II. THE MODEL

In this section, we present a model for calculating the correlation functions of operators in the quark-gluon plasma. The model is based on the assumption that the quark-gluon plasma is a field theory with a noncompact gauge symmetry. The model is described by a set of coupled differential equations, which are solved numerically to obtain the correlation functions.

The results of the model are compared with the data from the RHIC experiment, and it is found that the model is able to reproduce the data quite well. The model also provides a good description of the non-perturbative effects in the quark-gluon plasma.

Introduction

The introduction of the model is divided into two parts: part A and part B. Part A introduces the model and its applications, while part B presents the results of the model.

In this section, we discuss the origin of the model and its connection to the quark-gluon plasma. The model is based on the assumption that the quark-gluon plasma is a field theory with a noncompact gauge symmetry. The model is described by a set of coupled differential equations, which are solved numerically to obtain the correlation functions.

The results of the model are compared with the data from the RHIC experiment, and it is found that the model is able to reproduce the data quite well. The model also provides a good description of the non-perturbative effects in the quark-gluon plasma.

In the conclusion, we summarize the main results of the model and discuss its implications for the understanding of the quark-gluon plasma.
emergence of superconductivity. We follow ref. [3] and use a standard NJL model with massless quarks carrying two flavours and three colours. We leave for future work the study of more complex scenarios.

Following ref. [3] we model the attractive interaction responsible for the pairing of quarks at large densities with a contact four-quark interaction whose form is suggested by the instanton-induced interaction. The Hamiltonian then has the form $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$ with

$$
\begin{align*}
\hat{H}_0 &= -\int d^3x \left( \overline{\psi}_{L_\alpha}(x) i \alpha \cdot \nabla \psi_{L_\alpha}(x) + \overline{\psi}_{R_\alpha}(x) i \alpha \cdot \nabla \psi_{R_\alpha}(x) \right), \\
\hat{H}_{\text{int}} &= -K \sum_{k l ; \alpha \beta \gamma} \int d^3x \overline{\psi}_{R_{l \alpha}}(x) \psi_{L_k \beta}(x) \overline{\psi}_{R_l \beta}(x) \psi_{L_k \beta}(x) + \text{h.c.,}
\end{align*}
$$

where flavour and colour indices are Latin and Greek respectively, and the tensor $\sum_{k l ; \alpha \beta \gamma}$ is given by

$$
\sum_{k l ; \alpha \beta \gamma} \equiv \epsilon_{\alpha \beta} (3 \delta_{\alpha \gamma} \delta_{\beta \gamma} - \delta_{\alpha \gamma} \delta_{\beta \gamma}).
$$

At zero density the quarks acquire a dynamical mass, through the generation of a chiral condensate. This mass drops as the density increases, reaching zero at the chiral phase transition. At higher densities the colour condensate which signals the onset of colour superconductivity forms.

In order to regularise the contributions stemming from large momenta, the authors of Ref. [3] have supplemented the interaction (1) with a form factor, to be associated with each quark of a given momentum. We choose to use a form factor of the form

$$
F(k) = \frac{1 + e^{-\xi}}{1 + e^{(k^2 - \Lambda^2)/\Lambda^2}},
$$

The parameter $\xi$ determines the sharpness of the cutoff, and has been taken to be 10. This leaves two free parameters, $\Lambda$ and the coupling constant $K$ in the interaction. These are fixed at zero density by requiring the pion decay constant to have its physical value, and by choosing the dynamically generated quark mass to be 400 MeV; this gives $\Lambda = 700$ MeV and $K = 1.755 \times 10^{-65}$ MeV$^{-2}$.

The modes of the fields are quantised in the standard way, and we define the creation operators $\hat{a}_{L_\alpha}^\dagger(k)$, $\hat{b}_{L_\alpha}^\dagger(k)$, $\hat{c}_{L_\beta R_\alpha}^\dagger(k)$ for particles, antiparticles and holes relative to the Fermi sea, $|k_F\rangle$, which can thus be written as

$$
|k_F\rangle = \prod_{\alpha \beta} \theta(k_F - k) \hat{a}_{L_\alpha}^\dagger(k) \hat{a}_{R_\beta}^\dagger(k) |0\rangle.
$$

We choose to work with states in which the Fermi momenta $k_F$ for all three colours are equal. In fact the properties of the BCS state are independent of the initial Fermi momentum for the condensing quarks. However colour projection becomes involved if $k_F$ is different for the spectator quarks.

The Fermi sea is the natural choice of ground state at low to intermediate density. At large densities the interaction (1) is responsible for the condensation of Cooper pairs of quarks, and we get a new form for the wave function,

$$
\begin{align*}
\Psi &= G_L \quad G_R \quad |k_F\rangle, \\
\end{align*}
$$

where

$$
\begin{align*}
\bar{G}_L &= \prod_{(\alpha \beta)k} \left[ \cos \theta^L_{\alpha\beta}(k) + \epsilon_{\alpha \beta} \sin \theta^L_{\alpha\beta}(k) e^{i\xi^L_{\alpha\beta}(k)} \hat{a}_{L_\alpha}^\dagger(k) \hat{a}_{L_\alpha}^\dagger(k) \right] \\
\bar{G}_R &= \prod_{(\alpha \beta)k} \left[ \cos \theta^R_{\alpha\beta}(k) + \epsilon_{\alpha \beta} \sin \theta^R_{\alpha\beta}(k) e^{i\xi^R_{\alpha\beta}(k)} \hat{a}_{R_\beta}^\dagger(k) \hat{a}_{R_\beta}^\dagger(k) \right] \\
\end{align*}
$$

and similarly for $\bar{G}_R$, with the substitutions $L \leftrightarrow R$. The sum denoted $\langle \alpha \beta \rangle$ is taken over $\alpha, \beta = 1, 2$ and 2, 1 only; the BCS angles $\theta^L_{(A,B,C)}(k)$ and $\theta^R_{(A,B,C)}(k)$ are to be determined variationally.

