Noncommutative Quantum Mechanics from Noncommutative Quantum Field Theory

Pei-Ming Ho¹, Hsien-Chung Kao²

¹Department of Physics, National Taiwan University, Taipei, Taiwan, R.O.C.
²Department of Physics, Tamkang University, Tamsui, Taiwan, R.O.C.

pmho@phys.ntu.edu.tw
hckao@mail.tku.edu.tw

Abstract

We derive noncommutative multi-particle quantum mechanics from noncommutative quantum field theory in the nonrelativistic limit. Particles of opposite charges are found to have opposite noncommutativity. As a result, there is no noncommutative correction to the hydrogen atom spectrum at the tree level. We also comment on the obstacles to take noncommutative phenomenology seriously, and propose a way to construct noncommutative $SU(5)$ grand unified theory.


1 Introduction

Recently there has been a growing interest in noncommutative geometry as well as its phenomenological implications. This was motivated by the discovery in string theory that the low energy effective theory of D-brane in the background of NS-NS $B$ field lives on noncommutative space [1]-[6]. In the brane world scenario [7], our spacetime may be the worldvolume of a D-brane, and thus may be noncommutative. In fact, apart from string theory, it has long been suggested that the spacetime may be noncommutative as a quantum effect of gravity, and it may provide a natural way to regularize quantum field theories [8, 9].

In many proposals to test the hypothetical spacetime noncommutativity, one does not need the exact quantum field theory, but only its quantum mechanical approximation. Although noncommutative quantum mechanics (NCQM) has already been extensively studied [10]-[17], there are misunderstandings in the literature that we want to clarify. The main point is that the noncommutativity $\theta_{ab}$ is not the same for all particles in NCQM. The noncommutativity of a particle should be opposite to (differ by a sign from) that of its anti-particle; and the noncommutativity of a charged particle should be opposite to any other particle of opposite charge. Our basic assumption is that NCQM should be viewed as an approximation of a noncommutative field theory (NCFT) in which all fields live on the same noncommutative space.

We will always assume that the time coordinate $t$ is commutative. Otherwise the formulation of quantum mechanics may require drastic modification [18].

In Sec. 2 we illustrate some ambiguities in defining NCQM. To resolve these ambiguities we derive NCQM from NCFT in Sec. 3. We find that the noncommutativity of particle coordinates depends on the charge. This implies that there is no correction to the spectrum of the hydrogen atom due to noncommutativity at the tree level (Sec. 4). We generalize these results in Sec. 5. In the last section, we comment on the obstacles to a complete, consistent description of noncommutative phenomenology. Since there exist particles with electric charges other than $e = 0, \pm 1$, electromagnetic interaction can not be consistently described as a noncommutative $U(1)$ gauge theory. Hence we propose a way to construct noncommutative $SU(5)$ grand unified theory, where all charges are already properly quantized, as a better theoretical basis for noncommutative phenomenology.
2 Two-Particle System

Naively, to define a physical system on noncommutative space, we simply take the Lagrangian for ordinary space and replace all products by star products. For example, one tends to claim that the noncommutative Schrödinger equation for a Hydrogen atom is [19, 12]

$$i \frac{\partial}{\partial t} \psi = -\frac{\nabla^2}{2m_e} \psi + V(x) \ast \psi,$$

where

$$V = -\frac{e^2}{|x|}$$

is the electric potential of the proton, and the * product is defined by

$$f(x) \ast g(x) \equiv e^{i\frac{\theta}{2} \partial_x \partial_{x'} f(x) |_{x'=x}} g(x').$$

Here $x$ should be interpreted as the relative coordinate between the electron and the proton

$$x^a = x_e^a - x_p^a, \quad a = 1, 2, 3.$$

This means that the commutation relation for $x$ should be derived from those for $x_e$ and $x_p$. Suppose

$$[x_e^a, x_e^b] = i\theta_e^{ab}, \quad [x_p^a, x_p^b] = i\theta_p^{ab}, \quad [x_e^a, x_p^b] = 0,$$

then

$$[x^a, x^b] = i(\theta_e^{ab} + \theta_p^{ab}).$$

We will show below that we should take $\theta_e = -\theta_p$ and thus $x$ is actually commutative!

If we assume that the proton has infinite mass and is localized at the origin as a delta function, we can interpret $x$ as the coordinate of the electron. Then it would make sense to say that $x$ is a coordinate on the noncommutative space. However, as it was pointed out in [20], the use of delta function on noncommutative space invalidates the perturbative expansion in $\theta$. It is also unnatural to assume an extreme localization of proton on a noncommutative space.

