Minimal Higher-Dimensional Extensions of the Standard Model and Electroweak Observables

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ABSTRACT

We consider minimal 5-dimensional extensions of the Standard Model compactified on an $S^1/Z_2$ orbifold, in which the SU(2)$_L$ and U(1)$_Y$ gauge fields and Higgs bosons may or may not all propagate in the fifth dimension while the observable matter is always assumed to be confined to a 4-dimensional subspace. We pay particular attention to consistently quantize the higher-dimensional models in the generalized $R_\xi$ gauge and derive analytic expressions for the mass spectrum of the resulting Kaluza-Klein states and their couplings to matter. Based on recent data from electroweak precision tests, we improve previous limits obtained in the 5-dimensional Standard Model with a common compactification radius and extend our analysis to other possible 5-dimensional Standard-Model constructions. We find that the usually derived lower bound of $\sim 4$ TeV on an universal compactification scale may be considerably relaxed to $\sim 3$ TeV in a minimal scenario, in which the SU(2)$_L$ gauge boson is the only field that feels the presence of the fifth dimension.
1 Introduction

String theory provides the only known theoretical framework within which gravity can be quantized and so undeniably plays a central rôle in our endeavours of unifying all fundamental forces of nature. A consistent quantum-mechanical formulation of a string theory requires the existence of additional dimensions beyond the four ones we experience in our every-day life. These new dimensions, however, must be sufficiently compact so as to escape detection. In the original string-theoretic considerations [1], the inverse length $1/R$ of the extra compact dimensions and the string mass $M_s$ turned out to be closely tied to the 4-dimensional Planck mass $M_P = 1.9 \times 10^{16}$ TeV, with all involved mass scales being of the same order. More recent studies, however, have shown [2, 3, 4, 5, 6] that there could still be conceivable scenarios of stringy nature where $1/R$ and $M_s$ may be lowered independently of $M_P$ by several or many orders of magnitude. In particular, Ref. [5] considers the radical possibility that $M_s$ is of order TeV and represents the only fundamental scale in the universe at which unification of all forces of nature occurs. In this model, the compactification radius related to the higher-dimensional gravitational interactions lies in the sub-millimeter range, i.e. $1/R \lesssim 10^{-3}$ eV, so Cavendish-type experiments may potentially test the model by observing deviations from Newton’s law [5, 7] at such small distances. The model also offers a wealth of phenomenological implications for high-energy colliders [8].

The above low string-scale framework could be embedded within e.g. type I string theories [4], where the Standard Model (SM) may be described as an intersection of higher-dimensional $Dp$ branes [5, 6, 9]. As such intersections may naturally be higher dimensional, in addition to gravitons the SM gauge fields could also propagate independently within a higher-dimensional subspace, where the size of the new extra dimensions is of order TeV$^{-1}$ for phenomenological reasons [10, 11, 12, 13, 14, 15]. Since such low string-scale constructions may effectively result in different higher-dimensional extensions of the SM [9], the actual limits on the compactification radius are, to some extent, model dependent. Nevertheless, in the existing literature the derived phenomenological limits were obtained by assuming that the SM gauge fields propagate all freely in a common higher-dimensional space, in which the compactification radius is universal for all the extra dimensions.

In this paper we wish to lift the above restriction and extend the analysis to models which minimally depart from the assumption of a universal higher-dimensional scenario. Specifically, we will consider 5-dimensional extensions of the SM compactified on an $S^1/Z_2$ orbifold, where the SU(2)$_L$ and U(1)$_Y$ gauge bosons may not both live in the same higher-dimensional space, the so-called bulk. For example, one could imagine that the SU(2)$_L$
gauge field propagates in the bulk whilst the $U(1)_Y$ gauge boson is confined to our observable 4-dimensional subspace and vice versa. This observable 4-dimensional subspace is often termed 3-brane or simply brane and is localized at one of the two fixed points of the $S^1/Z_2$ orbifold, the boundary. In the aforementioned higher-dimensional scenarios, all SM fermions and the Higgs boson should necessarily be brane fields, such that an explicit breaking of the 4-dimensional gauge symmetry of the original (classical) Lagrangian is avoided.