The BCS wave function $\Psi$ of Eq. (5) describes the condensation of pairs of quarks with the same helicity, but antisymmetric in flavour and in colour, as can be seen by inspection of Eq. (6). The condensate carries a colour-3 order parameter. Only two colours, by convention 1 and 2 (or red and green), condense, leaving the third colour inactive. This corresponds to a spontaneous symmetry breaking.

The BCS angles are obtained by minimizing the thermodynamic potential

$$
F = \langle \Psi | H - \mu N | \Psi \rangle.
$$

The phases $\phi$ are fixed by requiring that the expectation value of $\hat{H}_{\text{int}}$ is maximally attractive; this gives

$$
\begin{align*}
\xi_C &= -\xi_A = -\xi_B = \pi/2 \\
\xi^L_{(A,B,C)} + \xi^R_{(A,B,C)} &= \pi.
\end{align*}
$$

Minimising with respect to the functions $\theta$ one finds the gap $\Delta$ and the angles $\theta_{(A,B,C)}(k)$ for particles, antiparticles and holes respectively to be given by the solutions
The figure shows the calculated distribution of the captured nucleon as a function of the proton number in the GCS. The peaks and valleys correspond to the different nuclear shells and magic numbers. The GCS was used to predict the number of protons in the GCS, while the dashed lines represent the predicted values. The figure highlights the importance of understanding these distributions for predicting nuclear properties.
(Section III) and then onto colour-singlet states (Section IV).

III. PROJECTION ONTO DEFINITE BARYON NUMBER

There are various ways to solve the problem of picking a component of fixed particle number from a BCS state. The most efficient ones are based on the calculation of a contour integral; we follow here the method of residues of Ref. [8]. This will be used to project the colour superconducting state considered in Eq. (5) onto a state with a fixed number of particles.

We start by defining the rotated BCS state

\[ |\Psi(\zeta)\rangle = e^{i(N-N_0)/\beta} |\Psi\rangle = \zeta^{(N-N_0)/\beta} |\Psi\rangle \] (11)  

where

\[ \hat{G}_L(\zeta) = \prod_{\langle \alpha \beta \rangle} \left[ \cos \theta_1^L(k) + \zeta \cos \theta_2^L(k) \sin \gamma(k) a_{1\alpha}^\dagger(k) a_{1\beta}^\dagger(-k) \right] \times \prod_{\langle \alpha \beta \rangle} \left[ \cos \theta_1^R(k) + \zeta^* \cos \theta_2^R(k) \sin \gamma(k) a_{1\alpha}^\dagger(k) a_{1\beta}^\dagger(-k) \right] \times \prod_{\langle \alpha \beta \rangle} \left[ \cos \theta_1^C(k) + \zeta \cos \theta_2^C(k) \sin \gamma(k) c_{1\alpha}^\dagger(k) c_{1\beta}^\dagger(-k) \right] \equiv \prod_{\langle \alpha \beta \rangle} \hat{G}_{LA\alpha\beta}(\zeta, k) \prod_{\langle \alpha \beta \rangle} \hat{G}_{LB\alpha\beta}(\zeta, k) \prod_{\langle \alpha \beta \rangle} \hat{G}_{LC\alpha\beta}(\zeta, k), \] (13)

and similarly for \( \hat{G}_R(\zeta) \). Because the creation of antiparticles or holes lowers the baryon number, the corresponding terms in \( \hat{G} \) contain \( \zeta^* \) or, equivalently, \( 1/\zeta \).

The number-projected wave function is produced from an appropriate superposition of rotated states:

\[ |\Psi_n\rangle = C_n \int \frac{d\zeta}{\sqrt{N+1}} |\Psi(\zeta)\rangle \] (14)

where \( n \) is the number of pairs.

The only operators whose expectation values in the number-projected state are non-vanishing are those which are particle-number conserving and so commute with the number operator \( N \). This enables us to write

\[ \langle \Psi_n | O | \Psi_n \rangle = \frac{\langle \Psi | O | \Psi \rangle}{\langle \Psi | \Psi \rangle} \] (15)

A convenient way to calculate these matrix elements is to find the Thouless operator \( \hat{S}(\zeta) \) which maps \( |\Psi\rangle \equiv |\Psi(1)\rangle \) into \( |\Psi(\zeta)\rangle \) [9]:

\[ \hat{S}(\zeta) |\Psi\rangle = |\Psi(\zeta)\rangle. \] (16)

The explicit expression for \( \hat{S} \) is given in Appendix A. There we also give the definitions of the quasi-particle operators \( a(\zeta, k), \beta(\zeta, k), \gamma(\zeta, k) \) which annihilate the state \( |\psi(\zeta)\rangle \).

The Thouless operator can be expanded in a series of terms containing zero, two, four, \ldots quasiparticle creation operators:

\[ \hat{S}(\zeta) = \sum_{n=0}^\infty S^{(n)}(\zeta). \] (17)

Depending on the form of the interaction, only a limited number of these terms will contribute to the matrix elements of the Hamiltonian. Luckily, the interaction (1) selects only the first two terms in the expansion. For
left-handed particles these are

\[ S^{(1)}_{L,(A)}(\zeta) = \prod_{(\alpha)\mathbf{k}} \left( \cos^2 \theta_A^\alpha(k) + \zeta \sin^2 \theta_A^\alpha(k) \right), \]

\[ S^{(1)}_{L,(A)}(\zeta) = \mathcal{S}^{(1)}_{L,(A)}(\zeta) \times \sum_{\gamma \neq k} (\zeta - 1) \varepsilon_{\gamma k} \sin^2 \theta_A^\gamma(k) \cos^2 \theta_A^\gamma(k) e^{i \phi(k)} \cos^2 \theta_A^\gamma(k) + \zeta \sin^2 \theta_A^\gamma(k) \times \alpha_{L,0}^{\dagger}(1, k) \alpha_{L,0}(1, k). \] (18)

The function \( S^{(1)}_{L,(A)}(\zeta) = \langle \Psi | \Psi(\zeta) \rangle \) can be expanded as a Taylor-Laurent series about \( \zeta = 0 \), with real coefficients \( d_\alpha \). These are related to the normalisation constant \( C_n \) in Eq. (14), by \( 2 \pi i C_n = 1/\sqrt{\sigma_n} \), and to the expansion of the BCS state \( |\Psi\rangle \) in terms of states of definite particle number:

\[ |\Psi\rangle = \sum_n \sqrt{d_n} |\Psi_n\rangle. \] (19)

Since \( |\Psi\rangle \) has unit norm, the sum of the \( d_n \) is unity.

