^6) + \Lambda^{-1} d(y^7)^2 + \Lambda^{-1/2} h.$$  \hspace{1cm} (26)

Analogously, the $N(2,5)$-brane gives rise to the $N(2,5)bs$-brane

$$\phi = \frac{1}{2} \log \Lambda \quad \text{and} \quad H = 2 dx^- \wedge d\text{vol}(\mathbb{E}^2),$$

with metric

$$g = \Lambda ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^5) + \Lambda^{1/2} h.$$  \hspace{1cm} (27)

whereas the $N(0,7)$-brane gives rise to the $N(0,7)bs$-brane

$$\phi = \log \Lambda \quad \text{and} \quad F^{(1)} = 2 dx^-,$$

with metric

$$g = \Lambda^{1/2} ds^2(\mathbb{E}^7) + \Lambda h.$$  \hspace{1cm} (28)

One can proceed systematically, uplifting the new type IIA configurations obtained on the process to get new M-theory configurations such as the $N(6,2)$-brane described by

$$F^{(4)} = F \wedge d\text{vol}(\mathbb{E}^2),$$

and metric

$$g = \Lambda^{1/3} ds^2(\mathbb{E}^6) + \Lambda^{-2/3} ds^2(\mathbb{E}^2) + \Lambda^{-1/6} h.$$  \hspace{1cm} (29)

There are many more configurations that go beyond the scope of our present analysis, but deserve further study since, among many other interests, they might lead to more general Ansatz to solve the eleven- and ten-dimensional supergravity equations of motion.

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