Another issue of particular interest to us is related to our ability of consistently quantizing the higher-dimensional models under study in the so-called $R_\xi$ gauge. In particular, it can be shown that higher-dimensional gauge-fixing conditions can always be found that reduce to the usual $R_\xi$ gauge after the compact dimensions have been integrated out. Such a quantization procedure can be successfully applied to both Abelian and non-Abelian theories that include Higgs bosons living in the bulk and/or on the brane. The $R_\xi$ gauge has the attractive theoretical feature that the unphysical sector decouples from the theory in the limit of the gauge-fixing parameter $\xi \to \infty$, thereby allowing for explicit checks of the gauge independence of physical observables, such as S-matrix elements.

After compactification of the extra dimensions, we obtain an effective 4-dimensional theory which is usually described by infinite towers of massive Kaluza–Klein (KK) states. In the 5-dimensional extensions of the SM under consideration, such infinite towers generically consist of KK excitations of the $W$-boson, the $Z$-boson and the photon. Since the mass of the first excited KK state is typically set by the inverse of the compactification radius $R$, one expects that the KK effect on high-precision electroweak observables will become more significant for higher values of $R$. Thus, if all SM gauge bosons live in the bulk, compatibility of this model with the present electroweak data gives rise to a lower bound [12] of $\sim 4 \text{ TeV}$ on $1/R$ at the $2\sigma$ level.

On the other hand, the possibilities that the $SU(2)_L$ gauge boson is a brane field with the $U(1)_Y$ gauge boson living in the bulk and vice versa are phenomenologically even more challenging. In such cases, we find that the lower limit on the compactification scale $1/R$ can become significantly weaker, i.e. $1/R \gtrsim 3 \text{ TeV}$. This new result emerges partially from the fact that some of the most constraining high-precision electroweak observables are getting differently affected by the presence of the KK states within these mixed brane-bulk scenarios. For example, the muon lifetime does not directly receive contributions from KK excitations if the $W$ boson lives on the brane, but only indirectly when the analytic result is expressed in terms of the $Z$-boson mass in the context of our adopted renormalization scheme. Most interestingly, unlike in the frequently investigated model with all SM gauge
fields in the bulk, other competitive observables, such as $A_{\text{FB}}^b$ and $A_{\text{LR}}^b$ [16], do now possess additional distinct analytic dependences on the compactification scale $1/R$ within these novel brane-bulk models. As a consequence, the results of the performed global-fit analysis become substantially different for these scenaria.

The paper is organized as follows: in Section 2 we consider a 5-dimensional Abelian model compactified on an $S^1/Z_2$ orbifold, in which the gauge field propagates in the bulk. The model is quantized by prescribing the proper higher-dimensional gauge-fixing condition which leads to the usual class of $R_\xi$ gauges after the extra dimension has been integrated out. The same gauge-fixing procedure may successfully be implemented for Abelian models augmented by one Higgs boson which could either be a bulk or a brane field, or even for more general models with two Higgs bosons where the one Higgs boson can live on the brane and the other one in the bulk. In Section 2 we also present analytic expressions for the masses of the physical KK gauge bosons and for their mixings with the corresponding weak eigenstates. In Section 3 we extend our gauge-fixing procedure to a higher-dimensional non-Abelian theory and discuss the basic structure of the gauge sector after compactification. In Section 4 we study 5-dimensional extensions of the SM, in which the SU(2)$_L$ and U(1)$_Y$ gauge fields and Higgs bosons may or may not all feel simultaneously the presence of the compact dimension while the fermionic matter is always assumed to be confined on the brane. In fact, we distinguish three cases: (i) both SU(2)$_L$ and U(1)$_Y$ gauge bosons are bulk fields, (ii) only the U(1)$_Y$ gauge boson is a bulk field while the SU(2)$_L$ one is a brane field, and (iii) only the SU(2)$_L$ gauge boson resides in the bulk while the U(1)$_Y$ one is restricted to the brane. Technical details of our study have been relegated to the Appendices A and B. In Section 5 we perform a fully-fledged global-fit analysis to the aforementioned 5-dimensional extensions of the SM, based on recent data on high-precision electroweak observables. Section 6 summarizes our conclusions.

