Light Cone QED in a Homogeneous Electric Background†

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ABSTRACT

I present an exact solution for the Heisenberg picture, Dirac electron in the presence of an electric field which depends arbitrarily upon the light cone time parameter $x^+ = (t+x)/\sqrt{2}$. This is the largest class of background fields for which the mode functions have ever been obtained. The solution applies to electrons of any mass and in any spacetime dimension. The traditional ambiguity at $p^+ = 0$ is explicitly resolved. It turns out that the initial value operators include not only $(I + \gamma^0\gamma^1)\psi$ at $x^+ = 0$ but also $(I - \gamma^0\gamma^1)\psi$ at $x^- = -L$. Pair creation is a discrete and instantaneous event on the light cone, so one can compute the particle production rate in real time. In $D = 1+1$ dimensions one can also see the anomaly. Another novel feature of the solution is that the expectation value of the currents operators depends non-analytically upon the background field. This seems to suggest a new, strong phase of QED.

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1 Introduction

I will be reporting on work done with my good friends Nikolaos Tsamis and Theodore Tomaras, both from the University of Crete. Our study concerns a generalization of Schwinger’s classic treatment of QED in the presence of an external electromagnetic field [1]. Recall that Schwinger computed what is now known as the one loop effective action for two special classes of background fields: the case of constant $F_{\mu\nu}$ and the case of a plane wave solution of the free Maxwell equations. What we did is to solve the Dirac equation for the electron field operator in the presence of an electric background field $\vec{E} = E(x^+)\hat{x}$ which points in the $\hat{x}$ direction and which depends arbitrarily upon the light cone time parameter $x^+ \equiv (t + x)/\sqrt{2}$ [2, 3]. We were originally seeking a system in which back-reaction could be studied without the complications that attend the analogous problem in the presence of a background metric. However, the solution turns out to have a number of interesting applications in its own right which are the subject of this report.

I will first give the solution and then devote a section to each application.

We define the light cone coordinates as $x^\pm \equiv (t \pm x)/\sqrt{2}$. The remaining, $(D-2)$ transverse coordinates are denoted thusly: $x_\perp$. We work in the gauge where $A_+ = 0$ and,

$$A_-(x^+) = -\int_0^{x^+} du E(u) .$$

The transverse components of the vector potential vanish, $A_\perp = 0$. It is useful as well to define ± spinor components,

$$\psi_\pm(x^+, x^-, x_\perp) \equiv \frac{1}{2}(I \pm \gamma^0\gamma^1)\psi(x^+, x^-, x_\perp) .$$

With these conventions the Dirac equation reduces to the following system,

$$i\partial_+ \psi_+ = \frac{1}{2}(m + i\nabla_\perp \cdot \gamma_\perp)\gamma^- \psi_- ,$$  

$$i\partial_- - eA_- \psi_- = \frac{1}{2}(m + i\nabla_\perp \cdot \gamma_\perp)\gamma^+ \psi_+ .$$

The Fourier transform on $x^-$ does not properly exist, but it is possible to transform on the transverse coordinates as usual,

$$\tilde{\psi}_\pm(x^+, x^-, k_\perp) \equiv \int d^{D-2}x_\perp e^{-ik_\perp \cdot x_\perp} \psi_\pm(x^+, x^-, x_\perp) .$$

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In this notation the general solution for $\tilde{\psi}_+$ is [3],

$$
\tilde{\psi}_+(x^+, x^-, k_\perp) = \int_{-L}^{\infty} dy^- \int_{-\infty}^{+\infty} \frac{dk^+}{2\pi} e^{i(k^+ + i/L)(y^- - x^-)}
\left\{ E[A_-](0, x^+; k^+, k_\perp) \tilde{\psi}_+(0, y^-, k_\perp) - \frac{i}{2} (m - k_\perp \cdot \gamma_\perp) \gamma^- \times \int_0^{x^+} dy^+ e^{-iA_-} e^{-A_-} (y^+) \int_{x^-}^{x^+} \frac{du}{k^+ - eA_- (u) + i/L} \right\},
$$

(6)

where the $E$-dependent mode function is,

$$
E[A_-](y^+, x^+; k^+, k_\perp) \equiv \exp \left[ -\frac{i}{2} \omega_\perp^2 \int_{y^+}^{x^+} \frac{du}{k^+ - eA_- (u) + i/L} \right],
$$

