I. INTRODUCTION

Recently there has been an interest of models in higher dimensions with Lorentzian vacuum which generalize the

Supersymmetry Phenomenology in Various Dimensions
II. GENERAL Fp-BRANES

In this section we analyse equations governing Fp-branes associated with the flux of a q-form field strength. The system contains a graviton, a q-form field strength, \( F_{[q]} \), and a dilaton scalar, \( \phi \), coupled to the form field with the coupling constant \( a \). This is a general framework which encompasses the bosonic sector of various supergravity theories, coming from a truncation of the low energy limit of M-theory and string theories, by a certain choice of the dimension \( D \), the rank of form field \( q \), and the dilaton coupling \( a \). In the Einstein frame, the action is given by

\[
S = \int d^D x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2q!} \partial_{\mu} \phi F^{\mu}_{[q]} \right).
\]

(1)

This action is invariant under the following discrete S-duality:

\[
g_{\mu\nu} \to g_{\mu\nu}, \quad F \to e^{-a\phi} s F, \quad \phi \to -\phi,
\]

(2)

where \( s \) denotes a \( D \)-dimensional Hodge dual. This may be used to construct electric versions of magnetic fluxbranes and vice versa. The equations of motion, derived from the variation of the action with respect to the individual fields, are

\[
R_{\mu\nu} - \frac{1}{2} \partial_{\rho} \partial_{\nu} \phi - \frac{e^{a\phi}}{2(q - 1)!} \left[ F_{\mu\alpha_1 \cdots \alpha_{q-1}} F^\alpha_{\alpha_1 \cdots \alpha_{q-1}} - \frac{1}{q(D-2)} R_{[q]} g_{\mu\nu} \right] = 0,
\]

(3)

\[
\partial_{\mu} \left( \sqrt{-g} e^{a\phi} F^{\mu\nu} \right) = 0,
\]

(4)

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} e^{a\phi} F^{\mu\nu} \right) = 0.
\]

(5)

We study fluxbranes with world volumes possessing the \( p + 1 \) dimensional Poincaré invariance and with a transverse space having the \( \text{SO}(k) \), \( k \leq q - 1 \), rotational symmetry, the remaining \( q - k - 1 \) dimensions being (conformally) flat. Obviously, in \( D \) dimensions, \( p = D - q - 1 \). For \( k = q - 1 \) the transverse space is spherically symmetric, while for lower \( k \) one deals with cylindrical symmetry (smeared fluxbranes). With this in mind we choose the metric

\[
d s^2 = e^{2A} (-dt^2 + dx_1^2 + \cdots + dx_{p+1}^2) + e^{2B} dr^2 + e^{2C} \left( r^2 d\Omega_k^2 + dr_1^2 + \cdots + dr_{q-k-1}^2 \right),
\]

(6)

parameterized by three \( r \)-dependent functions \( A(r) \), \( B(r) \) and \( C(r) \). An alternative gauge \( B = 0 \) can be used in the spherically-symmetric case \( k = q - 1 \), but we prefer \( B \) the curvature coordinates for the cylindrical transverse space, in which case \( B \neq 0 \).

With this ansatz, the equation for the form field (5), can easily be solved giving

\[
F_{[q]} = 2b e^{H-2(p+1)A+2B-a\phi} dr \wedge c_{[q]} \wedge dy_1 \wedge \cdots \wedge dy_{q-k-1},
\]

(7)

where \( b \) is the field strength parameter, \( c_{[q]} \) denotes the unit volume form of the round sphere \( S^k \), and

\[
H \equiv (p+1)A + B + (q-1)C + \ln r.
\]

(8)

Substituting this into the Eq.(5) one obtains the following dilaton equation

\[
\phi'' + H' \phi' = 2ab^2 e^{-2(p+1)A+2B-a\phi} = 0,
\]

(9)

where primes denote derivatives with respect to \( r \).

