Reheating and turbulence

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We show that the "turbulent" particle spectra found in numerical simulations of the behavior of matter fields during reheating admit a simple interpretation in terms of hydrodynamic models of the reheating period. We predict a particle number spectrum $n_k \propto k^{-\alpha}$ with $\alpha \sim 2$ for $k \to 0$.

Principal PACS No 98.80.Cq; additional PACS nos: 04.62.+v, 11.10.Wx, 47.27

I. INTRODUCTION

The reheating period in the Early Universe stands out as a challenge to theorists due to the close interrelationship of nonlinear and gravitational phenomena in its unfolding (see [1–4], henceforth "papers 1-4"). The observation that, due to the high occupation numbers produced during preheating, most of the physics of reheating may be understood in terms of the behavior of nonlinear classical waves, has been the key to substantial progress. The authors of papers 1-4 have undertaken systematic numerical simulations of the behavior of matter fields during reheating, finding spectra of occupation numbers reducing to simple power laws both in the infrared and ultraviolet limits. They have remarked that this behavior shows a connection between the physics of reheating and the phenomena of weak turbulence [5], but to the best of our knowledge they have offered no theoretical prediction for the exponents involved during the turbulent phase. Our goal is to provide these theoretical estimates.

In this paper we shall follow this same trend of ideas, by observing that, from the macroscopic point of view, a stochastic ensemble of classical waves may be described by a conserved energy momentum tensor subject to the Second Law of thermodynamics. There is therefore an equivalent fluid description, consisting of a fluid whose energy momentum tensor and equation of state reproduce the observed ones for the microscopic fluctuations. Solving the dynamics of this equivalent fluid yields answers to all relevant questions concerning the behavior of the actual fluctuations.

An immediate consequence of energy momentum conservation and the Second Law is that when velocities are low, the phenomenological fluid may be described within the Eckart theory of dissipative fluids [6] (for an analysis of the limitations of Eckart’s theory see [7]). It follows that it obeys a continuity equation and a curved space time Navier-Stokes one. The "turbulent" spectra found in numerical simulations correspond to the self-similar solutions discussed long ago by Chandrasekhar [8]. The Chandrasekhar solutions are built on the Heisenberg closure hypothesis [9] (see [10–12] for a general discussion of turbulence). They were generalized to Friedmann - Robertson - Walker (FRW) backgrounds by Tomita et al [13]. These solutions agree with the Kolmogorov 1941 theory in the inertial range [12], failing to reproduce observations for very small eddies. Fortunately, we are most interested in the opposite limit of very large eddies, where it is trustworthy (we wish to point out that the applicability of Kolmogorov’s spectrum to large scale turbulence should not be taken for granted [14]). With minor adjustments, Tomita’s analysis of turbulence decay in FRW space times also provides a solution to the evolution of our equivalent fluid.

The rest of the paper is organized as follows. In next section we provide a brief summary of hydrodynamics in flat and expanding universes, in order to set up the language for the rest of the paper, and introduce the self similar solutions. In section III we proceed to discuss the equivalent fluid description of field fluctuations, and how to extract the particle spectrum therefrom. In section IV we place the self-similar solutions in the context of reheating. We state our main conclusions in the final section.

II. HYDRODYNAMIC FLOWS

A. Flows in flat space time

The equations governing the dynamics of a fluid in local thermodynamic equilibrium are the continuity and Navier-Stokes ones, which, in the case of flat space time, read:

$$\frac{\partial \rho}{\partial t} + (\mathbf{U} \cdot \nabla) \rho = 0$$  (1)
\[
\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{U}
\]  
(2)

where we have assumed incompressibility, valid when typical velocities are much smaller than the sound velocity; \( \nu = \eta/\rho \) is the kinematic shear viscosity. The transition from laminar to turbulent motion can be universally described by the dimensionless "Reynolds" number:

\[
R = \frac{UL}{\nu}
\]  
(3)

where \( U \) is a typical velocity and \( L \) a typical length scale. This number represents the order of magnitude of the ratio of the inertial to the viscous term. Low Reynolds numbers correspond to laminar motion, while high ones suggest turbulent behavior.