To clarify this problem, we note that a complete description of the Hydrogen atom should be given by the total wave function $\Psi(x_e, x_p)$. On classical space, the Schrödinger

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1There is an ambiguity in the ordering of the last term. It could as well be $\psi \ast V$. However, replacing $V \ast \psi$ by $\psi \ast V$ is equivalent to replacing $\theta$ by $-\theta$. Without specifying $\theta$, we can choose either case without loss of generality.
equation is

\[ i \frac{\partial}{\partial t} \Psi = \left( -\frac{\nabla^2}{2m_e} - \frac{\nabla^2}{2m_p} + V(x_e, x_p) \right) \Psi, \quad (7) \]

where

\[ V(x_e, x_p) = -\frac{e^2}{|x_e - x_p|}. \quad (8) \]

When we try to modify this equation to the noncommutative case, we have to face the following ambiguities. First, we need to find \( V \) on noncommutative space, and specify the ordering of \( V \) and \( \Psi \) in the Schrödinger equation. Although this was not a problem for the wave function of a single particle, it is a problem for multi-particle states. The reason is that there is a new possibility for which the coordinate \( x_e \) in \( V \) is multiplied from the left, and \( x_p \) in \( V \) from the right, to the wave function \( \Psi \). This can be written as

\[ V \star_{++} \Psi \]

by defining \( \star_{\epsilon_1 \epsilon_2} \) as

\[ f(x_e, x_p) \star_{\epsilon_1 \epsilon_2} g(x_e, x_p) \equiv e^{i\theta_{ab}(\epsilon_1 \frac{\partial}{\partial x_e} \frac{\partial}{\partial x'_e} + \epsilon_2 \frac{\partial}{\partial x_p} \frac{\partial}{\partial x'_p})} f(x_e, x_p) g(x'_e, x'_p) |_{x' = x}. \quad (10) \]

The star product (10) assumes that \( x_e \) commutes with \( x_p \), although it is mathematically consistent to assume that \( x_e \) does not commute with \( x_p \).

To fix these ambiguities, the basic assumption in our discussion below is that NCQM is a nonrelativistic approximation of NCFT in which all fields live on the same noncommutative space. Without this assumption, the proton and electron coordinates may have arbitrary independent noncommutativity.

### 3 NCQM from NCFT

Consider the NCFT of some charged particles and a \( U(1) \) gauge field. The action is of the form

\[ S = \sum_\alpha S_\alpha + S_A. \quad (11) \]

\( S_\alpha \) is the action for a charged particle. For instance, for a fermion in the fundamental representation of the gauge group, it is

\[ S_\alpha = \int d^4x \bar{\psi}_\alpha \star (iD + m_\alpha) \star \psi_\alpha, \quad (12) \]

where \( m_\alpha \) is the mass of the particle \( \alpha \) and

\[ D_\mu = \partial_\mu + A_\mu. \quad (13) \]
The action for the $U(1)$ gauge field is

$$S_A = \int d^4 x F_{\mu \nu} * F^{\mu \nu}, \quad (14)$$

where

$$F_{\mu \nu} = [D_\mu, D_\nu]. \quad (15)$$

On a noncommutative space, even the $U(1)$ gauge group is non-Abelian. Therefore all fields must have the same charge: particles have charge +1 and anti-particles have charge −1.

In order to derive NCQM from NCFT, we repeat what we do for the commutative case. First, we collect those terms in the action involving $A_\mu$

$$\int d^4 x (J^\mu * A_\mu) + S_A, \quad (16)$$

where $J = \sum_\alpha j_\alpha$, and the current density for the fermion in (12) is

$$j^\mu_\alpha = i \bar{\psi}_\alpha (\gamma^0 \gamma^\mu)^T \psi^\dagger_\alpha. \quad (17)$$

Now we can integrate out $A_\mu$ and find the effective interaction between the charged particles. In the weak coupling limit or weak field limit where we can ignore the self-interaction of $A_\mu$, one finds the effective interaction

$$S_I = \int d^4 x d^4 x' J^\mu(x) * G_{\mu \nu}(x, x') *' J^\nu(x') = \int dt H_I, \quad (18)$$

where $G$ is the photon propagator in a certain gauge, and *' means *-product with respect to $x'$.

In fact, we can ignore the *'s between $J$ and $A_\mu$ in (16) because

$$\int d^4 x f(x) * g(x) = \int d^4 f(x) g(x). \quad (19)$$

We can also drop the *'s in (18), and $G$ is simply the usual propagator on commutative space.