From Eq. (15) we obtain the expectation value of the Hamiltonian in the number projected state:

\[ \langle \Psi_n | H | \Psi_n \rangle = -2 \pi i |C_n|^2 \int \frac{d\zeta}{\zeta_{n+1}} \langle \Psi | H | \Psi(\zeta) \rangle. \] (20)

Explicitly, we obtain for the non-interacting part of the Hamiltonian:

\[ \langle \Psi_n | H_{\text{int}} | \Psi_n \rangle = \frac{4}{d_n} \sum_k \left\{ \theta(k_F - k) \left[ d_n + 2 \cos^2 \theta_C(k) I_{\alpha,\beta}(\theta_C(k)) \right] \right. \]

\[ + 2 \theta(k - k_F) \sin^2 \theta_A(k) I_{\alpha,\beta}(\theta_A(k)) \]

\[ + 2 \sin^2 \theta_B(k) I_{\alpha,\beta}(\theta_B(k)) \right\}. \] (21)

In this expression the first term describes the hole contribution, the second the particle one, and the last the antiparticle. The first term \( \{d_n \} \) in the square brackets describes the spectator colour; the second term reduces to twice this in the absence of pairing (when \( \theta_{(A,B,C)} = 0 \)). The interaction Hamiltonian leads to the more complicated form

\[ \langle \Psi_n | H_{\text{int}} | \Psi_n \rangle = \frac{4}{d_n} \sum_k \left\{ \theta(k_F - k) \left[ d_n + 2 \cos^2 \theta_C(k) I_{\alpha,\beta}(\theta_C(k)) \right] \right. \]

\[ + 2 \theta(k - k_F) \sin^2 \theta_A(k) I_{\alpha,\beta}(\theta_A(k)) \]

\[ + 2 \sin^2 \theta_B(k) I_{\alpha,\beta}(\theta_B(k)) \right\}. \] (22)

The volume of the box is \( \Omega \equiv L^3 \). The contour integrals are included in the definitions of \( I_{(\alpha,\beta,\gamma),n} \) and \( J_{(\alpha,\beta,\gamma,n)} \) (see Appendix A); \( \theta_{kk'} \) is the angle between \( k \) and \( k' \). The form factors in the expression (22) suppress the contributions coming from large momenta.

The expectation value of the number operator is obtained from Eq. (21) by dropping the single-particle energy \( k \) which appears immediately after the summation sign and swapping the sign of the antiparticle (\( B \)) term.

These expressions (21) and (22) reduce to the expectation values in the unprojected BCS state if the integrals \( I \) and \( J \) and the constant \( d_n \) are set to one. The infinite-volume gap equations (9) and (10) are easily obtained by minimising \( \langle H - \mu N \rangle \) with respect to the angles \( \theta \), if the sums are replaced by integrals. However in the finite-volume projected case, the equations are much more complicated. For a start, in the cubic box there is no reason to expect the angles to depend only on the length and not on the direction of \( k \). This introduces a dependence on the specific geometry that one might be quite happy to ignore. However even then the dependence of the \( I \)’s and \( J \)’s on the \( \theta \)’s renders the equations very
complicated. We do not attempt to solve these. Instead we assume that the form of the $\theta$s is given as before by Eq. (9), and minimise the expression for $\langle H - \mu N \rangle$ obtained from Eqs (21) and (22) with respect to the gap $\Delta$ only.

This process of restricting the BCS angles to have the same form as in the unprojected state will presumably introduce fewest errors where the projected state has a high overlap with the unprojected state. Thus although in principle we can project a state of a given baryon number from any BCS state, in practice we choose values of $\mu$ where the expectation value of the number of pairs in the unprojected BCS state has the value desired in the projected state (see Fig. 3).

In the BCS state there are non-vanishing expectation values for diquark condensates such as $\psi_{R1}(x)\psi_{R2}(x)$. However as this operator changes the quark number by two units, it does not commute with the number operator $N$ and the expectation value in the number-projected state will vanish. None-the-less, the square of the diquark condensate operators do not vanish, and they act as order parameters in this case.

IV. COLOUR PROJECTION

We now turn to the issue of colour projection. As we have pointed out before, the colour-superconducting state which forms at large densities must be a colour singlet. However, the state in Eq. (5) does not fulfil this requirement, given the form of the operators $G(L,B)$. For this reason we want to project the colour-singlet state from Eq. (5). This requires us to integrate over the group manifold:

$$\langle \tilde{\psi}_n \rangle = \int d\Omega_2 U_2 |\psi_n\rangle .$$

Here $d\Omega_2$ is the volume element on the group manifold and $|\psi_n\rangle$ is the number projected BCS state. $U_2$ is a unitary operator which performs a rotation in the $SU(3)$ space.

Following [9] we parametrize an element $g$ of $SU(3)$ in the form

$$g = e^{i \frac{\theta}{2} \lambda_1 e^{i \frac{\phi}{2} \lambda_2 e^{i \frac{\beta}{2} \lambda_3}}},$$

where the $\lambda_i$ are the Gell-Mann matrices, and write the volume element as

$$d\Omega_2 = \frac{1}{2^6} \sin(\phi_0) \sin(\phi_1) \sin(\phi_2) \sin^2\left(\frac{\theta}{2}\right) \times d\theta d\phi_1 d\phi_2 d\phi_3 d\phi_4 d\phi_5 .$$

To obtain the realization of these rotation operators when acting on our quarks states, we replace the $\lambda_i$ with the transition operators $Q_2 = \int d^3x \psi_1(x) \frac{1}{2^6} \psi(x)$, which are associated with the $SU(3)$ group.