In general, the velocity profile displays variations in space and time. This implies that the flow must be described probabilistically. Thus, each quantity involved in (1-2) is divided in its mean value and a fluctuation from it; for example, we write \( \mathbf{U} = \bar{U} + u \), where \( u \) stands for the fluctuating part of the velocity. In the case where motion is isotropic, the mean value \( \bar{U} \) for the velocity must be zero, since otherwise there would be a preferred direction.

To analyze the system’s behavior, we define the two-point one-time correlation function for the velocity:

\[
R_{ij}(\mathbf{x}, \mathbf{x}', t) = \langle u_i(\mathbf{x}, t)u_j(\mathbf{x}', t) \rangle
\]  
(4)

In the case of homogeneous and isotropic motion, this correlation must be only a function of the time \( t \) and the distance between \( \mathbf{x} \) and \( \mathbf{x}' \), i.e. \( R_{ij}(\mathbf{x}, \mathbf{x}', t) = R_{ij}(r, t) \), where \( r = |\mathbf{x} - \mathbf{x}'| \). Observe that \( R_{ii}(0, t) \) (summation over repeated indices must be understood) is twice the average energy density of the flow at time \( t \). From (2) we obtain the equation that this correlation must obey, namely:

\[
\frac{\partial}{\partial t} R_{ij}(r, t) = T_{ij}(r, t) + P_{ij}(r, t) + 2\nu \nabla^2 R_{ij}(r, t)
\]  
(5)

where

\[
P_{ij}(r, t) = \frac{1}{\rho} \left( \frac{\partial}{\partial r_i} \langle p(x, t)u_j(x', t) \rangle - \frac{\partial}{\partial r_j} \langle p(x', t)u_i(x, t) \rangle \right)
\]  
(6)

and

\[
T_{ij}(r, t) = \frac{\partial}{\partial r_k} \langle u_i(x, t)u_k(x, t)u_j(x', t) - u_i(x, t)u_k(x', t)u_j(x', t) \rangle
\]  
(7)

The tensor \( T_{ij} \) comes form the inertia term in Navier-Stokes equation and, as it involves a product of third order in the velocity, reflects the fact that there is not a close set of equation for the correlations of successive orders but there is a hierarchy of equations instead. The problem of closing that hierarchy is known as the "moment closure problem" [15]. Let us call \( \Phi_{ij}(k, t) \) the Fourier transform of \( R_{ij}(r, t) \). Then the energy density becomes

\[
\frac{1}{2} R_{ii}(0, t) = \int E(k, t) \; dk,
\]

where

\[
E(k, t) = \frac{1}{2} \int \Phi_{ii}(k, t) \; k^2 \; d\Omega(k)
\]  
(8)

is the energy density stored in eddies of size \( k^{-1} \). Defining \( \Gamma_{ij} \) as the Fourier transform of \( T_{ij} \), we obtain from (5) the equation of balance of the energy spectrum:

\[
-\frac{\partial}{\partial t} E(k, t) = T(k, t) + 2\nu k^2 E(k, t)
\]  
(9)

where

\[
T(k, t) = -\frac{1}{2} \int \Gamma_{ii}(k, t) \; k^2 \; d\Omega(k)
\]  
(10)

The inertia term \( T(k, t) \) is the one that contains the mode-mode interaction, and its effect is to drain energy from the more energetic modes -typically the bigger ones- to the ones where there is major viscous dissipation -the smaller ones-. 
B. Flows in expanding universes

For a curved space-time, in particular a Friedmann - Robertson - Walker (FRW) Universe with zero spatial curvature \((ds^2 = -dt^2 + a^2(t) \,(dx^2 + dy^2 + dz^2))\), the generalization of the above arguments has been considered by many authors [16–20]. We follow Tomita et al.’s analysis [13], in which they obtain the solution for the energy spectrum in the case of homogeneous, isotropic and incompressible turbulence.

In a generic space time, we describe fluid flow from the energy density \(\rho\), pressure \(p\) and four velocity \(U\). The symmetries of the FRW solution suggest using instead the comoving three velocity \(u^i = U^i/U^0\); if \(U^i \ll U^0\) the flow is non relativistic, and if \(\nabla u = 0\), it is incompressible \((u = (u^1, u^2, u^3))\). Later on, we shall also use the physical three velocity \(v = a(t)u\).