(7)

and $\omega_\perp \equiv m^2 + k_\perp \cdot k_\perp$. The field $\tilde{\psi}_-$ is obtained by differentiating $\tilde{\psi}_+$,

$$
\tilde{\psi}_-(x^+, x^-, k_\perp) = \left( \frac{m - k_\perp \cdot \gamma_\perp}{\omega_\perp^2} \right) \gamma^- i \partial_\perp \tilde{\psi}_+(x^+, x^-, k_\perp).
$$

(8)

Fig. 1: The domain of solution is shaded.
The domain of our solution is depicted in Fig. 1. It is defined by $x^+ > 0$ and $x^- > -L$, for arbitrary $L$. The initial value operators consist of

$$\psi_+(0, x^-, x_\perp), \text{ for all } x^- > -L, \text{ and } \psi_-(x^+, -L, x_\perp) \text{ for all } x^+ > 0.$$  

Canonical quantization reveals the nonvanishing portion of the anti-commutation algebra to be,[2]

$$\{\psi_+(0, x^-, x_\perp), \psi^\dagger_-(0, y^-, y_\perp)\} = \frac{P_+}{\sqrt{2}} \delta(x^- - y^-)\delta^{D-2}(x_\perp - y_\perp),$$  \hspace{1cm} (9)

$$\{\psi_-(x^+, -L, x_\perp), \psi^\dagger_-(y^+, -L, y_\perp)\} = \frac{P_-}{\sqrt{2}} \delta(x^- - y^-)\delta^{D-2}(x_\perp - y_\perp),$$  \hspace{1cm} (10)

where $P_\pm \equiv (I \pm \gamma^0 \gamma^1)/2$. We assume that the initial value operators act upon “the vacuum” in the same way as they do for zero electric field. Computing the VEV of any operator therefore consists of first employing (6-8) to express that operator in terms of the initial value operators, and then taking the expectation value of these in the free theory. It is often useful to take the large $L$ limit as well.

2 Particle production

By simply deleting the $k^+$ integration from (6) we obtain a finite $L$ analog of the Fourier transform on $x^-$,

$$\psi_+(x^+, k^+, k_\perp) \equiv \int_{-L}^{x^+} dy^- e^{-i(k^+ + i/L)y^-} \left\{ \mathcal{E}(0, x^+; k^+, k_\perp)\psi_+(0, y^-, k_\perp) - \frac{i}{2}(m - k_\perp \cdot \gamma_\perp)\gamma^- \right\}
\times \int_0^{x^+} dy^+ e^{-ieA_-(y^+ + L)}\mathcal{E}(y^+, x^+; k^+, k_\perp)\psi_-(y^+, -L, k_\perp) \right\}. \hspace{1cm} (11)$$

This operator comes very close to being an eigenfunction of the light cone Hamiltonian,

$$-i\frac{\partial}{\partial x^+}\psi_+(x^+, k^+, k_\perp) = -\frac{\omega^2_+}{2}\frac{\psi_+(x^+, k^+, k_\perp)}{k^+ - eA_-(x^+) + i/L} - \frac{1}{2}(m - k_\perp \cdot \gamma_\perp)\gamma^- \psi_-(x^+, -L, k_\perp) + \frac{ie^{-i(k^+ + i/L)L}}{k^+ - eA_-(x^+) + i/L}. \hspace{1cm} (12)$$
In the limit of large $L$ the second term contributes only at $k^+ = eA_-(x^+)$,
\[
\lim_{L \to \infty} \frac{ie^{-i(k^- - eA_-(x^+)) + i/L}L}{k^+ - eA_-(x^+)} = 2\pi \delta \left(k^+ - eA_-(x^+)\right). \tag{13}
\]

In the limit of infinite $L$ this means that, for $k^+ > eA_-(x^+)$, the field (11) annihilates electrons whereas, for $k^+ < eA_-(x^+)$, it must create positrons. To find the amplitude we evaluate its anti-commutator,
\[
\{\hat{\psi}_+(x^+, k^+, k_\perp), \hat{\psi}^+_\dagger(x^+, q^+, q_\perp)\} = \frac{P_+ ie^{-i(k^- - q^+ + 2i/L)L}}{\sqrt{2} k^+ - q^+ + 2i/L} (2\pi)^D - 2\delta^{D-2}(k_\perp - q_\perp). \tag{14}
\]

From (13-14) we see that, in the large $L$ limit, $\hat{\psi}_+(x^+, k^+, k_\perp)$ creates or destroys particles with amplitude $2^{-1/4}$ within the $P_+$ spinor subspace.