Because of the residual colour symmetry, the operators associated with the red-green $SU(2)$ subgroup annihilate the BCS state, and also commute with the number projection operation. Thus

$$\hat{Q}_{1,2} |\psi_n\rangle = 0.$$ (26)

We can simplify the algebra considerably by noting that

$$Q_8 = \int d^3 x \psi^d(x) \frac{\lambda_8}{2} \psi(x)$$

$$= \frac{1}{2} \left[ N_{11} + N_{22} - 2 N_{33} \right] ,$$ (27)

where 1, 2 and 3 are the values taken by the the colour index. Since we are working with states built on Fermi seas which have equal numbers of red, green and blue quarks, we see that $Q_8$ annihilates only the number-projected BCS state $|\psi_0\rangle$ which contains zero net pairs (i.e. the number of pairs of holes and antiparticles is the same as the number of pairs of particles). We therefore restrict our projection to this state.

Thus the only operator which acts non-trivially is $Q_5$, which is non-diagonal in the colour space and connects one of the two paired colours with the remaining one. As a result we are able to reduce the integration over the eight different angles $\phi_i$ in Eq. (24) down to an integration over the single angle $\phi$ for the rotation generated by $Q_5$. The residual volume element is therefore

$$d\Omega_5 = \frac{1}{2^6} \sin(\phi) \sin^2\left(\frac{\phi}{2}\right) d\phi$$

$$= \sin^3\left(\frac{\phi}{2}\right) d\phi ,$$ (28)

where $0 \leq \phi \leq \pi$ is the angle associated with $\hat{Q}_5$.

As before, we want to project states with zero pairs from BCS states which have zero pairs on average. So we are restricted to working only at certain values of $\mu$. These are the values at which the two curves of Fig. 3 cross. For small boxes, this is a significant restriction on the densities we can consider.

Just as in the case of number projection particle-number-conserving operators have non-zero expectation values, when we project onto a colour singlet only colour-singlet operators will have non-vanishing expectation values. Such operators commute with the colour rotation, and so we can write (cf. Eq. (15))

$$\langle \tilde{\psi}_0 | \hat{Q} | \tilde{\psi}_0 \rangle = \frac{\int d\Omega_2 <\psi_0 | U_2 | \psi_0>}{\int d\Omega_2 <\psi_0 | U_2 | \psi_0>}$$

$$= \frac{\int d\Omega_5 <\psi_0 | e^{i Q_5} | \psi_0>}{\int d\Omega_2 <\psi_0 | U_2 | \psi_0>}$$ (29)

We can write the colour-number projected state in
terms of the colour rotated operators \( \tilde{G}_{(L,R)}(\zeta) \),

\[
\begin{align*}
\tilde{\Psi}_0 &= \hat{C}_0 \int \frac{d\zeta}{\zeta} \int d\Omega_5 e^{i \tilde{\lambda} \tilde{Q}_5} |\Psi(\zeta)\rangle \\
&= \hat{C}_0 \int \frac{d\zeta}{\zeta} \int d\Omega_5 \tilde{G}_{(L,R)}(\zeta, \phi) \tilde{G}_{R}(\zeta, \phi) |k_F\rangle,
\end{align*}
\]

\( \tilde{G}_{LA12}(k, \zeta, \phi) = \cos \theta_A(k) + \zeta \sin \theta_A(k) e^{i \tilde{\lambda}_A(k)} \left( \cos \frac{\phi}{2} a_{12}^\dagger(k) - \sin \frac{\phi}{2} a_{12}^\dagger(-k) \right) a_{L22}^\dagger(-k), \)

\[ \tilde{G}_{LA21}(k, \zeta, \phi) = \cos \theta_A(k) - \zeta \sin \theta_A(k) e^{i \tilde{\lambda}_A(k)} a_{L12}^\dagger(k) \left( \cos \frac{\phi}{2} a_{12}^\dagger(-k) - \sin \frac{\phi}{2} a_{23}^\dagger(-k) \right), \]

(31)

We can expand \( W(\zeta, \phi) \) as a Laurent series in \( \zeta \), with coefficients \( d_n(\phi) \). The normalization constants \( \hat{C}_n \) are related to integrated coefficients \( \hat{d}_n \), as in the pure number-projection case the \( \hat{C}_n \) are to the Laurent coefficients \( d_n \) of \( S(\zeta) \).

The forms of the expectation values of \( H_3, H_{\text{int}} \), and \( N \) are as given in the previous section, in Eqs (21) and (22), but setting \( n = 0 \), and with the replacement of \( \hat{d}_n \) with \( d_n \) and of the integrals \( I \) and \( J \) with the new integrals \( \hat{I} \) and \( J \) which are given in Appendix B. Once again a one-parameter variation with respect to the gap \( \Delta \) is performed.

\[ \text{V. RESULTS} \]

In this section we present our final results for the projected states.

First however we look at the effect of finite size on the unprojected BCS state. As the size of the box grows the fractional fluctuations in the baryon number will drop as \( 1/\sqrt{N} \), and we expect that the value of the gap and of the energy should converge toward their values in the infinite

![Figure 4](image_url)
volume. This behaviour is evident in Fig. 4, which shows the superconducing gap and total energy per particle. Convergence is obtained quickly and continuously.

Now we turn to the effects of number and colour projection. As we have explained in Section IV, the projection over colour is carried out on BCS states having the same total baryon number as the underlying Fermi gas they are built on. This restricts us to certain values of $\mu$, as we saw from Fig. 3. In Eq. (16) we introduced the function $S^{(n)}(\zeta)$ whose Laurent coefficients $d_n$ are related to the amount of the projected state with $n$ pairs contained in the unprojected state (see Eq. (19)).

Fig. 5 shows the distribution of the Laurent coefficients for the function $S^{(n)}(\zeta)$ at these points. Because we are working at our chosen values of $\mu$, the coefficients peak at $n = 0$, and we see that the unprojected BCS state has a substantial overlap with the projected state with zero pairs. The distributions are easily approximated with Gaussian functions.

In Fig. 6 we show the superconducting gap as a function of chemical potential for a range of box sizes. The dashed line is the finite-volume unprojected gap, and for comparison the continuous line shows the result for the infinite-volume case. We remind the reader that because we perform only a limited variation after projection, as explained above, the projection can only be done for certain discrete values of $\mu$ for which the average net number of pairs in the unprojected BCS state is zero. At these points we show results for the unprojected BCS circles, which must coincide with the dashed line, and for the number-projected (triangle up) and combined number-colour projected BCS (triangle down).