The corresponding continuity and Navier-Stokes equations for a Robertson-Walker background are obtained by the condition of conservation of the energy-momentum tensor [6]. For a non relativistic incompressible fluid, with shear viscosity \(\eta = \nu (p + \rho)\) (but no bulk viscosity), these reduce to:

\[
\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a} (p + \rho) = 0
\]

\[
\frac{\partial u}{\partial t} + \left[ (u \cdot \nabla) - \frac{\partial \ln ((p + \rho)a^3)}{\partial t} \right] u = - \frac{\nabla p}{a^2 (p + \rho)} + \frac{1}{a^2} \nu \nabla^2 \mathbf{u}
\]

where we have assumed that \(p + \rho\) depends only on time. For the physical three velocity \(v\), the corresponding Navier-Stokes equation reads:

\[
\frac{\partial v}{\partial t} + \left[ \frac{1}{a} (v \cdot \nabla) + \frac{\partial \ln ((p + \rho)a^4)}{\partial t} \right] v = - \frac{\nabla p}{a (p + \rho)} + \frac{1}{a^2} \nu \nabla^2 \mathbf{v}
\]

In obtaining (11)-(13) we have neglected possible perturbations to the FRW metric. The corresponding equations considering fluctuations in the metric \((g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu})\) have been obtained by Weinberg [6]. The continuity equation is not corrected by gravitational perturbations, while in the Navier-Stokes equation the metric fluctuations appear explicitly only within the shear viscosity term. It can be demonstrated that these terms involving metric fluctuations are negligible for scales that are inside the horizon [21]. For scales bigger than the Hubble radius, since dissipation through viscosity is not effective anyway, we may still use the unperturbed Navier-Stokes equation.

The operation of Fourier transforming in the case of a Robertson-Walker cosmology is done in terms of comoving wave-numbers. In doing so, the following equation for the energy spectrum is obtained:

\[
- \frac{\partial}{\partial t} E(k, t) = T(k, t) + 2 \left\{ \frac{\nu k^2}{a^2} + \frac{\partial \ln ((p + \rho)a^4)}{\partial t} \right\} E(k, t)
\]

where the relationship between \(E(k, t)\) and \(\Phi_{ij}(k, t)\) as well as between \(T(k, t)\) and \(\Gamma_{ij}(k, t)\) is the same as that for a flat space-time, if we define \(R_{ij}\) and \(T_{ij}\) from correlations of physical quantities, as follows:

\[
R_{ij}(r, t) = a^2 \langle u_i(x, t)u_j(x + r, t) \rangle,
\]

\[
T_{ij}(r, t) = a^4 \frac{\partial}{\partial r_k} \left( \langle u_i(x, t)u_k(x, t)u_j(x + r, t) \rangle - \langle u_i(x, t)u_k(x + r, t)u_j(x + r, t) \rangle \right)
\]

C. Self similar flows in flat and expanding universes

As we have seen in the previous section, the key element in the description of the flow is the energy spectrum \(E(k)\) (Eq.(8)), which is the solution of the balance equation (Eq.(9)). In it, the right hand side contains the viscous dissipation as well as the inertial force \(T(k, t)\). The overall effect of this term is to transfer energy from a given scale to smaller ones through mode - mode coupling; thus it is natural to model the action of the inertia term as a source of viscous dissipation, where the effective turbulent viscosity for a given mode depends on the motion of all smaller eddies [11]. By providing closure, that is, writing this effective viscosity in terms of the spectrum itself, a closed evolution equation for \(E(k)\) is obtained. Concretely, Heisenberg [9] proposed the ansatz
\[ \int_0^k T(k', t) \, dk' = 2\nu(k, t) \int_0^k E(k', t) \, k'^2 \, dk' \quad (17) \]

where

\[ \nu(k, t) = A_{flat} \int_k^\infty \frac{E(k', t)}{k'^3} \, dk' \quad (18) \]

and \( A_{flat} \) is a dimensionless constant. With this hypothesis (known as the Heisenberg hypothesis) as the solution to the closure problem, Chandrasekhar \[8\] has obtained the energy spectrum for decaying turbulence, assuming that there is a stage in the decay where the bigger eddies have sufficient amount of energy to maintain an equilibrium distribution, thus requiring that the solution for the spectrum should be self similar. With this consideration into account he obtained an energy spectrum:

\[ E(k, t) = \frac{1}{A_{flat} k_0^2 t_0^2} \sqrt{\frac{t}{t_0}} F\left( \frac{k\sqrt{t}}{k_0}\right) \quad (19) \]

where \( k_0 \) and \( t_0 \) are initial conditions (namely, the wave number corresponding to the bigger eddy and its typical time of evolution). The function \( F \) obeys the equation

\[ \frac{1}{4} \int_0^x \left[ F(x') - x' \frac{dF(x')}{dx'} \right] \, dx' = \left\{ \nu k_0^2 t_0 + \int_x^\infty \frac{\sqrt{F(x')}}{x'} \, dx' \right\} \int_0^x F(x') x'^2 \, dx' \quad (20) \]

which predicts a Kolmogorov type behavior for an inviscid fluid \((R \to \infty, R = (\nu k_0^2 t_0)^{-1})\) in the ultraviolet limit:

\[ F(x) \to const \, x^{-5/3} \, (\nu = 0, \, x \to \infty) \quad (21) \]

While for nonzero viscosity:

\[ F(x) \to const \, x^{-7} \, (\nu \neq 0, \, x \to \infty) \quad (22) \]

In the infrared limit, \( F \) has the universal behavior \( F(x) = 4x \, (x \ll 1) \), and thus we find a linear energy spectrum for \( k\sqrt{t} \ll k_0\sqrt{t_0} \).

Chandrasekhar’s self similar solutions are easily generalized to flows in expanding Universes. The dependence on time and wave-number for the self similar energy spectrum is \[13\]

\[ E(k, t) = \frac{1}{2} v_t^2 \left( \frac{\lambda(t)}{\lambda} \right) F(\lambda k) \quad (23) \]

where \( \lambda \) and \( v_t \) are respectively the Taylor’s microscale and an average turbulent velocity, defined as:

\[ \lambda^2(t) = 5 \frac{\int E(k, t) \, dk}{\int E(k, t) \, k^2 \, dk} \quad \frac{1}{2} v_t^2(t) = \int E(k, t) \, dk \quad (24) \]

The second equation implies the normalization condition

\[ \int_0^\infty F(x') \, dx' = 1 \quad (25) \]

To obtain a self similar flow reducing to Eq. (19) in the flat space limit, we must require the time evolution laws:

\[ \lambda^2(t) = \lambda_i^2 + 10 \int_t^\infty \frac{\eta}{(p+i\rho)a^2} \, dt \quad v_t = v_t(i) \left( \frac{(p+i\rho)a_i^2}{(p+i\rho)a^2} \right) \frac{\lambda_i}{\lambda(t)} \quad (26) \]

The equation which determines the function \( F(\lambda k) \) in (23) turns out to be

\[ \int_0^x \left[ F(x') - x' \frac{dF(x')}{dx'} \right] \, dx' = \left\{ \frac{2}{5} + A \int_x^\infty \frac{\sqrt{F(x')}}{x'^{3/2}} \, dx' \right\} \int_0^x F(x') x'^2 \, dx' \quad (27) \]
where $A$ is a constant. This equation has the same structure as in flat space time, Eq. (20), which means that assuming Heisenberg’s hypothesis the spectrum is linear in $k$ for length scales much bigger than the Taylor’s microscale.

We wish to point out an ambiguity concerning the meaning of Heisenberg’s hypothesis in the case of curved spaces. For flat space time, the proportionality between the integral up to a certain wave number $k$ of the inertia and the viscous forces is given by (17) and (18). In the case of a FRW space time, the autosimilar solution required by Tomita et al. (23) needs a time dependent dimensionless constant $A_{\text{curv}} = 5A\eta a^2$ for the consistency of the solution. This product does remain constant only if the dynamic shear viscosity evolves in time proportional to $a^{-2}$. Thus unless this is the case, the solution we described looks like a natural curved space generalization of the Heisenberg - Chandrasekhar solution, but does not admit the same physical interpretation.