2 5-Dimensional Abelian Model

To describe as well as motivate our higher-dimensional gauge-fixing quantization procedure, it is very instructive to consider first a simple Abelian 5-dimensional model, such as 5-dimensional Quantum Electrodynamics (5D-QED) where the extra spatial dimension is compactified on an $S^1/Z_2$ orbifold. Then, we shall extend our quantization procedure to more general Abelian models with bulk and/or brane Higgs fields.
As a starting point, let us consider the 5D-QED Lagrangian given by

\[ \mathcal{L}(x, y) = -\frac{1}{4} F_{MN}(x, y) F^{MN}(x, y) + \mathcal{L}_{GF}(x, y) + \mathcal{L}_{FP}(x, y), \]  

(2.1)

where

\[ F_{MN}(x, y) = \partial_M A_N(x, y) - \partial_N A_M(x, y) \]  

(2.2)

denotes the 5-dimensional field strength tensor, and \( \mathcal{L}_{GF}(x, y) \) and \( \mathcal{L}_{FP}(x, y) \) are the gauge-fixing and the induced Fadeev–Popov ghost terms, respectively. In a 5-dimensional Abelian model, one may neglect the Fadeev–Popov ghost term \( \mathcal{L}_{FP} \) induced by \( \mathcal{L}_{GF} \), as the Abelian ghosts are non-interacting and hence they cannot occur in S-matrix elements. We shall return to this point in Section 3, when discussing quantization of higher-dimensional non-Abelian theories.

Throughout the paper, Lorentz indices in 5 dimensions are denoted with capital Roman letters, e.g. \( M, N = 0, 1, 2, 3, 5 \), while the respective indices pertaining to the ordinary 4 dimensions are symbolized by Greek letters, e.g. \( \mu, \nu = 0, 1, 2, 3 \). Furthermore, we use the abbreviations \( x = (x^0, \vec{x}) \) and \( y = x^5 \) to denote the coordinates of the usual 1+3-dimensional Minkowski space and the coordinate of the fifth compact dimension, respectively.

In a 5-dimensional theory, the gauge-boson field \( A_M \) transforms as a vector under the Lorentz group \( \text{SO}(1,4) \). In the absence of the gauge-fixing and ghost terms \( \mathcal{L}_{FP} \) and \( \mathcal{L}_{GF} \) in (2.1), the 5D-QED Lagrangian is invariant under a U(1) gauge transformation:

\[ A_M(x, y) \to A_M(x, y) + \partial_M \Theta(x, y). \]  

(2.3)

Being consistent with the above property of gauge symmetry, we can compactify the theory on an \( S^1/Z_2 \) orbifold, such that the following equalities are satisfied:

\[ A_M(x, y) = A_M(x, y + 2\pi R), \]
\[ A_\mu(x, y) = A_\mu(x, -y), \]
\[ A_5(x, y) = -A_5(x, -y), \]  

(2.4)
\[ \Theta(x, y) = \Theta(x, y + 2\pi R), \]
\[ \Theta(x, y) = \Theta(x, -y). \]

As we will see below, the fact that \( A_\mu(x, y) \) is taken to be even under \( Z_2 \) results in the embedding of conventional QED with a massless photon into our 5D-QED. Notice that all other constraints on the field \( A_5(x, y) \) and the gauge parameter \( \Theta(x, y) \) in (2.4) follow automatically if the theory is to remain gauge invariant after compactification.
Given the periodicity and reflection properties of $A_M$ and $\Theta$ under $y$ in (2.4), we can expand these quantities in a Fourier series as follows:

$$
A^\mu(x, y) = \frac{1}{\sqrt{2\pi R}} A^\mu_{(0)}(x) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} A^\mu_{(n)}(x) \cos\left(\frac{ny}{R}\right),
$$

$$
A^5(x, y) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} A^5_{(n)}(x) \sin\left(\frac{ny}{R}\right),
$$

$$
\Theta(x, y) = \frac{1}{\sqrt{2\pi R}} \Theta_{(0)}(x) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} \Theta_{(n)}(x) \cos\left(\frac{ny}{R}\right).
$$

The Fourier coefficients $A^\mu_{(n)}(x)$, also called KK modes, turn out to be the mass eigenstates in 5D-QED. However, this is not a generic feature of higher-dimensional models, namely the Fourier modes cannot always be identified with the KK mass eigenstates. Below we will encounter examples, in which the Fourier modes will mix to form the KK mass eigenstates.