The most striking finite-volume effect is the disappearance of the gap for values of $\mu$ which are far from shells.

These regions are confined to very low $\mu$ for boxes above 3 fm in length. In these regions the effects of projection on the gap can be striking, as can be seen even at 7 fm. However, in all other regions, where the unprojected gap does not disappear, projection has very limited effect. In general the results for colour and number projection lie above those for number projection alone; the latter show almost no difference from the unprojected case unless the unprojected gap vanishes.

The effect of finite size on the energy of the condensate has already been shown in Fig. 4. The effect of projection is to lower the energy by less than 1 MeV even for the smallest boxes considered. In the approach to the infinite-volume limit, shell effects for the unpaired quarks vanish more slowly than finite-size or projection effects on the condensate.

VI. CONCLUSIONS

In this work we have studied the effects of finite size over the colour superconducting state, by considering quark matter at finite density and zero temperature enclosed in a cubic box. We have considered the 2SC superconducting state of ref. [3] and we have calculated the effects of the projection over definite baryon number and over a colour singlet state. The most significant finite size effect is the vanishing of the gap for values of the chemical potential between widely spaced shells. For these cases, projection has a significant effect, restoring a sizeable gap. Otherwise the effects of projection are slight, though in all cases each projection increases the gap to some degree.

This behaviour points to the interesting possibility that the interplay between the chiral and the superconducting phase can also change in finite size. This hypothesis can be verified by extending the present calculation, allowing a comparison between the two phases. This topic is currently under investigation [10].

There are other aspects which require future investigation as well: it would be interesting to extend our results to more general form of pairing interactions, rather than the simple “instanton-motivated” interaction of Eq. (1).

For technical reasons, the number projection which was carried out exactly in the present work, would require more effort in a general case, where the interaction contributions will contain not only terms which mix the helicity, such as $\bar{\psi}_L \psi_R$, but also terms which connect fermions with the same helicity, such as $\bar{\psi}_L \gamma_5 \psi_L$. It would be more appropriate in such cases to apply approximate number projection techniques, which have been developed in the context of nuclear theory [9].

Another aspect of interest, which we plan to investigate in the future, is the effect of the colour projection on different superconducting states, and in particular on the colour-flavour locked phase (CFI), which was originally studied in [4]. Because in this state the colour and flavour degrees of freedom are “locked”, the colour projection
FIG. 6: Gap for the finite box at a fixed size of $L = 3, 3.5, 4, 5, 7, 8$ fm: unprojected (circles) and dashed line, colour projected (cross), number projected (triangle up) and number-colour projected (triangle down). The solid line is the gap for the infinite volume unprojected BCS state.

will affect both colour and flavour, and could possibly yield larger effects than the one observed in the case of the 2SC.

We also remark that in the present analysis we have not performed a full variation after projection. Instead we have followed a simpler approach, assuming for the solution the same functional form as in the unprojected case and obtaining the numerical solution by performing a minimization of the total energy with respect to the gap $\Delta$.

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APPENDIX A: NUMBER PROJECTION: SOME USEFUL FORMULAS

1. Quasi-particle operators

We give here the explicit expressions for the quasi-particle operators annihilating the state $|\psi(\zeta)\rangle$ of Eq. (14). After some algebra we find that these can be written as follows:

\[
\begin{align*}
\hat{a}_{1,1}(\zeta, k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{a}_{1,1}(k) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{a}_{2,2}(k), \\
\hat{a}_{1,2}(\zeta, k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{a}_{1,2}(k) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{a}_{2,1}(k), \\
\hat{b}_{1,1}(\zeta, k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{b}_{1,1}(k) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{b}_{2,1}(k), \\
\hat{b}_{1,2}(\zeta, k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{b}_{1,2}(k) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{b}_{2,2}(k), \\
\hat{c}_{1,1}(\zeta, k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{c}_{1,1}(k) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{c}_{2,1}(k), \\
\hat{c}_{1,2}(\zeta, k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{c}_{1,2}(k) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{c}_{2,2}(k).
\end{align*}
\]

(A1)

Notice that these operators fulfill canonical anticommutation relations. The inverse equations, which allow us to express the original operators in terms of the quasi-particle ones, are also useful and read:

\[
\begin{align*}
\hat{a}_{1,1}(k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{a}_{1,1}(\zeta, k, \bar{\zeta}) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{a}_{2,2}(\zeta, k), \\
\hat{a}_{1,2}(k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{a}_{1,2}(\zeta, k, \bar{\zeta}) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{a}_{2,1}(\zeta, k), \\
\hat{b}_{1,1}(k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{b}_{1,1}(\zeta, k, \bar{\zeta}) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{b}_{2,1}(\zeta, k), \\
\hat{b}_{1,2}(k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{b}_{1,2}(\zeta, k, \bar{\zeta}) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{b}_{2,2}(\zeta, k), \\
\hat{c}_{1,1}(k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{c}_{1,1}(\zeta, k, \bar{\zeta}) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{c}_{2,1}(\zeta, k), \\
\hat{c}_{1,2}(k) & = \cos \frac{\theta_A}{\alpha}(k) \hat{c}_{1,2}(\zeta, k, \bar{\zeta}) + \zeta \epsilon_{\alpha\beta\gamma} \sin \theta_A(k) e^{i \xi(k)} \hat{c}_{2,2}(\zeta, k).
\end{align*}
\]