D. Solving for the spectrum

Let us analyze in more detail the solutions of eq. (27). We assume the normalization eq. (25). By taking the $x \to \infty$ limit in eq. (27) we find

$$
\int_0^\infty F(x')x'^2dx' = 5
$$

(28)

Taking a derivative of eq. (27) we get

$$
1 - x \frac{F'}{F} = x^2 \left\{ \frac{2}{5} + A(G - H) \right\}
$$

(29)

where

$$
G = \int_x^\infty \sqrt{\frac{F(x')}{x'}^{3/2}} dx'
$$

(30)

$$
H = \frac{1}{\sqrt{F(x)}} \int_0^x F(x')x'^2dx'
$$

(31)

Let us consider first the $x \to 0$ limit. Assume $F \propto x^\alpha$. The left hand side of eq. (29) tends to a finite limit $1 - \alpha$. On the right hand side, $G$ and $H$ behave as $x^{(\alpha - 1)/2}$, so if $\alpha > 0$, this side goes to zero. We must therefore have $\alpha = 1$, and

$$
F \sim Cx, \quad x \to 0
$$

(32)

were $C$ is some constant.

In the $x \to \infty$ limit, assume again a power law behavior $F \propto x^{-\beta}$. Now $G \to 0$, so we must have $H \to 2/5A$. From eq. (28) we know that in this limit $H \sim 5/\sqrt{F(x)}$, so we must have $\beta = 7$ and

$$
F \sim \left( \frac{25A}{2} \right)^2 x^{-7}, \quad x \to \infty
$$

(33)

Taking into account both limiting behaviors and eq. (25), the function $F$ may be approximated as

$$
F[x] = \frac{x}{[\alpha + \beta x^4]^2}, \quad \alpha = \left[ \frac{25A\pi^2}{120} \right]^{1/3}, \quad \beta = \frac{2}{25A}
$$

(34)

III. EQUIVALENT FLUID FOR FIELD FLUCTUATIONS

After establishing the basic necessary notions for the description of hydrodynamic flows, our goal is to associate an equivalent fluid description to field fluctuations, and to derive the particle spectrum therefrom. Our first step is to obtain the energy density, pressure and velocity of this fluid as functionals of the quantum state of the field.
For simplicity, we shall consider the theory of a single, self-interacting scalar field \( \phi \), minimally coupled to gravity. The action is

\[
S = \int d^4x \sqrt{-g} \left\{ \left( \frac{1}{2} \right) \partial_\mu \phi \partial^\mu \phi - V[\phi] \right\}
\]

where \( V[\phi] \) is renormalized effective potential. The energy-momentum tensor is associated to the Heisenberg operator

\[
T_Q^{\mu\nu} = \frac{(-2)}{\sqrt{-g}} \delta S \delta g_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g^{\mu\nu} \left\{ \left( \frac{1}{2} \right) \partial_\rho \phi \partial^\rho \phi + V[\phi] \right\}
\]

(35)

The macroscopic behavior of the field, however, may be described in terms of a c-number energy-momentum tensor

\[
T^{\mu\nu} = T_C^{\mu\nu} + T_S^{\mu\nu}
\]

where

\[
T_C^{\mu\nu} = \langle T_Q^{\mu\nu} \rangle_Q
\]

(36)

and \( T_S^{\mu\nu} \) is a stochastic component with zero mean and self-correlation

\[
\langle T_S^{\mu\nu} T_S^{\rho\sigma} \rangle_S = \frac{1}{2} \left\{ \langle T_Q^{\mu\nu} T_Q^{\rho\sigma} \rangle_Q - T_C^{\mu\nu} T_C^{\rho\sigma} \right\}
\]

(37)

In these equations, \( \langle \rangle_S \) denotes a stochastic average, while \( \langle \rangle_Q \) is the average with respect to the quantum state of the field. Following Landau, we define the fluid four-velocity \( U^\mu \) and energy density \( \rho \) as the (only) time-like eigenvector of \( T^{\mu\nu} \) and (minus) its corresponding eigenvalue

\[
T^{\mu\nu} U_\nu = -\rho U^\mu
\]

(38)

Introducing the pressure \( p = p(\rho) \) as given by the equilibrium equation of state (our theory does not have a conserved particle number current, and therefore the equation of state is barotropic), we may decompose

\[
T^{\mu\nu} = \rho U^\mu U^\nu + p \Delta^{\mu\nu} + \tau^{\mu\nu}
\]