Decomposing each field into positive and negative frequency modes

$$\psi = \int d^3 k (b_{ks}(t) u_{ks} e^{ik_i x_i} + d^\dagger_{ks}(t) v_{ks} e^{-ik_i x_i}), \quad (20)$$

where $b$ is the annihilation operator for the particle, $d^\dagger$ is the creation operator for its anti-particle, and the particle index $\alpha$ is suppressed. We will ignore the spinor index $s$ as it will not play a role in our problem. In the operator formulation, one can define the field operators

$$\hat{\psi}_- \equiv \int d^3 k b_{ks}(t) e^{ik_i x_i} \quad (21)$$
for the particle $\alpha$, and
\[
\hat{\psi}_- \equiv \int d^3k e^{ik\cdot x(t)}
\] (22)
for its anti-particle $\bar{\alpha}$. The quantum mechanical wave function for a two-particle state $|\xi\rangle$ in the NCFT is
\[
\Psi_{(\alpha\epsilon_1)(\beta\epsilon_2)}(x_1, x_2) \equiv \langle 0|\hat{\psi}_{\alpha\epsilon_1}(x_1)\hat{\psi}_{\beta\epsilon_2}(x_2)|\xi\rangle.
\] (23)

Here $x_1, x_2$ are viewed as commutative coordinates in the star product representation. Thus the coordinates for different particles in the wavefunction $\Psi$ always commute with one another by definition. Similarly, one can define the wave function for a state of an arbitrary number of particles and anti-particles.

The Schrödinger equation is a result of the fact that $\psi_\alpha$ satisfies its equation of motion, which can be written as
\[
i\dot{\psi}(x) = [H, \psi(x)]
\] (24)
in terms of the Hamiltonian $H$. For the effective action, $H$ is
\[
H = H_0 - H_I,
\] (25)
where $H_0$ is the kinetic term and $H_I$ is given by (18). Thus, for example,
\[
i\frac{\partial}{\partial t} \Psi_{AB} = \langle 0|[H, \hat{\psi}_A\hat{\psi}_B]|\xi\rangle,
\] (26)
where $A = (\alpha\epsilon_1)$ and $B = (\beta\epsilon_2)$.

Straightforward derivation shows that, in the non-relativistic approximation where the interaction is dominated by the Coulomb potential, the Schrödinger equation is given by
\[
i\frac{\partial}{\partial t} \Psi_{AB}(x_1, x_2) = \left(-\frac{\nabla_1^2}{2m_\alpha} - \frac{\nabla_2^2}{2m_\beta} + V(x_1, x_2)\right)\ast_{\epsilon_1\epsilon_2} \Psi_{AB}(x_1, x_2),
\] (27)
where $\ast_{\epsilon_1\epsilon_2}$ is defined in (10), and $V$ given by (8).

While the above prescription applies to generic interactions, for our special case of a gauge field, the result above (27) can be easily obtained by demanding gauge symmetry. For a field in the fundamental representation,
\[
\hat{\psi}_+ \rightarrow U \ast \hat{\psi}_+,
\] (28)
\[
\hat{\psi}_- \rightarrow \hat{\psi}_- \ast U^\dagger
\] (29)
under a gauge transformation. This implies that the covariant derivative must act on the wave function \( \Psi \) from the left for particles and from the right for anti-particles. Since the electric potential \( V \) is just the time component of the gauge potential \( A_\mu \), we immediately reach the same conclusion as in (27).

Since all fields must have the same charge in a noncommutative gauge theory, it is equivalent to say that the coordinate of each particle of positive (negative) charge in \( V \) is multiplied to the wave function by the star product with parameter \( \theta (-\theta) \).

In the context of string theory, for an open string ending on a D-brane, the two endpoints appear as opposite charges to the D-brane gauge field. In a \( B \) field background, the two endpoints also observe opposite noncommutativity [3]. It was first argued in [3] that the NCFT on a single noncommutative space automatically takes care of this effect. In this section we provided a rigorous derivation.

It is interesting to note that although it is a matter of pure convention whether one uses \( \psi \) to represent, say, the electron or positron, once we have made the choice, there is no more freedom in making this choice for any other fields living on the same noncommutative space. Here we also see that the charge conjugation results in a change of noncommutativity \( \theta \rightarrow -\theta \) [21].

4 Separation of Variables

To solve the Schrödinger equation for multi-particle wave functions, we use the technique of separation of variables. For the Hydrogen atom, the Schrödinger equation is

\[
i \frac{\partial}{\partial t} \Psi(x_e, x_p) = \left( -\frac{\nabla_x^2}{2m_e} - \frac{\nabla_p^2}{2m_p} + V(x_e, x_p) \right) \ast -+ \Psi(x_e, x_p), \tag{30}\]

where we choose the convention that the noncommutativity parameter \( \theta (-\theta) \) is associated with positive (negative) charges.

Since the kinetic term is not modified, we take the ansatz

\[
\Psi(x_e, x_p) = \Phi(X)\psi(x), \tag{31}\]

where

\[
X = \frac{m_e x_e + m_p x_p}{m_e + m_p} \tag{32}\]

is the center of mass (COM) coordinate, and

\[
x = x_e - x_p \tag{33}\]
is the relative coordinate. The noncommutativity for these coordinates is given by

\[ [X^i, X^j]_{--} = \frac{m_p - m_e}{m_p + m_e} \theta^{ij} \equiv i \theta^{ij}_p, \quad (34) \]
\[ [x^i, x^j]_{--} = 0, \quad (35) \]
\[ [x^i, X^j]_{--} = i \theta^{ij} \equiv i \theta^{ij}_q. \quad (36) \]

The kinetic term can be rewritten as

\[ \frac{\nabla_e^2}{2m_e} + \frac{\nabla_p^2}{2m_p} = \frac{\nabla_X^2}{2M} + \frac{\nabla_x^2}{2m}, \quad (37) \]

where

\[ M = m_e + m_p \quad (38) \]

is the total mass and

\[ m = \frac{m_em_p}{m_e + m_p} \quad (39) \]

is the reduced mass.

For the Fourier mode of \( X \),

\[ \Psi(X) = e^{-iEt+iK_iX^i} \psi(x), \quad (40) \]

(30) is reduced to

\[ \left( E - \frac{K^2}{2M} \right) \psi(x) = \left( -\frac{\nabla_x^2}{2m} + V(x - \frac{1}{2} \theta_{ep} K) \right) \psi(x). \quad (41) \]

Note that translational invariance implies that \( V \) can only depend on the relative coordinate \( x \). Let \( \psi(x) = \psi'(x - \frac{1}{2} \theta_{ep} K) \). Since (41) contains no star product, it is exactly the same equation for classical space in terms of \( \psi' \). Unless we include self-interactions of the gauge field, the whole spectrum is exactly the same as the commutative case! The shift in the relative coordinate is easy to understand from the D-brane picture, where space non-commutativity is resulted from background \( B \) field.

Therefore, for example, the noncommutative correction to Lamb shift should be much smaller than the one given in [19]. There is no correction at tree level. The lowest order contribution of \( \theta \) comes from the one-loop diagrams and is negligible.

## 5 Generalization

In [22] it was shown that a matter field in the fundamental representation is not allowed to couple to more than two different gauge fields on noncommutative space. So far we
have only considered the case of one gauge field. It is straightforward to include another
gauge field.

Suppose that there are \( m \) particles. Let the charges of particle \( \alpha \) \((\alpha = 1, \cdots, m)\) be \( q^\alpha \), where \( q^\alpha = 1, 0, -1 \). If \( q^\alpha = 1 \), it means that the field operator for particle \( \alpha \), which was denoted as \( \hat{\psi}_+ \) before, transforms from the left

\[
\phi_\alpha \rightarrow U \ast \phi_\alpha. \quad (42)
\]

It transforms from the right

\[
\phi_\alpha \rightarrow \phi_\alpha \ast U \quad (43)
\]

if \( q^\alpha = -1 \). The covariant derivative of a field operator is

\[
D_\mu \phi_\alpha = \partial_\mu \phi_\alpha + q^\alpha A_\mu \ast q^\alpha \phi_\alpha, \quad (44)
\]

where \( \ast \) is the star products with the parameter \( \pm \theta \).

If we repeat the derivation in the previous sections, the Schrödinger equation for \( N \) particles is

\[
i \frac{\partial}{\partial t} \Psi(x_1, \cdots, x_N) = - \sum_{\alpha=1}^{N} \frac{\nabla_i^2}{2m_i} \Psi + \frac{1}{2} \sum_{\alpha \neq \beta} q_\alpha q_\beta V(x_\alpha, x_\beta) \ast q_\alpha q_\beta \Psi, \quad (45)
\]

where \( V \) is the (00) component of the Green’s function for the gauge field \( A_\mu \).

The COM coordinates \( X^\mu \) of the system satisfies

\[
[X^\mu, X^\nu]_{(q_i)} = i \frac{\sum_{\alpha=1}^{N} q_\alpha^2 m_i^2}{\sum_{\beta} m_\beta^2} \theta^{\mu\nu}. \quad (46)
\]

It is easy to see that the magnitude of the noncommutativity is never larger than \( |\theta| \).