(A2)

\[2^+\] Thouless operator

We now consider the operator defined in Eq. (16). For the part corresponding to $G_{L}^{1}(\zeta, k)$ we look for solutions satisfying

\[
\begin{align*}
\left[ w^{1}_{L}(\zeta) + \epsilon_{\alpha\beta\gamma} w_{L}^{2}(\zeta) \right] \hat{a}_{1,1}(k) + \hat{a}_{2,2}(\zeta, k, \bar{\zeta}) \right] G_{L,\alpha\beta}(1, k) & = G_{L,\alpha\beta}(1, k), \\
\left[ w^{1}_{L}(\zeta) + \epsilon_{\alpha\beta\gamma} w_{L}^{2}(\zeta) \right] \hat{b}_{1,1}(k) + \hat{b}_{2,2}(\zeta, k, \bar{\zeta}) \right] G_{L,\alpha\beta}(1, k) & = G_{L,\alpha\beta}(1, k), \\
\left[ w^{1}_{L}(\zeta) + \epsilon_{\alpha\beta\gamma} w_{L}^{2}(\zeta) \right] \hat{c}_{1,1}(k) + \hat{c}_{2,2}(\zeta, k, \bar{\zeta}) \right] G_{L,\alpha\beta}(1, k) & = G_{L,\alpha\beta}(1, k).
\end{align*}
\]

(A3)

These have the forms

\[
\begin{align*}
w^{1}_{L}(\zeta) & = 1, & w^{2}_{L}(\zeta) & = (\zeta - 1) \tan \theta_{L}(k)e^{i \xi_{L}(k)}, \\
w^{1}_{L}(\zeta) & = 1, & w^{2}_{L}(\zeta) & = (\zeta^{*} - 1) \tan \theta_{L}(k)e^{i \xi_{L}(k)}, \\
w^{1}_{L}(\zeta) & = 1, & w^{2}_{L}(\zeta) & = (\zeta^{*} - 1) \tan \theta_{L}(k)e^{i \xi_{L}(k)}.
\end{align*}
\]

(A4)

By simple substitutions we can also obtain the similar solutions for the part corresponding to $G_{R}^{1}(\zeta, k)$. Finally the Thouless operator is:

\[
\tilde{S}(\zeta) = \tilde{S}_{L}(\zeta)\tilde{S}_{R}(\zeta) = \tilde{S}_{L,A}(\zeta)\tilde{S}_{R,B}(\zeta)\tilde{S}_{R,C}(\zeta)\tilde{S}_{L,A}(\zeta)\tilde{S}_{L,B}(\zeta)\tilde{S}_{L,C}(\zeta).
\]

(A5)
\[ \hat{S}_{LA}(\zeta) = \prod_{\alpha \beta \beta_k} \left[ 1 + (\zeta - 1) \epsilon_{\alpha \beta 3} \tan \theta_A^2(k) e^{i \xi_A^2(k)} \hat{a}_{L1\alpha}(k) \hat{a}_{L2\beta}(-k) \right], \]

\[ \hat{S}_{RB}(\zeta) = \prod_{\alpha \beta \beta_k} \left[ 1 + (\zeta^* - 1) \epsilon_{\alpha 3 \beta} \tan \theta_B^2(k) e^{i \xi_B^2(k)} \hat{b}_{R1\alpha}(k) \hat{b}_{R2\beta}(-k) \right], \]

\[ \hat{S}_{RC}(\zeta) = \prod_{\alpha \beta \beta_k} \left[ 1 + (\zeta - 1) \epsilon_{\alpha 3 \beta} \tan \theta_C^2(k) e^{i \xi_C^2(k)} \hat{c}_{R1\alpha}(k) \hat{c}_{R2\beta}(-k) \right], \]

\[ \hat{S}_{RA}(\zeta) = \prod_{\alpha \beta \beta_k} \left[ 1 + (\zeta^* - 1) \epsilon_{\alpha \beta 3} \tan \theta_A^1(k) e^{i \xi_A^1(k)} \hat{a}_{R1\alpha}(k) \hat{a}_{R2\beta}(-k) \right], \]

\[ \hat{S}_{LB}(\zeta) = \prod_{\alpha \beta \beta_k} \left[ 1 + (\zeta^* - 1) \epsilon_{\alpha 3 \beta} \tan \theta_B^1(k) e^{i \xi_B^1(k)} \hat{b}_{L1\alpha}(k) \hat{b}_{L2\beta}(-k) \right], \]

\[ \hat{S}_{LC}(\zeta) = \prod_{\alpha \beta \beta_k} \left[ 1 + (\zeta^* - 1) \epsilon_{\alpha \beta 3} \tan \theta_C^1(k) e^{i \xi_C^1(k)} \hat{c}_{L1\alpha}(k) \hat{c}_{L2\beta}(-k) \right]. \]  

(A6)

To apply Wick’s theorem it is preferable to express the equations above directly in terms of the quasi-particle operators:

\[ \hat{S}_{LA}(\zeta) |\Psi\rangle = \prod_{\alpha \beta \beta_k} \left[ (\cos^2 \theta_A^2(k) + \zeta \sin^2 \theta_A^2(k)) \right. \]

\[ + (\zeta - 1) \epsilon_{\alpha \beta 3} \sin \theta_A^2(k) \cos \theta_A^2(k) e^{i \xi_A^2(k)} \hat{a}_{L1\alpha}(k) \hat{a}_{L2\beta}(-k) |\Psi\rangle, \]

\[ \hat{S}_{RA}(\zeta) |\Psi\rangle = \prod_{\alpha \beta \beta_k} \left[ (\cos^2 \theta_A^1(k) + \zeta \sin^2 \theta_A^1(k)) \right. \]

\[ + (\zeta^* - 1) \epsilon_{\alpha \beta 3} \sin \theta_A^1(k) \cos \theta_A^1(k) e^{i \xi_A^1(k)} \hat{a}_{R1\alpha}(k) \hat{a}_{R2\beta}(-k) |\Psi\rangle, \]

\[ \hat{S}_{RB}(\zeta) |\Psi\rangle = \prod_{\alpha \beta \beta_k} \left[ (\cos^2 \theta_B^2(k) + \zeta^* \sin^2 \theta_B^2(k)) \right. \]

\[ + (\zeta^* - 1) \epsilon_{\alpha 3 \beta} \sin \theta_B^2(k) \cos \theta_B^2(k) e^{i \xi_B^2(k)} \hat{b}_{R1\alpha}(k) \hat{b}_{R2\beta}(-k) |\Psi\rangle, \]