(39)

where \( \Delta^{\mu\nu} = g^{\mu\nu} + U_\mu U_\nu \) and by construction \( \tau^{\mu\nu} U_\nu = 0 \). Since \( \tau^{\mu\nu} \) vanishes by definition in the equilibrium state, it may be parametrized in terms of deviations from equilibrium. Remaining within the so-called first order formalism [22,23], we may write

\[
\tau^{\mu\nu} = -\eta H^{\mu\nu} - \zeta U^\rho \Delta^{\mu\nu}; \quad \eta, \zeta \geq 0
\]

where

\[
H^{\mu\nu} = \frac{1}{2} \Delta^{\mu\lambda} \Delta^{\nu\sigma} \left[ U_{\lambda,\sigma} + U_{\sigma,\lambda} - \frac{2}{3} \Delta_{\lambda\sigma} U^\rho \right]
\]

(40)

and \( \eta \) and \( \zeta \) are the shear and bulk viscosity coefficients, respectively.

Let us decompose each quantity in a mean component (denoted by a \( C \) subscript) and a fluctuation (denoted by a \( S \)). If the quantum state shares the symmetries of the FRW background, then \( U^i_0 = 0 \). Since \( U^2 = -1 \) holds identically (as opposed to “in the mean”) we must have

\[
(U_C^0)^2 + \langle (U_S^0)^2 \rangle - a^2 \langle U_S^i U_S^i \rangle_S = 1
\]

(41)

The second equation shows that \( U_S^0 \) is a higher order fluctuation with respect to \( U_S^i \). If we remain at linear order, then, we may approximate \( U^0 = U^0_C \). Observing that all mean values are homogeneous and isotropic, we see that \( \tau^{0i} \) is also a higher order fluctuation. We find

\[
2U^0_C U^0_S - \langle U^i_S U^i_S \rangle_S = 0
\]
\[ T^{0i} = T^{0i}_S = (\rho + p) C U^i_S \]  

and therefore the velocity correlation

\[ R^{0j}(\vec{r}, t) = \frac{a^2(t)}{2(\rho + p) C(t)^2} \langle \left\{ T^{0i}_Q(\vec{r}, t) , T^{0j}_Q(0, t) \right\} \rangle_Q \]  

This is the key equation linking the quantum and stochastic descriptions. To estimate the velocity correlation, let us assume that, after integrating out the hard modes, the soft modes of interest may be described in terms of quasi free, long lived excitations with an effective mass \( M^2(t) \). Then

\[ T^{0i}_Q = \left( \frac{-1}{a^2(t)} \right) \partial_i \phi \partial_t \phi \]  

The fluctuations are Gaussian to a very good approximation, and therefore

\[ \langle \left\{ T^{0i}_Q(\vec{r}, t) , T^{0j}_Q(0, t) \right\} \rangle_Q = \left( \frac{1}{a^4(t)} \right) \left\{ \partial_{ij} G^+ \partial_{ij} G^+ + \partial_{ij} G^+ \partial_{ij} G^+ + (\vec{r} \rightarrow -\vec{r}) \right\} \]  

where \( G^+ (x, x') \) is the positive frequency propagator

\[ G^+ ((\vec{r}, t) , (0, t)) = \langle \phi (\vec{r}, t) \phi (0, t) \rangle_Q \]  

(\( \partial_{i,t} \) stand for derivatives with respect to the first argument, while \( \partial_{i',t'} \) stand for derivatives with respect to the second argument of \( G^+ \)). Let us decompose the soft field into modes

\[ \phi (\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{r}} \phi_k(t) \]  

At any time \( t_0 \) we may introduce positive frequency adiabatic modes defined by [24]

\[ f_k(t) = \frac{1}{\sqrt{2\omega_k(t)}} \exp \left\{ -i \int_{t_0}^{t} dt' \omega_k(t') \right\} \]  

where

\[ \omega_k^2(t) = \frac{k^2}{a^2(t)} + M^2(t) \]  

and decompose the mode amplitude \( \phi_k(t) \) into positive and negative frequency components

\[ \phi_k(t) = f_k(t) A_k(t) + f_k^*(t) A_k^+(t) \]  

\[ \partial_t \phi_k(t) = -i \omega_k(t) \left\{ f_k(t) A_k(t) - f_k^*(t) A_k^+(t) \right\} \]  