To derive the equations for the metric functions \( A, B, C \) one calculates first the Ricci tensor for the metric (6), the non-vanishing components being

\[
- R_{tt} = R_{xx} = -e^{2A-2B}(A'' + H'A'),
\]

(10)

\[
- R_{tr} = (p+1)(A'' + A'^2 - A'B') + (q-1)(C'' + C'^2 - B'C') + kr^{-1}(2C' - B'),
\]

(11)

\[
R_{yy} = -e^{2C-2B}(C'' + H'C'),
\]

(12)

\[
R_{ww} = [r^2 R_{yy} + k - 1 - e^{2C-2B}(rH' - 1)] \tilde{g}_{ww}.
\]

(13)

Here \( \tilde{g}_{ww} \) denotes the metric of the sphere \( S^k \). \( d\Omega_k^2 = \tilde{g}_{ww} dw_i dw_i \). In the case of the full spherical symmetry of the transverse space \( k = q - 1 \) there is no \( R_{yy} \) component of the Ricci tensor, while the \( R_{yy} \) term in (13) has to be
understood as a shorthand for the expression on the right hand side of (12). Finally, the Einstein equations (4) reduce to the following set of equations

\begin{align}
A'' + H''A' - \frac{2(q-1)b^2}{D-2} e^{-2(p+1)A + 2B - \phi} &= 0, \\
C'' + H'C' + \frac{2pk^2}{D-2} e^{-2(p+1)A + 2B - \phi} &= 0, \\
(p + 1)(A'' + A'^2 - A'B') + (q - 1)(C'' + C'^2 - B'C') + kr^{-1}(2C' - B') \\
+ \frac{1}{2} \phi'^2 + \frac{2pk^2}{D-2} e^{-2(p+1)A + 2B - \phi} &= 0, \\
rH' - 1 - (k-1)e^{2B - 2C} &= 0. 
\end{align}

(14) (15) (16) (17)

Obviously the equations (9), (14) and (15) are similar in structure; moreover, the physical conditions to be imposed on the solutions are also similar, therefore the function $C$ and the dilaton $\phi$ have to be related to the function $A$ as follows

\[ \phi = \frac{a(D-2)}{q-1} A, \quad C = -\frac{p}{q-1} A. \]

(18)

Consequently, the equations of motion reduce to a second order differential equation for $A$

\[ A'' + A'^2 - A'B' + kr^{-1}A' - \frac{2(q-1)b^2}{D-2} e^{-\lambda A + 2B} = 0, \]

(19)

and two equations in terms of $B'$ and $B$

\begin{align}
B' &= A' + (k-1)r^{-1} - (k-1)r^{-1} e^{2B + \frac{2pk^2}{D-2} A}, \\
e^{2B} &= \frac{2k(k-1)(q-1) + 4k(D-2)A - (D-2)(\lambda - 2)r^2A'^2}{(q-1)[2k(k-1)e^{(\lambda + \frac{2pk^2}{D-2})A} + 4b^2r^2]},
\end{align}

(20) (21)

where the parameter $\lambda$ is defined as

\[ \lambda = 2(p+1) + \frac{q^2(D-2)}{q-1}. \]

(22)

In fact, the three equations (19), (20) and (21), are not independent. Using the Eq. (21), $B$ can be expressed in terms of $r, A$ and $A'$. Once we substitute either (21) into (20), or (21) and (20) into (19), we obtain the following non-linear second order differential equation for the function $A$:

\[ (k-1)(D-2)[2k(q-1)rA'' + 2k^2(q-1)A' + 4k(D-2)rA'^2 - (D-2)(\lambda - 2)r^2A'^2] e^{(\lambda + \frac{2pk^2}{D-2})A} \\
+ 2(q-1)b^2r^2 [2(D-2)r^2A'' - 2(2k-1)(D-2)rA' + (D-2)(\lambda - 2)r^2A'^2 - 2k(k-1)(q-1)] = 0. \]

(23)