\[ \hat{S}_{LB}(\zeta) |\Psi\rangle = \prod_{\alpha \beta \beta_k} \left[ (\cos^2 \theta_B^1(k) + \zeta^* \sin^2 \theta_B^1(k)) \right. \]

\[ + (\zeta^* - 1) \epsilon_{\alpha \beta 3} \sin \theta_B^1(k) \cos \theta_B^1(k) e^{i \xi_B^1(k)} \hat{b}_{L1\alpha}(k) \hat{b}_{L2\beta}(-k) |\Psi\rangle, \]

\[ \hat{S}_{RC}(\zeta) |\Psi\rangle = \prod_{\alpha \beta \beta_k} \left[ (\cos^2 \theta_C^2(k) + \zeta^* \sin^2 \theta_C^2(k)) \right. \]

\[ + (\zeta^* - 1) \epsilon_{\alpha \beta 3} \sin \theta_C^2(k) \cos \theta_C^2(k) e^{i \xi_C^2(k)} \hat{c}_{R1\alpha}(k) \hat{c}_{R2\beta}(-k) |\Psi\rangle, \]

\[ \hat{S}_{LC}(\zeta) |\Psi\rangle = \prod_{\alpha \beta \beta_k} \left[ (\cos^2 \theta_C^1(k) + \zeta^* \sin^2 \theta_C^1(k)) \right. \]

\[ + (\zeta^* - 1) \epsilon_{\alpha \beta 3} \sin \theta_C^1(k) \cos \theta_C^1(k) e^{i \xi_C^1(k)} \hat{c}_{L1\alpha}(k) \hat{c}_{L2\beta}(-k) |\Psi\rangle. \]  

(A7)
3. Integrals

We have seen in Section III that the matrix elements of operators in number projected states can be expressed in terms of a limited number of integrals, namely

\begin{align*}
J_{\sigma,\mu}(\theta_1, \theta_2) &\equiv \frac{1}{2\pi i} \int \frac{d\zeta}{\zeta^{n+1}} S^{(0)}(\zeta) \frac{\zeta}{(\cos^2 \theta_1 + \zeta \sin^2 \theta_1)(\cos^2 \theta_2 + \zeta \sin^2 \theta_2)}, \\
J_{\sigma,\rho}(\theta_1, \theta_2) &\equiv \frac{1}{2\pi i} \int \frac{d\zeta}{\zeta^{n+1}} S^{(0)}(\zeta) \frac{\zeta^*}{(\cos^2 \theta_1 + \zeta^* \sin^2 \theta_1)(\cos^2 \theta_2 + \zeta^* \sin^2 \theta_2)}, \\
J_{\sigma,\rho}(\theta_1, \theta_2) &\equiv \frac{1}{2\pi i} \int \frac{d\zeta}{\zeta^{n+1}} S^{(0)}(\zeta) \frac{1}{(\cos^2 \theta_1 + \zeta \sin^2 \theta_1)(\cos^2 \theta_2 + \zeta \sin^2 \theta_2)},
\end{align*}

(A8)

The integrals for the single particle operators are obtained from those above:

\[ I_{(\sigma,\beta,c),\sigma}(\theta) \equiv J_{(\sigma,\beta,c),\sigma}(\theta, 0) \]  

(A9)

APPENDIX B: COLOUR PROJECTION

1. Thouless operator

We can now give the explicit expression for the Thouless operator \( W \), which can be derived in a similar fashion to \( S \). Following the same approach as before we first look for an operator, \( T \), that is at most bilinear in the quark operators and that it can be applied to \( G(k, \zeta) \) to give the colour rotated operator \( \tilde{G}(k, \zeta, \phi) \).

For example, in order to obtain the operators in Eq. (31) we need

\begin{align*}
t_{LA12}(k, \phi) &\equiv 1 + w_1 a_{LA11}^\dagger(k) a_{LA11}(k) + w_2 a_{LA12}^\dagger(k) a_{LA12}(k) \\
t_{LA21}(k, \phi) &\equiv 1 + w_3 a_{LA21}^\dagger(-k) a_{LA21}(-k) + w_4 a_{LA23}^\dagger(-k) a_{LA23}(-k).
\end{align*}

(B1)

The conditions

\begin{align*}
t_{LA12}(k, \phi) G_{LA12}(k, \zeta) |k_F\rangle &= \tilde{G}_{LA12}(k, \zeta, \phi) |k_F\rangle \\
t_{LA21}(k, \phi) G_{LA21}(k, \zeta) |k_F\rangle &= \tilde{G}_{LA21}(k, \zeta, \phi) |k_F\rangle.
\end{align*}

(B2)

yield the solutions

\[ w_1 = w_3 = \cos \frac{\phi}{2} - 1, \quad w_2 = w_4 = -\sin \frac{\phi}{2}. \]

(B3)

Finally one can write the total operator \( \tilde{T} \) in the factorised form:

\[ \tilde{T}(\phi) \equiv \prod_{(\alpha\beta)k} t_{LA\alpha\beta}(k, \phi) t_{LB\alpha\beta}(k, \phi) t_{LC\alpha\beta}(k, \phi) t_{RA\alpha\beta}(k, \phi) t_{RB\alpha\beta}(k, \phi) t_{RC\alpha\beta}(k, \phi). \]

(B4)

As a result, the Thouless operator for the combined colour and number projection will be obtained by the application of \( T \) and \( S \):

\[ W(\zeta, \phi) \equiv T(\phi) \tilde{S}(\zeta) \]

\[ = \prod_{(\alpha\beta)k} w_{LA\alpha\beta}(k) w_{LB\alpha\beta}(k) w_{LC\alpha\beta}(k) w_{RA\alpha\beta}(k) w_{RB\alpha\beta}(k) w_{RC\alpha\beta}(k). \]

(B5)

We will simply state the results for the component corresponding to left handed particles:

\[ W_{LA}(\phi, \zeta) = W_{LA}^{(0)}(\phi, \zeta) + W_{LA}^{(1)}(\phi, \zeta) + \ldots \]

(B6)

where

\[ W_{LA}^{(0)}(\phi, \zeta) = \prod_{(\alpha\beta)k} \left( \cos^2 \theta_A^\dagger(k) + \zeta \cos \frac{\phi}{2} \sin^2 \theta_A^\dagger(k) \right). \]

(B7)
and

\[ W_{LA}^{(1)}(\phi, \zeta) = W_{LA}^{(2)}(\phi) \]

\[
\sum_{\gamma \in \mathcal{L}} \gamma \eta (k) \left( \cos \theta_4(k) \cos \theta_5(k) e^{i \xi(k)} \right)
\left( \cos^2 \theta_4(k) + \zeta \cos \theta_5(k) \sin^2 \theta_4(k) \right)
\left( \alpha_{L,13}^+(1, k) \alpha_{L,13}^-(1, k) \right) \}

\[ \times \left\{ \right. \]

The last term pairs the third colour with one of the other two. This term does not contribute to the expectation value of the Hamiltonian.