Let us define the spectrum

\[ n_k(t) = \langle A_k^+(t) A_k(t) \rangle_Q \]  

(because the quantum state is isotropic, the spectrum depends only on \( k \)) and assume that

\[ \langle A_{k'}(t) A_k(t) \rangle_Q = \langle A_{k'}^+(t) A_k^+(t) \rangle_Q = \langle A_{k'}^+(t) A_k(t) \rangle_Q \bigg|_{k' \neq k} = 0 \]  

(this happens, for example, if the different modes acquire random phases through interaction with an environment). Then
\[ G^+((\vec{r}, t), (0, t')) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{r}} \left\{ f_k(t) f_{k'}^*(t') (1 + n_k(t)) + f_k^*(t) f_k(t') n_k(t) \right\} \] (58)

And

\[ \partial_t G^+((\vec{r}, t), (0, t')) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{r}} \left(-i\omega_k(t)\right) \left\{ f_k(t) f_{k'}^*(t') (1 + n_k(t)) - f_k^*(t) f_k(t') n_k(t) \right\} \] (59)

Observe that only the vacuum part contributes in the coincidence limit \( t' \to t \). In the large occupation numbers regime we are interested in, this is negligible, and we get

\[ \langle \{ T_Q^{ij}(\vec{r}, t), T_Q^{ij}(0, t) \} \rangle_Q \sim \left(\frac{1}{a^4(t)}\right) \left\{ \partial_{ij}^2 G^+ \partial_{ii'}^2 G^+ + (\vec{r} \to -\vec{r}) \right\} \] (60)

where

\[ \partial_{ij}^2 G^+ = \frac{1}{3} \delta_{ij} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{r}} \left( \frac{k^2}{\omega_k(t)} \right) n_k(t) \] (61)

\[ \partial_{ii'}^2 G^+ = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{r}} \omega_k(t) n_k(t) \] (62)

We may now write down the Fourier transform of the velocity self correlation

\[ \Phi^{ij}(k, t) = \frac{\delta^{ij}}{3a^2(t)(\rho + p)^2 C(t)} \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p(t)}{\omega_{|\vec{k} - \vec{p}|}(t)} \left| \vec{k} - \vec{p} \right|^2 n_p(t) n_{|\vec{k} - \vec{p}|}(t) \] (63)

In principle, this is an integral equation relating the spectrum to the energy self-correlation. It simplifies if we consider spectra which are strongly peaked towards the infrared. In this case, the integral is dominated by the small \( p \) region, and we get

\[ \Phi^{ij}(k, t) = \frac{\rho(t) \delta^{ij}}{3a^2(t)(\rho + p)^2 C(t)} \left( \frac{k^2}{\omega_k(t)} \right) n_k(t) \] (64)

Finally, we may relate the particle spectrum to the turbulent energy spectrum

\[ E(k, t) = \frac{2\pi \rho(t)}{a^2(t)(\rho + p)^2 C(t)} \left( \frac{k^4}{\omega_k(t)} \right) n_k(t) \] (65)

**IV. REHEATING**

In the previous sections, we have analyzed on one hand self-similar turbulent flows in expanding universes, and on the other hand have given the rule to translate the turbulent energy spectrum into a particle number distribution. We must now show that the foregoing analysis is relevant to plausible models of the reheating period, and use it to predict the likely shape of the final particle spectrum.

We have in mind a generic model of reheating were parametric resonance or some other similar mechanism dumps essentially all the energy density stored in the inflaton field during inflation into quantum fields, resulting in a spectrum centered around a comoving scale \( k_0 \) at times \( t_0 \). Our goal is to analyze the long time behavior of the spectrum.

