A black hole hologram
in de Sitter space

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Abstract

In this paper we show that the entropy of de Sitter space with a black hole in arbitrary dimension can be understood using a modified Cardy-Verlinde entropy formula. We also comment on the observer dependence of the de Sitter entropy and make a conjecture about the form of the conformal anomaly in arbitrary dimension.
1 Introduction

Physics is very different depending on the presence and sign of the cosmological constant. With a vanishing cosmological constant, space time is asymptotically flat and physics can be conveniently described by an S-matrix. This is a consequence of the presence of a light like infinity; if we wait long enough particles may be separated by an arbitrary spacial distance and interactions will be suppressed. In the case of a negative cosmological constant the universe is anti de Sitter (AdS) on large scales and there is no light like infinity. There is, however, a space like infinity which can be used as a holographic screen of Lorentzian signature, [1], [2], [3]. This discovery was the first rigorous implementation of the idea of holography, [4], [5], and has lead to many subsequent studies.

But it is very likely that our universe is neither flat nor AdS but instead is a de Sitter space (dS) on large scales with a positive cosmological constant, [6], [7]. While being the most realistic possibility, de Sitter space is at the same time theoretically the most challenging. To this date there is, in fact, no successful implementation of de Sitter space in string theory.

One of the intriguing features of de Sitter space is the presence of a horizon and an associated temperature and entropy, [8]. While string theory successfully has addressed the problem of entropy for black holes, dS entropy remains a mystery. One reason is that the finite entropy seems to suggest that the Hilbert space of quantum gravity for asymptotically de Sitter space is finite dimensional, [9]. A recent discussion of this and other connected issues can be found in [10]. Another, related, reason is that the horizon and entropy in de Sitter space have an obvious observer dependence. For a black hole in flat space (or even in AdS) we can take the point of view of an outside observer who can assign a unique entropy to the black hole. The problem of what an observer venturing inside the black hole experiences, is much more tricky and has not been given a satisfactory answer within string theory. While the idea of black hole complementarity provides useful clues, [11], rigorous calculations are still limited to the perspective of the outside observer. In de Sitter space there is no way to escape the problem of the observer dependent entropy. This contributes to the difficulty of de Sitter space.

Recently there has been some progress towards a holographic understanding of de Sitter space, [10], [12]. A review with references may be found in [13]. The main observation is that there is a possibility of introducing a holographic screen at either time like past infinity, $I^-$, or time like future infinity, $I^+$. The theory on the screen will, just as in the case of AdS, be a conformal field theory with a scale that encodes the dimension transverse to the screen. But in contrast to the AdS case the holographic theory of de Sitter space will be Euclidean and it is time that is given a description through the scale. Large scales on $I^+$ will correspond to early times, while late times will correspond to small scales. The central charge of the theory is given by the area of the horizon. Very recently it was suggested, [14], (see also [15]) that our universe is described by a RG flow in the Euclidean theory on $I^+$. In the IR
the theory has a fixed point of relatively low central charge corresponding to an early phase of inflation with a horizon a few orders of magnitude larger than the Planck scale. In the UV, on the other hand, there is a fixed point of large central charge corresponding to the universe we now are approaching where a cosmological constant again will be dominating.

In this paper we will continue the study of a holographic description of de Sitter space. In particular we will investigate the properties of black hole holograms in de Sitter space. We will show that the entropy of the cosmological horizon in the presence of a black hole can be understood using a Cardy-Verlinde formula, [16], in a way very similar to the entropy of black holes in AdS. We will also make some speculations on how to address the problem of observer dependence.

2 Constructing the hologram

2.1 The metric

To construct the hologram we must find appropriate coordinates to use when we approach the space like boundary. We will concentrate on \( I^+ \) which seems to be the most reasonable choice in a realistic cosmology. While our universe is expected to become exactly de Sitter in the future, there may very well have been a messy beginning before the inflationary de Sitter space in the early universe which precludes the existence of a well defined \( I^- \). Clearly there are many possible coordinate systems to choose among, but we will concentrate on two important and interesting possibilities: static and cosmological coordinates respectively. Both can be used to approach \( I^+ \) and will lead to different coordinates for the boundary theory. The static choice will lead to what we will denote as cylindrical coordinates, while the cosmological choice will lead to planar coordinates. We will now consider these two possibilities in turn.

