In addition to the above, the fundamental interaction giving rise to the

$$\tau = \frac{g}{\sqrt{8\pi G}} \left( \frac{\phi}{\phi_0} - \sqrt{\frac{\phi}{\phi_0}} \right) + \frac{\eta}{\sqrt{8\pi G}} \left( \frac{\phi}{\phi_0} \right)$$

of the fundamental interaction.

A fundamental question that arises in the


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mix of field and metric fluctuations in an arbitrary gauge [10], where as the tensor modes are purely metric.

It is not clear that a short-distance cutoff will affect fluctuations of the metric in the same way as fluctuations in an arbitrary scalar field. However, general coordinate invariance implies that we can transform (for example) to a gauge in which even the scalar fluctuations are purely “metric”, and if this property is preserved at short distances then the effect of new physics on the scalars and tensors should be identical. Even in the existing literature, metric (gravitational wave) fluctuations can be treated as a generic scalar field at short distances. This is also reasonable, but not inevitable.

Following [6], tensor fluctuations \( v_k \) obey:

\[
\ddot{v}_k^\prime + \frac{\nu^\prime}{\nu} \dot{v}_k^\prime + \left( \mu - \frac{d^\prime}{a} - \frac{d^\prime \nu^\prime}{a \nu} \right) v_k = 0
\]

where \( a \) is the scale factor, the prime denotes differentiation with respect to conformal time \( \eta \), while \( k^2 = a \rho^2 e^{-\beta \eta^2/2} \) with \( \rho^2 \) being the Fourier transform of the physical coordinates \( x^i \), and

\[
\mu(\eta, \rho) \equiv \frac{a^2 \rho^2}{(1 - \beta \rho^2)^2}, \quad \nu(\eta, \rho) \equiv \frac{e^{\frac{1}{2} \beta \eta^2}}{(1 - \beta \rho^2)^2}.
\]

When evaluating the derivatives of \( v_k \) with respect to \( \eta \), we are looking \( k \) (and not the usual comoving momentum \( k \)) fixed with time. It is therefore convenient to express \( \mu \) and \( \nu \) in terms of \( k \) by introducing the Lambert W function [12], which is defined so that \( W(x e^x) = x \):

\[
\mu = -\frac{a^2}{\beta} \frac{W(\zeta)}{(1 + W(\zeta))^2}, \quad \nu = \frac{\nu'}{\nu} = \frac{a' W(\zeta) (5 + 3W(\zeta))}{a (1 + W(\zeta))^2}.
\]

where \( \zeta = -\beta k^2 / a^2 \).

For power law inflation

\[
a(t) = t^p, \quad a(\eta) = \left( \frac{\eta}{\eta_0} \right)^q, \quad q = \frac{p}{1 - p}.
\]

For inflation to occur, we need \( p > 1 \).

The cutoff is introduced by requiring that \( \rho^2 \leq 1 / \beta \), motivated by the notions of a minimum distance in string theory and the so-called “stringy uncertainty principle” [13]. Fluctuations with comoving wavenumber \( k \) reach the cutoff \( \rho^2 = 1 / \beta \) at \( \eta_k \), where

\[
\eta_k = \eta_0 \left( \frac{\epsilon \beta k^2}{1} \right)^{1/2q} = \frac{1}{1 - p} \left( \frac{\epsilon \beta k^2}{1} \right)^{1/2q}
\]

where the implicit definition of \( \eta_0 \) comes from setting \( a(t) = 1 \) when \( t = 1 \), and \( t \) is the usual physical time.

Writing \( \eta = \eta_0 (1 - y) \) in order to extract the \( k \) dependence and abbreviating \( W(\zeta) \) as \( W \), we have

\[
\ddot{v}_k - \frac{q(q - 1)}{1 - y} \frac{W(5 + 3W)}{(1 + W)^2} \dot{v}_k + \left( \frac{1 - y}{1 - y} \right)^{2q} \frac{W(5 + 3W)}{(1 + W)^2} \frac{\beta \eta_k^q (1 + W)}{\beta W(\zeta) (1 + W(\zeta))^2} v_k = 0
\]

In the de Sitter case, \( k \) can be eliminated from equation of motion, but this cannot be done here. During power-law inflation, different modes sample a different value of the Hubble constant as they cross the horizon, and this will be reflected in the scale-dependent modifications to the spectrum we will observe. Despite this, our analysis of the de Sitter case [8] can easily be adapted to the power-law problem.