Let us assume that the electric field is positive, in which case $eA_-(x^+)$ is a monotonically increasing function of $x^+$. Hence evolution in $x^+$ carries more and more electron annihilation operators through the singular point at which $k^+ = eA_-(x^+)$, after which they become positron creation operators. The Heisenberg picture vacuum does not change, but since the meaning of which operator creates a particle changes, so too must our interpretation of the vacuum state’s occupation number. To find the probability that the vacuum contains a positron of momentum $k^+$ and $k_\perp$ in one of the two spin states we compute,
\[
\lim_{L \to \infty} \sqrt{2} \left\langle \Omega \left| \hat{\psi}_+(x^+, k^+, k_\perp) \hat{\psi}^+_\dagger(x^+, q^+, q_\perp) \right| \Omega \right\rangle = \left[1 - \text{Prob}(x^+, k^+, k_\perp)\right] \frac{P_+ 2\pi \delta(k^+ - q^+)(2\pi)^D - 2\delta^{D-2}(k_\perp - q_\perp)}{(2\pi)^D - 2\delta^{D-2}(k_\perp - q_\perp)}. \tag{15}
\]

The result can be expressed in terms of the functions $X(k^+)$ and $\lambda(k^+, k_\perp)$,
\[
k^+ \equiv eA_-(X(k^+)) \quad , \quad \lambda(k^+, k_\perp) \equiv \frac{\omega_\perp^2}{2|eE(X(k^+))|}. \tag{16}
\]

The positron creation probability turns out to be [2, 3],
\[
\text{Prob}(x^+, k^+, k_\perp) = \theta(k^+)\theta \left(eA_-(x^+) - k^+\right) e^{-2\pi \lambda(k^+, k_\perp)}. \tag{17}
\]
From the theta functions in (17) we see that particle creation is an instantaneous and discrete event on the light cone. Why this is so follows from the fact that evolving in $x^+$ can be regarded as the infinite boost limit of evolving in time [4]. Suppose we consider a primed frame in which the empty state is specified at $t' = 0$. The homogeneous electric field will result in particle creation, but the created particles will possess finite, nonzero $p^\pm$. Now consider the light cone momenta in the frame obtained by boosting to velocity $\beta$ in the $\hat{x}$ direction,

$$p^+ = \sqrt{\frac{1-\beta}{1+\beta}} p'^+ \quad , \quad p^- = \sqrt{\frac{1+\beta}{1-\beta}} p'^-.$$  \hspace{1cm} (18)

As $\beta$ approaches unity we see that $p^+$ goes to zero. Therefore any particle created in the light cone problem comes out with $p^+ = 0$. But the physical momentum in a background vector potential is the minimally coupled one, $p^+ = k^+ - eA_-(x^+)$. Hence pair creation occurs at the instant when $k^+ = eA_-(x^+)$. Note also that $p^- \to \infty$ at creation, in conformity with the eigenvalue we read off from (12).

We can also compute the instantaneous rate of particle production in $D = 3 + 1$ dimensions. The probability of still being in vacuum at $x^+$ is [2],

$$P_{\text{vac}}(x^+) = \prod_{0<k^+<eA_-} \prod_{k_{\perp}} \left[ 1 - e^{-2\pi\lambda(k^+,k_{\perp})} \right]^2,$$

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### 3 The current operators

It might seem curious that we only see the creation of positrons. In fact one electron is created for each positron, however, the newly created electrons immediately leave the light cone manifold. To understand this consider the evolution of a virtual pair as depicted in Fig. 2. Electrons are accelerated to the speed of light in the negative $\hat{x}$ direction, which is parallel to the $x^-$ axis. They never evolve past a certain value of $x^+$. In contrast, positrons are accelerated to the speed of light in the positive $\hat{x}$ direction, which is parallel to the $x^+$ axis. We should therefore see the $J^+$ operator grow as the manifold fills up with positrons. We should also see $J^- \propto (L + x^-)$ from the flux of newly created electrons originating all the way back to $x^- = -L$.

![Fig. 2: The evolution of a virtual $e^+e^-$ pair.](image)

Explicit calculations verify both expectations. The light cone current density is $J^+$ and our result for it in $D = 3 + 1$ dimensions is [2],

$$
\lim_{L \to \infty} \left\langle \Omega \right\vert J^+(x^+, x^-, x_\perp) \left\vert \Omega \right\rangle = -2e \int_0^{eA_-(x^+)} \frac{dp^+}{2\pi} \int \frac{d^2p_\perp}{(2\pi)^2} e^{-2\pi\lambda(p^+, p_\perp)}.
$$

(23)
Once one accepts (17) for the creation probability, this result is simple to understand. Each positron carries charge \(-e\), the probability of creating either spin state with momenta \(k^+\) and \(k^\perp\) is \(\exp[-2\pi\lambda(k^+, k^\perp)]\), there are two spin states, and the number of modes per unit volume is \(dp^+/2\pi \times d^2p^\perp/(2\pi)^2\). So the differential increment in the light cone charge density should be,

\[
dJ^+ = (-e) \times (e^{-2\pi\lambda(k^+, k^\perp)}) \times (2) \times \left(\frac{dp^+}{2\pi}\right) \times \left(\frac{d^2p^\perp}{(2\pi)^2}\right). \tag{24}
\]

Our result (23) for \(J^+\) is just the integral of this over the modes which have undergone pair creation. In exact analogy, we expect the increment from each element \(dx^-\) to the electron flux to be,

\[
dJ^- = (+e) \times (e^{-2\pi\lambda(k^+, k^\perp)}) \times (2) \times \left(-eE(x^+)dx^-\right) \times \left(\frac{d^2p^\perp}{(2\pi)^2}\right). \tag{25}
\]

The total result is the integral from \(-L\) to \(x^-\), plus the ultraviolet divergent integration constant. The expectation values of the transverse currents are zero, so we see that current is indeed conserved.

### 4 Axial vector anomaly in \(D = 1+1\)

To get the current operator VEV’s in \(D = 1+1\) dimensions it is only necessary to drop the transverse coordinates, change the number of spin states from two to one, and drop the ultraviolet divergent integration constant for \(J^-\). The results are [3],

\[
\left\langle \Omega \left| J^+ (x^+, x^-) \right| \Omega \right\rangle = -e \int_0^{eA^-} \frac{dp^+}{2\pi} e^{-2\pi\lambda(p^+)} + O \left(\frac{\ln(L)}{L}\right), \tag{26}
\]

\[
\left\langle \Omega \left| J^- (x^+, x^-) \right| \Omega \right\rangle = -(L + x^-) \frac{e^2E(x^+)}{2\pi} e^{-2\pi\lambda(eA^-)} + O \left(\frac{\ln(L)}{L}\right). \tag{27}
\]

In \(D = 1 + 1\) the components of the axial vector current operator are proportional to those of the vector current operator,

\[
J_5^\pm(x^+, x^-) = \pm \frac{1}{e} J^\pm(x^+, x^-). \tag{28}
\]
Hence the divergence of the axial vector current is,

\[ \lim_{L \to \infty} \langle \Omega \left| \partial_+ J_5^+ + \partial_- J_5^- \right| \Omega \rangle = \frac{e^2 E(x^+)}{\pi} e^{-2\pi \lambda(eA_-)} . \]  \hfill (29)

To complete the computation we need the VEV of the pseudo-scalar, which is \( J_5 = \bar{\psi} \gamma_5 \psi \) up to technicalities of ordering and regularization. After some straightforward manipulations we obtain [3],

\[ \lim_{L \to \infty} \langle \Omega \left| J_5(x^+, x^-) \right| \Omega \rangle = \frac{ieE(x^+)}{2\pi m} \left[ 1 - e^{-2\pi \lambda(eA_-)} \right] . \]  \hfill (30)

Combining with (29) we find,

\[ \lim_{L \to \infty} \langle \Omega \left| \partial_\mu J_5^\mu (x^+, x^-) - 2iemJ_5(x^+, x^-) \right| \Omega \rangle = \frac{e^2}{\pi} E(x^+) . \]  \hfill (31)

As far as I know, this is the first time the axial vector anomaly has been checked for massive QED in \( D = 1 + 1 \) dimensions.

To make the same check in \( D = 3 + 1 \) dimensions one must extend the background to include at least a constant magnetic field which is parallel to the electric field. The relevant mode functions have been worked out by my graduate student, Marc Soussa. Evaluation of the various fermion bilinears in the presence of this background is already far advanced.

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**References**