A composite particle is a system of \( N \) particles which has a bound state with a small spatial extension. The COM coordinates of the system will be taken as the coordinates of the composite particle. If the size of the composite particle is larger than \( \sqrt{\theta} \), it is meaningless to talk about its noncommutativity. In the case of a Hydrogen atom, the relative coordinates \( x \) is commutative, thus its size can be arbitrarily small. On the other hand, if some relative coordinates for the constituents of the composite particle are noncommutative, which is always the case as long as there are three or more charged constituent particles, the size of the composite particle must be larger than the order of \( \sqrt{\theta} \), and hence the noncommutativity of the composite particle can be neglected for most purposes.
6 Discussion

On noncommutative (NC) space, charges are always quantized, even for $U(1)$ gauge field. However, in the standard model, there are particles of electric charges $1/3, 2/3$ etc. It implies that the electromagnetic interaction cannot be a NC $U(1)$ gauge theory. Similarly, the $U(1)$ gauge group for hypercharges cannot be noncommutative, either [23]. In the $SU(5)$ Grand Unified Theory (GUT), on the other hand, all charges are already quantized. There are fractional hypercharges only because the $U(1)$ group is embedded in $SU(5)$ with a generator $T = \text{diag}(1/3, 1/3, 1/3, -1/2, -1/2)$. But there are other problems for NCGUT. The first problem is to define NC $SU(5)$ gauge symmetry.

A possible resolution of this problem [24] is to define NC $SU(N)$ gauge symmetry as the image of the classical $SU(N)$ via Seiberg-Witten (SW) map [6]
\[
\hat{A} = \hat{A}(A),
\]
where quantities without (with) hats are commutative (noncommutative) fields. It is consistent with gauge transformations to restrict $A_\mu$ to the Lie algebra of $SU(N)$. The same idea can be used to define the noncommutative version of any classical group [24].

It is also possible to define NC $SU(5)$ theory directly in terms of the noncommutative variable $\hat{A}$ without mentioning the commutative $A$. We can simply take the NC $U(N)$ gauge field $\hat{A}$ and impose the following constraint
\[
C_{\mu\nu}(k) \equiv \text{Tr} F_{\mu\nu}(\hat{A})(k) = 0,
\]
where $F_{\mu\nu}(\hat{A})$ is the inverse SW map. (An exact expression for the inverse SW map was given in [25, 26].) This implies that the $U(1)$ part of $A_\mu$ can be gauged away, and the result is equivalent to the approach of [24].

It is interesting to note that another constraint with a much simpler expression
\[
C'_{\mu\nu}(k) \equiv \text{Tr} \int d^4x \hat{F}_{\mu\nu}(x) \ast e^{ik_\mu(x_\mu + i\theta^\mu\nu A_\nu)} = 0
\]
is also gauge invariant and has the same classical limit $\text{Tr} F_{\mu\nu} = 0$. At this moment we do not know if these two constraints are exactly the same.

Recently, a similar idea was proposed independently in [27], where the constraint was imposed on $\hat{A}$ instead. Another constraint on gauge transformations $U$ has to be imposed simultaneously for consistency [27]. It would be of interest to know if all such constraints are equivalent under field redefinitions.
Another problem about NCGUT is that there are matter fields in the symmetric representation of $SU(5)$, while the gauge field $\hat{A}$ is defined in the fundamental representations.

This problem can also be solved by using the SW map. For a $D$ dimensional representation of $SU(5)$, we consider the SW map for the NC $U(D)$ gauge field $\hat{A}^{(D)} = A^{(D)}(A^{(D)})$. Since it is consistent with gauge transformations to restrict $A^{(D)}$ to the subgroup $SU(5)$ embedded in $U(D)$, its image under the SW map can be defined as the $D$ dimensional representation of NC $SU(5)$. The covariant derivative of a matter field in this representation is

$$D_\mu \phi = (\partial_\mu + \hat{A}_\mu^{(D)}(A))\phi,$$

where $A$ is the commutative $SU(5)$ gauge potential.

Finally, due to the UV-IR mixing, the UV divergences of NC quantum field theories result in new IR poles nonperturbative in $\theta$ [28, 29]. For a comprehensive discussion on this problem see [30]. In order to give a reliable, consistent description of NC electromagnetic interactions, or any other low energy phenomena on NC space, it is necessary to properly address all these problems. We leave these issues for future study.

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**References**


12