2.1.1 Cylindrical coordinates

The d-dimensional de Sitter black hole in static coordinates is given by

\[
d s^2 = -V d t^2 + V^{-1} d r^2 + r^2 d \Omega^{2}_{D-2},
\]

where \( V = 1 - \frac{r^2}{R^2} - \frac{2M}{r^{D-3}} \). The advantage of these coordinates is their obvious simplicity and time independence. The disadvantage is that the expansion of the universe is not manifest. As is clearly seen there are in general two kinds of horizons, one associated with the black hole and one cosmological. \( I^+ \), which is in the focus of our interest, is located outside the future cosmological horizon where \( r \) is time like and \( t \) space like. For \( r \gg R \) the metric becomes

\[
d s^2 = -\frac{R^2}{r^2} d r^2 + \frac{r^2}{R^2} \left( d t^2 + R^2 d \Omega^{2}_{D-2} \right).
\]
It is clear that $I^+$ is approached for large $r$ and we note that the boundary is mapped on to an Euclidean cylinder $R \times S^{d-2}$ where $t$ is the coordinate along the cylinder. This kind of coordinates were used in [15].

2.1.2 Planar coordinates

Another convenient coordinate system is obtained by introducing

$$
\begin{cases}
\rho = \rho(t, r) = re^{-\tau} \\
\tau = \tau(t, r) = t + \frac{R}{2} \ln \left(\frac{r^2}{R^2} - 1\right).
\end{cases}
$$

With no black hole this becomes an inflating cosmology according to

$$-d\tau^2 + e^{2\tau/R} \left(d\rho^2 + \rho^2 \Omega_{D-2}^2\right),$$

i.e. the cosmological form of the de Sitter space. These coordinates have the obvious advantage of making the expansion of the universe explicit with a Hubble constant given by $1/R$. Without a black hole the geodesics are simply given by constant comoving spacial coordinates. One should note, however, that this metric only covers half of the space time. (Another half, bounded by $I^-$, is covered by changing $e^{\frac{2\tau}{R}}$ to $e^{-\frac{2\tau}{R}}$.) $I^+$ is now approached for large $\tau$ and it is clear that it will be described by spherical coordinates on the plane. Applying the same coordinate change to the metric with a black hole leads to the slightly more complicated metric

$$ds^2 = -N^2 d\tau^2 + h(d\rho + N_S d\tau)^2 + r^2 \Omega_{D-2}^2
\tag{3}$$

where, with $V_0 = 1 - \frac{r^2}{R^2}$, we have

$$h = \frac{r^2}{V} \left(1 - \frac{r^2 V^2}{R^2 V_0^2}\right) \rho^{-2}, \quad N_S = \frac{1 - \frac{V^2}{V_0^2}}{1 - \frac{r^2 V^2}{R^2 V_0^2}} R \quad \text{and} \quad N_\tau = \frac{\sqrt{V}}{\sqrt{1 - \frac{r^2 V^2}{R^2 V_0^2}}}. $$

If we fix $r$ and make a translation in $t$ we see from (2) that this corresponds to a scaling of $\rho$ that leaves the metric invariant even in the presence of the black hole. The crucial property that makes this possible is that the metric can be written in the static, time independent form above. Note however, that the corresponding Killing vector is not globally time like.

2.2 The Brown-York tensor

We have now obtained coordinates which can be used to describe the holographic theory. We have also seen the relation between conformal transformations in the hologram and time translations in the de Sitter space. To proceed we must find out more about the generators of the transformations and their corresponding conserved
quantities. To this end, let us follow the recipe of [17][15] and calculate the Brown-York tensor, [18], of the boundary which is the generator of coordinate changes in the bulk. The subtracted Brown-York tensor is given by

\[
T_{ij} = -\frac{1}{8\pi} \left( K_{ij} - Kh_{ij} - \frac{d-2}{R} h_{ij} - \frac{R}{d-3} G_{ij} + \ldots \right),
\]

where the last term is lacking in \( d = 3 \), and the \( \ldots \) refers to terms needed when \( d > 5 \). The terms added to the \( K_{ij} - Kh_{ij} \) are counter terms in the boundary theory needed to render \( T_{ij} \) finite. We have that

\[
K_{ij} = -\frac{1}{2} \left( h_{il} g^{l\mu} \nabla_\mu u_j + h_{jl} g^{l\mu} \nabla_\mu u_i \right)
\]

where \( u^\mu \) is a forward pointing, time like, unit normal to surfaces of constant \( \tau \). \( g_{\mu\nu} \) is the metric of the full space time while \( h_{ij} \) is the piece induced on the boundary. The conserved quantity associated with the conformal symmetry is given by

\[
Q = \oint_\Sigma d^{d-2} \sqrt{h} n^i \xi^j T_{ij},
\]

where \( n^i \) is an outward pointing unit normal to surfaces of constant \( \rho \). \( \xi^j \) is proportional to \( n^i \) with the constant of proportionality given by the appropriate lapse function.

Let us now calculate the Brown-York tensor in the two coordinate systems that we have chosen.

### 2.2.1 Planar coordinates

The calculation in planar coordinates involves the non diagonal metric (3). A straightforward calculation gives, in particular,

\[
T_{\rho\rho} = -\frac{1}{8\pi} (d - 2) \frac{M}{\rho^{d-3}} \frac{1}{\rho^2}
\]

at large \( \tau \). The \( G_{ij} + \ldots \) counter terms are not necessary to render the expressions finite in this case and we expect the above result to be true for arbitrary dimension. To obtain the conserved charge we use \( \xi^j = \frac{\sqrt{h}}{\sqrt{\rho}} n^i \), where the coefficient is the proper lapse function to make sure that our conserved quantity is conjugate to translations in \( t \). After some calculations we find that

\[
Q = -\frac{d-2}{8\pi} \Omega_{d-2} M,
\]

where \( \Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d}{2}+1)} \) is the surface area of the unit \( d - 1 \) sphere.

But how do we make contact with the CFT? From (2) it follows that

\[
\rho \frac{d}{d\rho} = -R \frac{d}{dt},
\]
and we therefore conclude that the energy $E$ of the CFT, conjugate to $\rho$, is given by

$$E = -\frac{d-2}{8\pi} \Omega_{d-2} \frac{MR}{\rho},$$

in planar coordinates. The sign follows provided that the conformal generator acts in the direction of decreasing $\rho$, (i.e. increasing $t$). Let us now proceed to perform the corresponding calculation in cylindrical coordinates.

### 2.2.2 Cylindrical coordinates

In this case the metric is a lot simpler than in the planar case, and is just given by (1). The result of the calculation is of the form

$$E = E_{C,A} - \frac{d-2}{8\pi} \Omega_{d-2} M,$$

where $E_{C,A}$ corresponds to an anomalous Casimir contribution. It vanishes for even $d$, while we have

$$E_{C,A} = \frac{1}{8} \quad \text{for} \quad d = 3$$

$$E_{C,A} = \frac{3\pi R^2}{32} \quad \text{for} \quad d = 5.$$

The appropriate expressions for higher dimensions need the precise form of the higher order counter terms. We will get back to this a little bit later. Similar formulae can also be obtained in the case of AdS space. The only differences are that the $M$ dependent term comes with a positive sign in all dimensions, while the anomaly is negative in $d = 3$.\(^1\)

### 3 Interpreting the hologram

We have now completed the first steps towards constructing the black hole hologram. In order to investigate it further we need an additional tool in the form of the Cardy-Verlinde entropy formula, [16]. We will begin by recalling how the formula is applied to the case of AdS before we proceed with the generalization to de Sitter.

#### 3.1 Black holes in AdS

In [16] it was conjectured that a CFT at temperature $T$ on a cylinder $\mathbb{R} \times S^{d-2}$ (where $S^{d-2}$ has constant radius $R$) with a central charge $c$, has an energy given by

$$E = \frac{(d-2)c}{48\pi} \frac{V}{L^{d-1}} \left(1 + \frac{L^2}{R^2}\right) \equiv E_E + E_C. \quad (4)$$

\(1\)The signs are quite subtle in the case of de Sitter. Our signs agree with those of [15] but disagrees with those of [21]. For more comments on this, see the last footnote of the paper.
where $V = \Omega_{d-2} R^{d-2}$ is the volume of the $S^{d-2}$. The energy is the sum of an extensive contribution $E_E$ and an intensive Casimir contribution $E_C > 0$.\footnote{Our convention for $E_C$ differs by a factor of 2 from [16].} $L$ is a parameter related to the temperature given by

$$T = \frac{1}{4\pi L} \left( d - 1 + (d - 3) \frac{L^2}{R^2} \right),$$

and the entropy, finally, is given by

$$S = \frac{c}{12 L^{d-2}} = \frac{4\pi R}{d - 2} \sqrt{E_C E_E}.$$

The form was argued on general grounds, while the detailed coefficients were deduced from the AdS/CFT correspondence applied to black holes in AdS. We will verify this in detail below after generalizing the expressions to de Sitter.

But let us see investigate what is going on a little bit more carefully. Depending on the coordinates that we use, we will find different vacua. If we use planar coordinates on the boundary\footnote{This corresponds, basically, to Poincare coordinates in the bulk. For more details on the choice of vacua in the case of AdS$_3$ see [19].}, we find that the AdS space has vanishing energy, while a black hole in AdS is an excitation of positive mass. Schematically we can write

$$E_{\text{plan}} \propto M.$$

If we instead use cylindrical coordinates, we find another vacuum where the AdS space has a shifted energy due to the anomaly (if $d$ is odd), and we have that

$$E_{\text{cyl}} \propto M + E_{C,A},$$

where $E_{C,A}$ is the anomalous contribution to the Casimir energy. (Note that $E_{C,A}$ is negative for $d = 3$ and positive for $d = 5$.) The analysis of [16] was performed in cylindrical coordinates based on the work of [20]. There the expressions for e.g. the action in the presence of a black hole, had been found to be divergent and was rendered finite by subtracting the equally divergent contribution from empty AdS. The remaining piece is what is given by equation (4). Hence we conclude that

$$E_E + E_C \propto M,$$

and the entropy is given by

$$S = \frac{4\pi R}{d - 2} \sqrt{E_C (E_{\text{plan}} - E_C)}.$$

This is the generalization of the Cardy formula proposed in [16]. For $d = 3$ one may note that $E_C = -E_{C,A} > 0$ and $E_{\text{cyl}} = E_E$. 

\footnote{This convention for $E_C$ differs by a factor of 2 from [16].}
3.2 Black holes in de Sitter

We will now proceed to the case of de Sitter space to see whether similar formulae are applicable to the relevant conjectured Euclidean CFT’s. According to our previous calculations the de Sitter black hole is an excitation of negative mass. Schematically we have

$$E_{\text{plan}} \propto -M.$$  

Using cylindrical coordinates, the new de Sitter vacuum has a shifted energy due to the anomaly (if $d$ is odd) given by

$$E_{\text{cyl}} \propto E_{C,A} - M.$$  

As was explained above, the anomaly $E_{C,A}$ is positive and hence has the opposite sign to AdS in the case of $d = 3$, while the sign is the same in $d = 5$. If we now subtract the contribution from empty dS, just as in the AdS case, we find

$$E_E + E_C \propto -M.$$  

To be able to account for the entropy in de Sitter space we now conjecture that the Casimir contribution is given by

$$E_C = - \frac{(d - 2)c}{48\pi} \frac{V}{L^{d-3}R^2},$$

which is negative and therefore has the opposite sign compared to the CFT relevant for AdS. This is also what to be expected from the naive continuation $R^2 \rightarrow -R^2$. The entropy is again given by

$$S = \frac{4\pi R}{d-2} \sqrt{|E_C| (E_{\text{plan}} - E_C)},$$

while the temperature becomes

$$T = \frac{1}{4\pi L} \left( d - 1 - \frac{(d - 3)L^2}{R^2} \right),$$

where again $R^2$ has been replace by $-R^2$. This has important consequences that we will come back to shortly. For $d = 3$ one may note that $E_C = -E_{C,A} < 0$ and $E_{\text{cyl}} = E_E$.

Let us now verify that the above indeed reproduces the correct result for black holes. We will do the AdS and the de Sitter cases at the same time. The prescription tells us to impose

$$\frac{(d - 2)c}{48\pi} \frac{V}{L^{d-1}} \pm \frac{(d - 2)c}{48\pi} \frac{V}{R^2L^{d-3}} = \pm \frac{d - 2}{8\pi} \Omega_{d-2}M,$$

where the + sign refers to AdS and the - sign refers to de Sitter. If we now put $L = R^2/r_s$ and choose

$$c = 3R^{d-2},$$
the equation becomes
\[ 1 \pm \frac{r_S^2}{R^2} - \frac{2M}{r_S^{d-3}} = 0, \]  
(6)

which is precisely the equation for the position of a horizon. From the formula for
the entropy we find
\[ S = \frac{c}{12} V = \frac{1}{4} \Omega_{d-2} r_S^{d-2} = \frac{1}{4} A, \]
and hence the generalized Cardy-Verlinde formula can account for the black hole
entropy in AdS as well as de Sitter space. The case \( d = 3 \) gives, using \( E_{C,A} = \frac{c}{24R} \),
\[ S = 4\pi R \sqrt{E_{C,A} E_E} = 2\pi \sqrt{\frac{c}{6} \left( RE_{\text{plan}} + \frac{c}{24} \right)}, \]
where \( E_{\text{plan}} < 0 \). In the case of AdS one has \( E_{\text{plan}} > 0 \) and the shift is by \(-\frac{c}{24}\).4

There is a slight puzzle with the above construction. In the case of de Sitter space
there are two solutions of equation (6), since there are two horizons. One corresponds
to the cosmological de Sitter horizon, while the other corresponds to the horizon of
the black hole. It is easily checked, however, that it is only the cosmological horizon
that leads to a positive temperature. The conclusion, then, is that the CFT that we
are studying on \( \mathcal{I}^+ \) is accounting for the entropy of the cosmological horizon only.
This is consistent with the fact that the energy and the entropy of the system is
decreased when the mass of the black hole is increased.5 For more on the entropy
of black holes in de Sitter space see [22]. The total entropy of both horizons as well
as the entropy of the cosmological horizon have this property, but our conclusion is
that it is only the entropy of the cosmological horizon that is accounted for by the
Euclidean CFT. Empty de Sitter space is also covered by the analysis and entails the
presence of a gas at nonzero temperature that can carry the entropy.

3.2.1 The Nariai black hole

The largest possible black hole in de Sitter space, the Nariai black hole [23], cor-
responds to a situation where the two horizons coincide. It is easy to check that this
occurs when
\[ M = \frac{1}{d-1} \left( \frac{d-3}{d-1} \right)^{\frac{d-3}{2}} R^{d-3}, \]
which corresponds to zero temperature. As observed in [21] [15] the energy in cylin-
drical coordinates vanishes for the Nariai black hole in \( d = 5 \). Is this just a coincidence
or is it true for all odd dimensional de Sitter black holes? If it is a general feature, and

\[ \text{Note that we are considering left and right movers together with } c = c_L + c_R. \]
\[ \text{One may note that if the expression for the energy of the system (and therefore also the tem-
}
\text{perature) is considered to have the opposite sign, the reasoning is reversed. Positive temperature
\text{now suggests that it is the entropy of the black hole horizon that is accounted for by the theory.} \]

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if an argument can be given why this is so, it provides a way of using the dS/CFT correspondence to calculate the conformal anomaly in arbitrary dimensions. The result is

\[ E_{C,A} = \frac{(d - 2) \pi^{\frac{d+3}{2}} c}{24 \Gamma \left( \frac{d+1}{2} \right)} \left( \frac{d - 3}{d - 1} \right)^{\frac{d-3}{2}} \frac{1}{R}, \]

for the CFT dual to de Sitter, while the CFT dual to AdS has an anomaly given by

\[ E_{C,A} = (-1)^{\frac{d+1}{2}} \frac{(d - 2) \pi^{\frac{d+3}{2}} c}{24 \Gamma \left( \frac{d+1}{2} \right)} \left( \frac{d - 3}{d - 1} \right)^{\frac{d-3}{2}} \frac{1}{R}. \]

The formulae are certainly true for \( d = 3 \) and \( d = 5 \), but to verify it for higher dimensions one must construct the necessary counterterms for the Brown-York tensor. It would be interesting to see how, and whether, this work out following, e.g., the analysis of [24].

4 Conclusions

In this paper we have found that a modified Cardy-Verlinde entropy formula reproduces the entropy of the cosmological de Sitter horizon in the presence of a black hole. But there are several unclear points. Why does the CFT only provide the entropy for the cosmological horizon? Is it correct to discard the negative temperature solution? Furthermore, the de Sitter horizon is an observer dependent construction. How does the CFT take this into account? Our analysis gives a hint. It is clear that different observers should correspond to the choice of different points of origin when we go to cylindrical coordinates. But these different choices will also give rise to different physics in the hologram. In particular, if we choose an origin that is offset from the black hole, we should be able to see how the entropy evolves. The corresponding observer would see how the black hole eventually leaves the horizon, while the area of the horizon, as a consequence, will increase corresponding to an increase in entropy. In the CFT the state will no longer be scale invariant. At large \( \rho \) the black hole will be enclosed by the sphere of radius \( \rho \) and the entropy will be the one corresponding to the cosmological horizon of a de Sitter space with a black hole. At small \( \rho \), i.e., late times, the black hole will be outside of the sphere and the entropy should be correspondingly larger.

In [14] it was indicated how a RG flow with a changing central charge may take us from an inflationary era in the past with a small \( R \) to a de Sitter space in the future with a large \( R \). But even with a constant number of degrees of freedom it is interesting to study how non trivial time evolution is encoded in various scales. What is the role of the second law in the dS/CFT correspondence? How is the Hawking evaporation of the black hole taken into account? There are clearly many interesting and important questions that need to be investigated.
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References