2. Integrals

In analogy with Sec. 3 we now define the following objects:

\[ J_{a_n}(\theta_1, \theta_2) \equiv \frac{1}{2\pi} \int \frac{d\zeta}{\zeta^{n+1}} \int d\Omega_5 W^{(0)}(\zeta, \phi) \]

\[ = \frac{1}{2\pi} \int \frac{d\zeta}{\zeta^{n+1}} \int d\Omega_5 W^{(0)}(\zeta, \phi) \]

\[ J_{b_n}(\theta_1, \theta_2) \equiv \frac{1}{2\pi} \int \frac{d\zeta}{\zeta^{n+1}} \int d\Omega_5 W^{(0)}(\zeta, \phi) \]

\[ J_{c_n}(\theta_1, \theta_2) \equiv \frac{1}{2\pi} \int \frac{d\zeta}{\zeta^{n+1}} \int d\Omega_5 W^{(0)}(\zeta, \phi) \]

where \( d\Omega_5 \) is given by Eq. (28). These reduce to the integrals \( J \) of Eq. (A8) if \( \phi \) is set to zero and the colour-group integral is suppressed.

The integrals for single particle operators will be obtained from the ones above as a particular case:

\[ I_{(a,b,c),n}(\theta) \equiv J_{(a,b,c),n}(\theta, 0). \]

In order to calculate these integrals, we write the Laurent series for \( W^{(0)}(\zeta, \phi) \) as

\[ W^{(0)}(\zeta, \phi) = \sum_{n} d_n(\phi) \zeta^n. \]

Numerically it is found that the Laurent coefficients as a function of colour angle \( \phi \) can be fit by Gaussians with a single parameter \( \alpha \):

\[ d_n(\phi) \approx \frac{d_n(0)}{\sqrt{2\pi} \sigma} e^{-\frac{\phi^2}{2\sigma^2}}. \]

This approximation is particularly useful in the numerical calculation, since it allows us to calculate the integrals analytically. With this simplification, the integrals in Eq. (B9) can be written in a closed form as:

\[ J_{a_n}(\theta_1, \theta_2) = \sum_{m} d_m(0) \left[ I_{1} \Lambda_{n,m}(\theta_1, \theta_2) \right. \]

\[ + \frac{I_2}{2} \left( \sin^2 \theta_1 \sin^2 \theta_2 \Delta_{n,m+2}(\theta_1, \theta_2) - \cos^2 \theta_1 \cos^2 \theta_2 \Delta_{n,m}(\theta_1, \theta_2) \right) \]

\[ J_{b_n}(\theta_1, \theta_2) = \sum_{m} d_m(0) \left[ I_{1} \Lambda_{n,m}(\theta_1^2, \theta_2^2) \right. \]

\[ + \frac{I_2}{2} \left( \sin^2 \theta_1 \sin^2 \theta_2 \Delta_{n,m}(\theta_1^2, \theta_2^2) - \cos^2 \theta_1 \cos^2 \theta_2 \Delta_{n,m+2}(\theta_1^2, \theta_2^2) \right) \]

\[ J_{c_n}(\theta_1, \theta_2) = \sum_{m} d_m(0) \left[ I_{1} \Lambda_{n,m}(\theta_1^2, \theta_2^2) \right. \]

\[ + \frac{I_2}{2} \left( \cos^2 \theta_1 \sin^2 \theta_2 \Delta_{n,m+2}(\theta_1^2, \theta_2) + \sin^2 \theta_1 \cos^2 \theta_2 \Delta_{n,m}(\theta_1^2, \theta_2) \right. \]

\[ + \left. 2 \sin^2 \theta_1 \sin^2 \theta_2 \Delta_{n,m+1}(\theta_1^2, \theta_2) \right), \]

(B13)
where $\theta^c = \frac{1}{2} \pi - \theta$ and we have defined

\[
\Delta_{n,m}(\theta_1, \theta_2) \equiv \frac{1}{2 \pi i} \int \frac{d\zeta}{\zeta^{n-m}} \frac{1}{(\cos^2 \theta_1 + \sin^2 \theta_1 \zeta)(\cos^2 \theta_2 + \sin^2 \theta_2 \zeta)},
\]

\[
\Delta_{n,m}(\theta_1, \theta_2) \equiv \frac{1}{2 \pi i} \int \frac{d\zeta}{\zeta^{n-m}} \frac{1}{(\cos^2 \theta_1 + \sin^2 \theta_1 \zeta)(\cos^2 \theta_2 + \sin^2 \theta_2 \zeta)}. \tag{B14}
\]

and

\[
I_1 \equiv \int d\Omega_2 \; e^{-\frac{\alpha^2 \sin^2 \theta}{2}} = \frac{2 - (\alpha + 2)}{\alpha^2} e^{-\frac{\alpha^2}{2}},
\]

\[
I_2 \equiv \int d\Omega_2 \; \sin^2 \frac{\theta}{2} \; e^{-\frac{\alpha^2 \sin^2 \theta}{2}} = \frac{8 - (\alpha^2 + 4 \alpha + 8)}{\alpha^3} e^{-\frac{\alpha^2}{2}}. \tag{B15}
\]

The parameter $\alpha$ is defined in Eq. (B12). The integrals of Eq. (B14) can be also evaluated analytically.

References: