We discuss the problems of dark matter, quantum gravity, and vacuum energy within the context of a theory for which Lorentz invariance is not postulated, but instead emerges as a natural consequence in the physical regimes where it has been tested.

In earlier work \(^1,^2\), we introduced a theory which implies violation of Lorentz invariance for (i) fermions at extremely high energy and (ii) fundamental scalar bosons which have not yet been observed. On the other hand, the theory appears to be in agreement with even the most sensitive experimental and observational tests of Lorentz invariance that are currently available, since many features of this symmetry are preserved, including rotational invariance, CPT invariance, and the same velocity \(c\) for all massless particles.

1 Dark Matter

Let us begin with the dark matter problem. It appears that conventional models of cold dark matter predict too much structure on small distance scales \(^3\). Since the dark matter almost certainly consists of particles of a new kind, let us allow for the possibility that \(v_0 \neq 0\), where \(v_0\) is the limiting value of the particle velocity \(v(p)\) as the 3-momentum \(\vec{p}\) goes to zero. Suppose that the particle energy \(\varepsilon\) is expanded as a Taylor series in the magnitude \(p\) of the 3-momentum:

\[
\varepsilon = \varepsilon(p) = \varepsilon_0 + pv_0 + p^2/2\tilde{m} + \ldots.
\]

(For conventional nonrelativistic particles, \(\tilde{m}\) is the particle mass, \(\varepsilon_0\) is the rest mass energy, and \(v_0 = 0\); for particles with zero rest mass, \(v_0 = c = 1\) and \(\varepsilon_0 = \tilde{m}^{-1} = 0\).) The particle velocity is then

\[
v = d\varepsilon/dp = v_0 + p/\tilde{m} + \ldots
\]

and the kinetic energy is

\[
T = \int v\, dp = \varepsilon(p) - \varepsilon_0 = pv_0 + p^2/2\tilde{m} + \ldots.
\]
The virial theorem implies that
\[ \langle pv \rangle = \langle \vec{p} \cdot \vec{v} \rangle = \langle \vec{F} \cdot \vec{r} \rangle = \langle r \, dV/dr \rangle = -\langle V \rangle \]  
(4)

where it has been assumed that \( V = -GMm/r \) with \( M \) constant. Since (3) can also be written as
\[ T = pv - \int p \, dv \]  
(5)

the binding energy \( -E \) of a particle with 3-momentum \( p \) is given by the simple expression
\[ -E = -\langle T + V \rangle = \left\langle \int p \, dv \right\rangle = \langle p^2 \rangle / 2\tilde{m} + \ldots \approx \langle p^2 \rangle / 2\tilde{m}. \]  
(6)

If \( v_0 = 0 \) (as for a conventional nonrelativistic particle), the momentum is determined by
\[ -\langle V \rangle = \langle pv \rangle = \langle pv_0 + p^2 / \tilde{m} + \ldots \rangle \approx \langle p^2 \rangle / \tilde{m} \quad \text{or} \quad \langle p^2 \rangle \approx \langle GMm\tilde{m}/r \rangle \]  
(7)

and the energy has the familiar form
\[ E \approx \langle V \rangle / 2. \]  
(8)

On the other hand, if \( v_0 \neq 0 \), the momentum is determined by
\[ -\langle V \rangle = \langle pv \rangle = \langle pv_0 + p^2 / \tilde{m} + \ldots \rangle \approx \langle p \rangle v_0 \quad \text{or} \quad \langle p \rangle \approx \langle GMm/v_0r \rangle \]  
(9)

and the binding energy is much smaller:
\[ E \sim -\frac{1}{2\tilde{mv}_0^2} \langle V \rangle^2. \]  
(10)

The specific form of \( \varepsilon(p) \) in the fundamental theory of Refs. 1-3 yields
\[ v_0 = c \left[ 1 + \left( \frac{2m}{\tilde{m}} \right)^2 \right]^{-1/2}, \quad \tilde{m}v_0^2 = mc^2 \left( \frac{\tilde{m}}{2m} \right)^3 \frac{c}{v_0}. \]  
(11)

It is interesting, however, that a general model with \( v_0 \neq 0 \) leads to the weaker binding (10), and thus to a weaker tendency to form both small-scale structure and cusps near the centers of galactic halos, apparently in agreement with the observations.
2 Gravity and Gauge Fields

Now let us turn to gravity and gauge fields, which have a radically new interpretation in the present theory:

The gravitational vierbein $e^a_\mu$ is identified with the “superfluid velocity” $v^a_\mu$ of a GUT Higgs field which condenses in the very early universe. The Euclidean action of this condensate initially has the form $I(3.3)$:

$$S_0 = \int d^Dx \bar{\Psi} \left( T + \frac{1}{2} V - \mu \right) \Psi_s \quad \text{with} \quad V = b \Psi^\dagger_s \Psi_s. \quad (12)$$

A local minimum in $S$ is given by $\delta S = 0$ for arbitrary variations $\delta \Psi_s$ and $\delta \bar{\Psi}^\dagger_s$. The arguments in Section 3 of Ref. 1 then lead to a Bernoulli equation $m v^2/2 + V + P = \mu$. For an additional bosonic or fermionic field, the Euclidean action initially has the form $I(4.1)$,

$$S_a = \int d^Dx \bar{\Psi}^\dagger_a (T + V - \mu) \Psi_a \quad (13)$$

if terms of order $(\Psi^\dagger_s \Psi_s)^2$ are neglected. When $\Psi_s$ satisfies its equation of motion, the Bernoulli equation holds, and it can be used in (13) to obtain the generalization of $I(9.5)$ given in Ref. 2:

$$L_a = \frac{1}{2} \bar{\psi} \left( -\bar{m}^{-1} \bar{g}^{\mu \nu} \partial_\mu \psi^\dagger_a \partial_\nu \psi_a + i \bar{\psi}^\dagger_a \sigma^\alpha \psi^\dagger_a \sigma^\alpha \partial_\mu \psi_a \right) + h.c. \quad (14)$$

Here $\psi_a$ is the field in a four-dimensional Lorentzian description, as observed in the frame of reference that is “moving with the condensate”. $L_a$ is actually an effective Lagrangian, which yields the same equation of motion for $\psi_a$ in a gravitational field as would be obtained from (13) (when $\Psi_s$ also satisfies its equation of motion). For fermions at low energy, this is exactly the same as in standard physics. The role of fermions and fundamental bosons as sources of gravity will be discussed elsewhere.

The Einstein field equations are also given by $\delta S = 0$, but this time for variations in the metric tensor $g^{\mu \nu}$. In a Euclidean picture, we search for a minimum in $S$ with respect to $g^{\mu \nu}$, while remaining on the minimum with respect to $\bar{\Psi}^\dagger_s$ that is represented by the equation of motion. (This is analogous to searching for the state of a particle with minimum energy $\varepsilon_k$, while requiring that $\psi_k$ always satisfy the Schrödinger equation.) In a Lorentzian picture, we search for an extremum in the Lorentzian action $S_L$, while again requiring that $\Psi_s$ always satisfy its equation of motion.

In the present theory, the curvature of gravitational and gauge fields can only result from topological defects, and it is these defects which also give
rise to the Einstein-Hilbert Lagrangian \( \mathcal{L}_G = \ell_p^{-2} g^{(4)} R \) and the gauge-field Lagrangian \( \mathcal{L}_g = -\left(4g_0^2\right)^{-1} g F_{\mu \nu}^I g^{\mu \rho} g^{\nu \sigma}. \) (Here \( g_0 \) is the coupling constant and \( g = |\det e^\mu_\alpha| = |\det g_{\mu \nu}|^{1/2}. \) In Ref. 1, we considered defects with point singularities. Suppose, however, that we assume (i) a short-distance cutoff \( a_0 \sim \ell_p \) (which is implied by the microscopic treatment of Ref. 2, with \( \ell_p \) the Planck length) and (ii) a long-distance cutoff \( R_0 \) (analogous to that in a superconductor) which results from screening. With these assumptions, defects with line singularities – i.e., vortex lines – will have finite action per unit area in four-dimensional Euclidean spacetime, in analogy with the finite energy per unit length for a vortex line in a superconductor. For example, consider a vortex line which has a length \( \ell \) in 3-space and a duration \( \Delta t \), so that the 4-dimensional volume is \( \sim \ell \Delta t \ell_p^2 \) in units with \( c = 1. \) (These vortex lines will ordinarily be extended vortex rings, which can arise in the condensate, expand, and shrink back to zero over some finite period of time.) The contribution to the Euclidean action is then given by

\[
\Delta S \propto \int d^4 x \, n_s \, m v_\theta^2 \propto \ell \Delta t \left( n_s / m \right) \int d^2 x \, v_\theta^2 \propto \ell \Delta t \ell_p^2 \log \left( R_0 / a_0 \right) \sim \ell \Delta t / \ell_p^2, \tag{15}
\]

where \( v_\theta = (m v^\mu) \) is the “superfluid velocity” around the vortex. Also, the contribution to the square of a gauge curvature is essentially given by

\[
\ell \Delta t \int d^2 x \left( \partial_1 m v^2 - \partial_2 m v^1 \right)^2 \propto \ell \Delta t \int d^2 x \, v_\theta^4 \propto \ell \Delta t \left( a_0^{-2} - R_0^{-2} \right) \sim \ell \Delta t / \ell_p^2. \tag{16}
\]

The contribution to the action is thus equal to the contribution to (16) multiplied by a dimensionless constant of order unity.

The simplest example of a vortex line producing curvature and action is one in which \( m v^\mu \) is identified with \( e A^\mu \), where \( e \) is the fundamental charge and \( A^\mu \) is the electromagnetic vector potential (in a convention which differs from that of Ref. 1 by the factor of \( e \)). For example, with \( B_z = \partial_x A_y - \partial_y A_x \), a vortex line with a Planck-scale core makes a discrete contribution to the magnetic flux \( \Phi \), as well as to the action (15) and to the quantity (16) which provides a measure of the action. In the present picture, the magnetic flux through a surface is a time average of the contributions from a large number of Planck-scale vortex lines. The flux contributed by one vortex line is a flux quantum \( \phi_0 \):

\[
\Phi = \int_\partial S \, d\sigma \, \cdot \, B_z = \int_C A_\theta \, r \, d\theta \propto \int_{C, \, e^{-1} r^{-1}} r \, d\theta = 2\pi / e = 2\pi \hbar c / e = \phi_0, \text{ with } \hbar \text{ and } c \text{ restored in the next to last expression. Due to rapid fluctuations in the number and positions of these topological defects, however, the contributions of individual defects cannot be easily resolved, and the field appears to be continuous on length and time scales which are large compared to } \ell_p.
\]
This picture can be rather straightforwardly extended to the full electromagnetic field, to nonabelian gauge fields with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, to gauge fields in curved spacetime, and to the gravitational field with

$$\omega^{\alpha\beta}_\mu = \frac{1}{2} e^{\nu\alpha} (\partial_\mu e^{\nu\beta}_\beta - \partial_\nu e^{\nu\beta}_\mu) - \frac{1}{2} e^{\nu\beta} (\partial_\mu e^{\nu\alpha}_\alpha - \partial_\nu e^{\nu\alpha}_\mu) - \frac{1}{2} e^{\rho\alpha} e^{\sigma\beta} (\partial_\rho e^{\sigma\gamma}_\gamma - \partial_\sigma e^{\rho\gamma}_\gamma) e^\gamma_\mu,$$

$$e^\mu_\alpha = w^\mu_\alpha, \quad R_{\mu\nu}^{\alpha\beta} = \partial_\mu \omega^{\nu\alpha}_\beta - \partial_\nu \omega^{\mu\alpha}_\beta + [\omega^{\mu\alpha}, \omega^{\nu\beta}]_\beta, \quad R_{\mu\nu}^{\alpha\beta} = e^{\alpha}_\sigma e^{\beta}_\tau R_{\mu\nu}^{\sigma\tau}. \quad (17)$$

A detailed treatment will be given in a longer paper.

Notice that quantum gravity is finite in the present picture, since the Einstein-Hilbert action is valid only on length scales that are large compared to $\ell_P$.

3 Vacuum Energy

Finally, consider the vacuum stress-energy tensor $T_{\mu\nu}^{\text{vac}} = -(2/g) \delta S_{\text{vac}}/\delta g^{\mu\nu}$. The equation of motion for $\Psi_s$, and the Bernoulli equation below (12), represent a local minimum in the Euclidean action $S$ ($\delta S = 0$ for arbitrary $\delta \Psi^1_s$), but they do not represent a minimum with respect to variations in $g^{\mu\nu}$ ($\delta S = 0$ as $g^{\mu\nu}$ is varied, with $\Psi_s$ always required to satisfy its equation of motion). There does not, in fact, appear to be any obvious reason why $T_{\mu\nu}^{\text{vac}}$ should be nearly zero within the simplest formulation of the present theory. The vacuum energy problem therefore remains just as big a mystery in the present theory as it is in standard field theory and in superstring/M theory.

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References

1. R. E. Allen, Int. J. Mod. Phys. A 12, 2385 (1997); hep-th/9612041. Equations from this paper will be distinguished with the prefix I – e.g., I(3.3).