Let us analyze the behavior of the quantum fields in terms of weakly-interacting excitations. The spectrum is characterized by its center \( k_0 \) and by the total initial energy density \( \rho_0 \). Mutual interaction will contribute to the field self-energy, so we shall have an effective mass \( M^2(t) \). Since we expect the end product of preheating will be the lightest fields, we shall assume \( M^2 \ll k_0^2 \). Integrating equation (65) over \( k \), we find the mean velocity in the equivalent fluid flow as

\[ v_{\text{tu}}^2 \sim \frac{\rho_0 p_0}{(\rho_0 + p_0)^2} \leq c_s^2 \] (66)
\[ c_s^2 \sim \frac{k_0^{2\text{phys}}}{k_0^{2\text{phys}} + M_0^2} \]  

\((k_0^{\text{phys}} = k_0/a(t))\) is the speed of sound, so the flow may be regarded as incompressible. We do not have a reliable estimate of the field viscosity, as the transport coefficients are usually investigated under near equilibrium conditions \[25\], so we shall take \(A\) in eq. (27) as a free parameter. As it turns out, agreement with the numerical results reported by papers 1-4 is obtained for \(A \sim 3\), which corresponds to the weakly turbulent regime.

The distribution of occupation numbers is found from eq. (65), where the energy spectrum is given by eq. (23). In the light field limit \(M^2 \ll k_0^2\), we find \(\omega_k \sim k\). Then eq. (32) implies \(n_k \sim k^{-2}\) for \(k \to 0\), and eq. (33) implies \(k^{-10}\) for \(k \to \infty\). This theoretical prediction for the exponents involved is the main result of this paper.

In Fig. 1 we show the full particle spectrum based on the approximation eq. (34) for the function \(F\). We have scaled the plot to make it easiest to compare with papers 1-4. Momentum is measured in units of \(a^{-1}10^{12}\text{GeV}\) (where \(a\) is the scale factor) \[3\], and we have chosen \(\lambda \sim a^{1.8} \times 10^{-13}\text{GeV}^{-1}\). Observing that in the ultrarelativistic limit the speed of flow \(v_t \sim 1\) and \(p \sim \rho\), integrating the particle density times \(k^3\) shows that the total energy density in the flow is \(\rho \sim 10^9 \times (10^{12}\text{GeV})^4\). The equivalent black-body temperature is then somewhat less than \(10^{15}\text{GeV}\), which is a reasonable value for the reheating era, and high enough to justify the neglect of all masses.

Comparison with the results in papers 1-4 is meaningful in the early time regime. Turbulence is necessarily a transient phenomenon. As time evolves, we expect the field will eventually thermalize, and the spectrum will get closer to a Rayleigh-Jeans tail, \(n_k \sim k^{-1}\) when masses are negligible. The several plots presented in ref. \[3\], where indexes go from 1.7 to slightly over 1, capture the transition from turbulence to equilibrium. Since the same plots show that earlier spectra are steeper (see also Fig. 1 in \[2\]) this is in satisfactory agreement with the prediction from self-similar flows.

Fig. 1: Log-Log plot of the particle spectrum, as given by eq. (65). The energy spectrum is given by (23), where the function \(F\) is given by eq. (34). We have chosen \(A = 3\), the Taylor microscale \(\lambda \sim 0.18 \times 10^{-13}\text{GeV}^{-1}\), and have scaled the spectrum to make it easiest to compare against the results presented in \[2\] and \[3\].

V. FINAL REMARKS

In this paper, we have shown that the self similar flows studied by Heisenberg, Chandrasekhar and Tomita may be used to provide an interpretation of the "turbulent" spectra found in papers 1-4. The hydrodynamic model predicts scale invariant spectra \(n_k \sim k^{-\alpha}\) both in the infrared and ultraviolet limits, with \(\alpha \sim 2\) in the former, and 10 in the latter regime. Agreement with the early time results presented in papers 1-4 is satisfactory.

The connection of hydrodynamics to the behavior of fluctuations during reheating has interest of its own, as it provides an alternative to brute force quantum field theoretic calculations, and also yields physical insight on the macroscopic behavior of quantum fields in the Early universe. The equivalent fluid method may be used to advantage also in other regimes, such as the inflationary period itself \[26\]. Moreover, it opens up a wealth of new phenomena, such as intermittence \[11\] and shocks \[27\], which are not apparent in the customary treatments. We will continue
our research in this field, which promises a most rewarding dialogue between cosmology, astrophysics, and nonlinear physics at large.

VI. ACKNOWLEDGMENTS

We thank Mario Castagnino for rekindling our interest in the subject.
EC has been partially supported by Universidad de Buenos Aires, CONICET, ANPCyT under project PICT-99 03-05229 and Fundación Antorchas. MG has been partially supported by National Science Foundation grant PHY97-22022.