When \( \zeta = -1 / \epsilon \), \( W(\zeta) \) has a branch point [12]. Physically, this is the moment when \( y = 0 \) and \( \eta = \eta_k \) and the fluctuation with wavelength \( k \) is “created”. As in [8], we solve for the leading behavior of \( v_k \) by extracting the most singular terms of the equation of motion,

\[
\ddot{v}_k - \frac{1}{2y} \dot{v}_k + \frac{\lambda_k}{y} v_k = 0,
\]

where dots denote derivatives with respect to \( y \), and

\[
\lambda_k = \frac{1}{q \frac{\beta k^2}{1 - \beta k^2} - \frac{q}{2}}.
\]

The solution to (8) is:

\[
v_k^{(0)}(y) = y^{3/4} \left( CH^{(2)}_{-\beta k^2/2} (2\sqrt{A_k y}) + DH^{(1)}_{-\beta k^2/2} (2\sqrt{A_k y}) \right).
\]

Here, \( H^{(2)}_{-\beta k^2/2} \) is the second Hankel function, \( C \) and \( D \) are constants. The first Hankel function is its complex conjugate \( H^{(1)}_{-\beta k^2/2} = \text{H}^{(2)*} \).

The solution is normalized by the Wronskian condition

\[
v_k^{(0)}(\eta) v_k^{(0)'}(\eta) - v_k^{(0)*}(\eta) v_k^{(0)'}(\eta) = i \left( 4 - \beta k^2 \right) e^{-\frac{1}{2} \beta \eta^2}.
\]

Using Hankel function identities [14] we deduce

\[
|C|^2 - |D|^2 = \eta_0 \pi \sqrt{-q e^{-3q/2}}.
\]

If \( D \neq 0 \), the spectrum never approaches the exact power-law form, and we thus make the “minimal” vacuum choice that \( D = 0 \). Setting \( q = -1 \) reproduces the de Sitter result, after reconciling the normalization factors. We solve the mode equations numerically [8, 11] and match the numerical solution to the approximate analytical form, including sub-leading corrections, near \( y = 0 \). We obtain the scalar spectrum by solving the mode equations for multiple values of \( k \), and then extracting the necessary late time limit to compute

\[
P_g^{1/2} = \frac{\beta}{2 \pi^2} \left( \frac{\eta_0}{\kappa - \epsilon H} \right)^{1/2} \eta^{1/2} \frac{P_s^{1/2}}{32 \pi^2 P_g^{1/2}}.
\]

where we have obtained the scalar spectrum from the tensor one, as outlined above.

The spectrum is only well defined if the ratio of the minimum length (\( \sqrt{3}H \)) is less than the horizon size (\( 1 / H \) ), or \( \sqrt{3}H \) is less than unity. The critical mode, \( k_{\text{crit}} \), that crosses the horizon at the moment when \( \sqrt{3}H = 1 \) is

\[
k_{\text{crit}} = \eta_0 \left( \frac{\epsilon \beta k^2}{1} \right)^{1/2q}.
\]
For large values of $p$, $\kappa_{\text{crit}}$ is enormous. This reflects the massive amount of inflation that takes place between the Planck time ($t = 1$ in natural units) and the moment at which $\sqrt{\beta} H = 1$; the numerical value of $k$ can always be rescaled by redefining $a_{0}$, the value of $a$ when $t = 1$.

Fig. 1 shows the spectrum for the longest modes, with $p = 100$. There is a large modulation in the spectrum, corresponding to the slow decrease in $H$ as the universe evolves. However, these modes have a much larger amplitude than those contributing to the CMB power spectrum and structure formation. Fig. 2 displays the spectrum with $p = 500$ and a “window” of $k$ values with amplitudes of the same order as the modes which are the precursors to structure formation. We have not carefully normalized this spectrum (which requires assumptions about the dark matter composition, $\Omega_{\Lambda}$, etc.), since any signal of trans-Planckian physics is much smaller than the uncertainty in current available data. Instead we assume that $10^{-6} \lesssim \Delta T/T \lesssim 10^{-5}$, which is given that $(\Delta T/T)^{2} \approx P_{s}/180$ [15] – corresponds to $1.5 \times 10^{-5} \lesssim P_{s}^{1/2} \lesssim 1.4 \times 10^{-4}$.

The standard power-law spectrum is modulated by an “oscillation” whose amplitude and wavelength depend on both the fundamental length $\sqrt{\beta}$ and the power-law parameter, $p$. The oscillations are attributable to successive modes undergoing increasing numbers of periods between the initial time and horizon exit, with a full extra period corresponding to a single oscillation in the spectrum.