Ultimately then, the construction of a fluxbrane in a general dilatonic theory is reduced to solving the master equation (23); the functions $B$, $C$ and $\phi$ being then straightforwardly calculated from the Eqs. (21) and (18), while the form field $F_{[5]}$ is readily given by (7). In the Table I, we list the values of parameters for eleven-dimensional supergravity and types IIA and IIB theories in ten dimensions. The corresponding fluxbranes are not entirely independent, the discrete S-duality (2) relates them in pairs; these electric/magnetic pairs are indicated in parentheses. The column for the F4-brane of IIB supergravity has been left blank because the five-form field strength should be self-dual, this is not ensured by our ansatz. Note that for all entries in this table the following relation is satisfied

\[ \lambda - 2 = \frac{4(D-2)}{q-1}. \]

(24)

We were unable to find globally regular solutions to the above equation analytically, except for $k = 1$ case, which is the generalized Melvin solution. There exists, however, an analytic solution singular at the origin which correctly reproduces the behaviour of regular solutions in the asymptotic limit. For $k = 1$ the global solution can be obtained in a closed form.

\[ \text{\footnotesize{\textsuperscript{1}}} \]

\[ \text{\footnotesize{\textsuperscript{1}}} \text{ In the case } k = q - 1 \text{ the equation (15) does not arise, while the relation between } C \text{ and } A \text{ in (18) can be treated as a gauge fixing for the radial coordinate } r. \text{ In the remaining cases, } k < q - 1 \text{, the metric ansatz (6) specify the gauge completely (this metric pattern is not preserved under a radial coordinate redefinition).} \]
TABLE I: Parameters of fluxbranes

<table>
<thead>
<tr>
<th>M-theory</th>
<th>Type IIA string theories</th>
</tr>
</thead>
<tbody>
<tr>
<td>F6</td>
<td>F3</td>
</tr>
<tr>
<td>D</td>
<td>11</td>
</tr>
<tr>
<td>q</td>
<td>0</td>
</tr>
<tr>
<td>α</td>
<td>3/2</td>
</tr>
<tr>
<td>p</td>
<td>3</td>
</tr>
<tr>
<td>λ</td>
<td>44/7</td>
</tr>
</tbody>
</table>

III. GENERALIZED MEILVIN

One solvable case is the $k = 1$ (maximally smeared) solution, which corresponds to a generalization of the Melvin flux tube. In this section, we give the general solution for this particular case. In the context of eleven-dimensional supergravity similar solutions have been discussed in [8]. The equation for $A$ becomes simply

$$2r^2 A'' - 2r A' + (\lambda - 2)r^2 A' = 0,$$

its exact solution reads

$$A = \frac{2}{\lambda - 2} \ln(1 + c_1 r^2) + \ln c_2,$$

where $c_1$ and $c_2$ are arbitrary constants. For the metric function $B$ one finds generally

$$B = A + \ln \left( \frac{c_1 c_2}{b} \right) + \frac{1}{2} \ln \left[ \frac{4(D - 2)}{(q - 1)(\lambda - 2)} \right].$$

In view of the relation (24) the argument of the constant under the second term logarithm is equal to one, therefore for all parameters listed in Table I one has $B = A + \ln(c_1 c_2/b)$. The form field strength reads

$$F_{[d]} = 2c_1 c_2 \sqrt{\frac{4(D - 2)}{(q - 1)(\lambda - 2)}} r e^{-(1 - \lambda) A} dr \wedge d\phi \wedge dy_1 \wedge \cdots \wedge dy_{q-2},$$

and again the relation (24) implies that the square root coefficient in the expression for the form field is equal to one for the tabulated list of parameters. Without loss generality, one can specify the values $c_1 = b$ and $c_2 = 1$ which recover the earlier known generalizations of Melvin flux tube.

One reason why the case $k = 1$ is integrable apparently lies in the enhanced U-duality symmetry of this family of solutions, in fact the above solution can be generated algebraically via some transformation in solution space [8].

IV. ATTRACTOR SOLUTION

For $k \neq 1$ we were unable to find a globally regular solution to the master equation (23), however a simple solution generalizing that of [10, 11] can be found which represents the asymptotic of the regular solutions with different degrees of smearing. Examining the Eq. (23) one can observe the scaling symmetry $r \rightarrow \Gamma r$, $e^A \rightarrow e^\alpha e^A$ for some $\alpha$. In view of this one can try the solution of the following form

$$A = \alpha \ln r + \ln \beta,$$

with constants $\alpha$ and $\beta$. This particular solution was presented in [10, 11] for $k = q - 1$, and, whilst by itself it represents a singular spacetime as $r \rightarrow 0$, it is the attractor solution for fluxbranes which are regular at their core. For $k \neq 1$, the value of $\alpha$ can be simply determined from (23) by comparing powers of $r$, this gives

$$\alpha = \frac{2(q - 1)}{2p + (q - 1)\lambda}. $$
The corresponding value of β is also easily determined as

$$\beta = \left[ \frac{2(q - 1)k^2}{\alpha(k - 1)(D - 2)} \right]^{1/2}. \quad (31)$$

Somewhat unexpectedly, the value of α is independent of the parameter k, that is the growth of the metric at infinity is the same for all smeared solutions $1 < k \leq q - 1$. From the Eq. (21) we obtain the following expression for $B$

$$B = -\frac{p}{q - 1} A + \frac{1}{2} \ln \gamma, \quad (32)$$

where

$$\gamma = \frac{2k(k - 1)(q - 1) + 4k\alpha(D - 2) - \alpha^2(D - 2)(\lambda - 2)}{2(k - 1)[k(q - 1) + \alpha(D - 2)]}. \quad (33)$$

It is worth noting that the asymptotic form of the generalized Melvin solution $k = 1$ is different from what we have found here. Using the Eq.(24) to express parameters in terms of the space-time dimension and the rank of the antisymmetric field strength, one finds for $k \neq 1$

$$e^{A} \sim r^{(q-1)/[3(D-2)]}, \quad (34)$$

while for $k = 1$ one has

$$e^{A} \sim r^{(q-1)/(D-2)}. \quad (35)$$

V. M- AND STRING THEORY FLUXBRANES

In eleven-dimensional supergravity one has a three-form potential with the standard kinetic term, but also a Chern-Simons term. The latter, however, plays no role for the purely electric or purely magnetic solutions we are dealing with, so our general framework remains valid for describing these solutions. The four-form field strength can support two different types of fluxbranes: the magnetic F6-brane ($q = 4$, $p = 6$), and the electric F3-brane, the S-dual of the F6-brane. The latter lies within our prescription when working with the seven-form dual to the four-form, $q = 7$, $p = 3$, see (2).

A. F6 solutions

The F6-branes correspond to parameters $D = 11$, $a = 0$, $q = 4$ and $p = 6$ in the Table I with k giving the level of smearing. The master equation for $A$ (23) reads

$$3(k - 1)e^{18A}[krA'' + k^2 A' + 6krA'^2 - 18r^2 A^6] + 2b^2 \{3r^2 A'' - 3(2k - 1)rA' + 18r^2 A'^2 - k(k - 1)\} = 0. \quad (36)$$

The other variables can be calculated as follows

$$e^{2B} = \frac{k(k - 1) + 6krA' - 18r^2 A'^2}{k(k - 1)e^{18A} + 2b^2 r^2 e^{14A}}, \quad (37)$$

$$C = -2A, \quad (38)$$

$$F_{[4]} = 2b^2 r^3 e^{13A+B} dr \wedge dy_1 \wedge dy_1 \wedge \ldots \wedge dy_{3-k}. \quad (39)$$

The attractor solution ($k \neq 1$) reads

$$A = \frac{1}{9} \ln r + \frac{1}{18} \ln \left( \frac{6b^2}{k - 1} \right), \quad (40)$$

$$B = -\frac{2}{9} \ln r - \frac{1}{9} \ln \left( \frac{6b^2}{k - 1} \right) + \frac{1}{2} \ln \left( \frac{3k - 2}{3k - 3} \right), \quad (41)$$

$$C = -\frac{2}{9} \ln r - \frac{1}{9} \ln \left( \frac{6b^2}{k - 1} \right). \quad (42)$$
FIG. 1: Solutions to (36) with varied initial conditions, showing the attractor nature of the solutions in the asymptotic region.