The amplitude and period (in log $k$) of the oscillations are roughly proportional to $\sqrt{\beta}$. In principle $\sqrt{\beta}$ is predicted by fundamental theory, but from our perspective here it is a free parameter. If $\sqrt{\beta}$ is identified with the string scale, it could conceivably be two orders of magnitude longer than the Planck length, and we use this value in the numerical plots. Observationally, the key parameter is the ratio of the fundamental length to the Hubble radius $\sqrt{\beta} H$. In power-law inflation (and any other non de Sitter model) $H$ is a slowly changing parameter. The observationally relevant range of $H$ is fixed by the amplitude of the power spectrum, which is deduced from observations of the CMB and large scale structure.

The rate of change of $H$ is determined by $p$, and as $p$ increases the wavelength of the fluctuations in the spectrum increases while their amplitude goes down. This accords with our physical understanding of the oscillations; for a fixed value of $H$ (and thus $P_{s}^{1/2}$), $H$ decreases with increasing $p$. Thus the wavelength of the oscillations (in log $k$) increases with $p$, since $H$ at horizon exit changes more slowly with $k$ at larger $p$. The variation in the amplitude arises because we are effectively holding $H$ fixed at horizon exit, but $H(\tau_{0})$ decreases as $p$ is increased. Consequently, the effective value of $\beta H^{2}$ for a given mode decreases as $p$ is increased, which accounts for the $p$ dependence of the amplitude of the oscillations. The oscillations do not vanish as $p$ becomes arbitrarily large, although their wavelength becomes arbitrarily long, and we approach the de Sitter limit where the spectrum is shifted by a constant multiplicative factor.
In Fig. 2, \( p = 500 \) the spectrum is almost flat, and the \( C_l \) values that would be measured by CMB experiments are modified by between -5 and 1%. A signal of this size lies at the limits of detectability, even with ideal experiments, and would be swamped by cosmic variance at all but the largest values of \( l \). Existing constraints on the spectral index put a weak lower bound on \( p \) of around 20. With this value, the oscillations’ wavelength is so short that the resulting spectrum appears to include a random noise term when plotted over the range of \( k \) values relevant to structure formation.

Despite the extreme challenge and perhaps near impossibility of detecting an effect of this size for reasonable values of \( \beta \), the conclusions of this letter are still much more optimistic than we might have otherwise expected. First, and in accord with our previous de Sitter calculations, we find that the magnitude of the modification to the spectrum is a function of \( (\beta H^2)^n \), where \( n \) appears to be slightly smaller than \( 1/2 \). This disagrees with [7], in which it is argued that \( n \) is roughly unity. However, [7] relies on a WKB approximation to the mode equation and chooses the vacuum to be the purely “-” WKB solution. We have decomposed our numerical solutions into the two WKB solutions at a time when the WKB approximation holds well, and the actual solution (using the initial conditions described above and also advocated by [7]) contains a mixture of both WKB solutions, whereas the result of Eq. 4 in [7] is not consistent with these initial conditions, perhaps explaining why the estimate of [7] for the impact of the fundamental length on the spectrum is significantly less than we find here.

We have assumed that the minimum length lies somewhat below the Planck scale. While this is justifiable from a stringy perspective, if we had put the fundamental length equal to the Planck length \( (\sqrt{\beta} = 1) \) the effect we see decreases significantly. Moreover, there is no guarantee that a fundamental length will modify the mode equations in precisely the manner that we have assumed here, and our results also hinge on this assumption, which will ultimately be checked by rigorous calculation in candidate unified theories. However, while the effects predicted here may be too small to be detected in even the most sophisticated measurements, it is reasonable to believe that they may be only one or two orders of magnitude below the threshold of observability. This stands in stark contrast to the 16 orders of magnitude between the Planck scale and conventional accelerator experiments, and gives us grounds for a modicum of optimism that astrophysical measurements may one day provide experimental tests of Planck scale physics.

In principle existing CMB measurements put experimental restrictions on a portion of the \((\beta, p)\) plane. Given the accuracy of current data, the constraints on \( \beta \) would be extremely weak, and we have not performed this calculation. As CMB data and surveys of Large Scale Structure (and our ability to work backwards from the observed to the primordial spectrum) improve, it may become possible to place meaningful restrictions on short scale physics using astrophysical and cosmological data.

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