The local series solution near the origin, for regular solutions, starts as follows

\[ A \approx A_0 + O(r^2), \]  

(43)

it contains (the only) one parameter \( A_0 \). We solved (36) using a fifth order Runge-Kutta technique, with \( A'(0) = 0 \) as the initial condition. Numerical solutions starting with different \( A_0 \) converge to the same attractor solution in the asymptotic region (Fig. 1). From (21) and (18) we see that \( B(0) = C(0) \), showing that there is no conical singularity at the core.

B. F3 solutions

The electric F3-branes correspond to \( D = 11, a = 0, q = 7 \) and \( p = 3 \) in the Table I; the master equation for \( A \) becomes

\[ 3(k - 1)e^{3A}[2k r A'' + 2k^2 A' + 6kr A'^2 - 9r^2 A'^3] + 4k^3 r^3[3r^2 A'' - 3(2k - 1)r A' + 9r^2 A'^3 - 2(k - 1)] = 0, \]  

(44)

other variables in terms of \( A \) read

\[ e^{2B} = \frac{2k(k - 1) + 6kr A' - 9r^2 A'^2}{2k(k - 1)e^{3A} + 4k^3 r^3} e^{8A}, \]  

(45)

\[ C = -\frac{1}{2} A, \]  

(46)

\[ F_{[7]} = 2k r^k e^{-7A} e^B d^7 \epsilon \wedge dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{6-k}. \]  

(47)

Similarly, the attractor solution for \( (k \neq 1) \) is

\[ A = \frac{2}{9} \ln r + \frac{1}{9} \ln \left( \frac{6k^2}{k-1} \right), \]  

(48)

\[ B = -\frac{1}{9} \ln r - \frac{1}{18} \ln \left( \frac{6k^2}{k-1} \right) + \frac{1}{2} \ln \left( \frac{3k-2}{3k-3} \right), \]  

(49)

\[ C = -\frac{1}{9} \ln r - \frac{1}{18} \ln \left( \frac{6k^2}{k-1} \right). \]  

(50)

We solved this system numerically for the same initial conditions \( A_0 = 1, A'(0) = 0 \), and the various \( k (k = 2, \ldots, 6) \) to observe the effect of smearing. The \( k = 1 \) solution, given in Sec. III, is known analytically and is shown as the dotted curve in Fig. 2. The lower curves show the behaviour of solutions with \( k = 2 \) (the top solid curve), 3, 4, 5, 6 (the bottom curve).
FIG. 2: Solutions to (44) with different levels of smearing, the upper (dashed) curve is for $k = 1$ with the lower curves describing $k = 2, 3, 4, 5, 6$.

C. Type II fluxbranes

The NS-NS sectors of type IIA and IIB theories are the same, with the three-form field strength coupled to the dilaton with the coupling constant $a = -1$. Therefore, there exist magnetic NS E6-branes with $k = 1, 2$. The dual, electric, NS F2-branes come through the discrete duality symmetry of (1), for them $k = 1, ..., 6$.

In the R-R sector there are different rank $q$-form fields: $q$ is even for IIA and odd for IIB. The dilaton coupling constant in the Einstein frame assumed here is $a = (5 - q)/2$ and we see from (2) that these fluxbranes come in dual pairs. There is one exceptional case, not covered by our treatment: the five-form field strength of type IIB supergravity is self-dual and our analysis does not cover this case of a dyon like solution. Table I shows a summary of the above discussion.

VI. CONCLUSION

We have shown that apart from the already known supergravity fluxbrane solutions with the spherically symmetric transverse space there exist a sequence of fluxbranes with a cylindrical transverse space being the product of a lower dimensional sphere and a flat space. All such solutions are shown to be governed by a single second order radial differential equation for the metric function. This improves on the previous analyses where solutions were described in terms of two unknown functions. An analytic solution was presented which has the correct asymptotic behaviour but diverges at the origin. Solutions which are regular at the origin were shown to depend on one parameter and tend to the analytic solution for sufficiently large radius. This gives numerical evidence for existence of globally regular solutions for all transverse topologies considered. The radial growth at infinity for the solution is the same for all degrees of smearing except from the exceptional case of maximal smearing. For this an analytic solution exists which can also be obtained algebraically using U-duality.

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