From (2.3) and (2.5), one can now derive the corresponding gauge transformations for the KK modes [6]

$$
A_{(n)\mu}(x) \rightarrow A_{(n)\mu}(x) + \partial_\mu \Theta_{(n)}(x),
$$

$$
A_{(n)5}(x) \rightarrow A_{(n)5}(x) - \frac{n}{R} \Theta_{(n)}(x).
$$

Integrating out the $y$ dimension yields the effective 4-dimensional Lagrangian

$$
\mathcal{L}(x) = -\frac{1}{4} F_{(0)\mu\nu} F^{\mu\nu}_{(0)} + \sum_{n=1}^{\infty} \left[ -\frac{1}{4} F_{(n)\mu\nu} F^{\mu\nu}_{(n)} 
+ \frac{1}{2} \left( \frac{n}{R} A_{(n)\mu} + \partial_\mu A_{(n)5} \right) \left( \frac{n}{R} A^\mu_{(n)} + \partial^\mu A_{(n)5} \right) \right] + \mathcal{L}_{GF}(x),
$$

where $\mathcal{L}_{GF}(x) = \int_0^{2\pi R} dy \mathcal{L}_{GF}(x, y)$. Note that the invariance of $\mathcal{L}(x)$ under the transformations (2.6) becomes manifest in the absence of the gauge-fixing term $\mathcal{L}_{GF}(x, y)$.

In addition to the usual QED terms involving the massless field $A^\mu_{(0)}$, the other terms in the effective 4-dimensional Lagrangian (2.7) describe two infinite towers of massive vector excitations $A^\mu_{(n)}$ and (pseudo)-scalar modes $A^5_{(n)}$ that mix with each other, for $n \geq 1$. The scalar modes $A^5_{(n)}$ play the rôle of the would-be Goldstone modes in a non-linear realization of an Abelian Higgs model, in which the corresponding Higgs fields are taken to be infinitely massive.

As in usual Higgs models, one may seek for a higher-dimensional generalization of 't-Hooft’s gauge-fixing condition, for which the mixing terms bilinear in $A^\mu_{(n)}$ and $A^5_{(n)}$ are
eliminated from the effective 4-dimensional Lagrangian (2.7). For instance, the covariant
gauge-fixing term \[6\]
\[
L_{GF}(x, y) = -\frac{1}{2\xi} \left( \partial_M A^M \right)^2
\]  
(2.8)
does not lead to a complete cancellation of the bilinear operators \(A_{(n)}^\mu \partial_\mu A_{(n)}^5\) in (2.7), with
exception the Feynman gauge \(\xi = 1\). Taking, however, advantage of the fact that orbifold compactification generally breaks \(SO(1,4)\) invariance \([17]\), one can abandon the requirement of covariance of the gauge fixing condition with respect to the extra dimension. In this
case, we are free to choose the following non-covariant generalized \(R_\xi\) gauge:
\[
L_{GF}(x, y) = -\frac{1}{2\xi} (\partial^\mu A_\mu - \xi \partial_5 A_5)^2.
\]  
(2.9)
Nevertheless, the gauge-fixing term in (2.9) is still invariant under ordinary 4-dimensional
Lorentz transformations. Upon integration over the extra dimension, it is not difficult
to see that all mixing terms involving \(A_{(n)}^\mu \partial_\mu A_{(n)}^5\) in (2.7) drop out up to irrelevant total
derivatives. As a consequence, the propagators for the fields \(A_{(n)}^\mu\) and \(A_{(n)}^5\) take on their
usual forms that describe massive gauge fields and their respective would-be Goldstone
bosons of an ordinary 4-dimensional Abelian-Higgs model in the \(R_\xi\) gauge:
\[
\mu \quad \quad \quad \quad \nu = \frac{i}{k^2 - \left(\frac{\xi}{\kappa}\right)^2} \left[ -g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi(\kappa)^2} \right]
\]  
(2.10)
\[
\quad = \frac{i}{k^2 - \xi(\kappa)^2}
\]  
(2.11)
Therefore, we shall often refer to the \(A_{(n)}^5\) fields as Goldstone modes, even though these
KK modes do not directly result from a mechanism of spontaneous symmetry breaking in
the usual sense.

Having defined the appropriate \(R_\xi\) gauge through the gauge-fixing term in (2.9), we
can recover the usual unitary gauge in the limit \(\xi \rightarrow \infty\). This limit is also equivalent to the
gauge-fixing condition \(A_5(x, y) = 0\) or equivalently to \(A_{(n)}^5(x) = 0\), where all unphysical KK
scalar modes are absent from the theory \([19]\). Thus, for the case at hand, we have seen how
starting from a non-covariant higher-dimensional gauge-fixing condition, we can arrive at
the known covariant 4-dimensional \(R_\xi\) gauge after compactification. As we will see below,
the above quantization procedure can be extended to more elaborate higher-dimensional
models that may include brane and/or bulk Higgs fields.

*For a related suggestion made recently, see \([18]\).
2.1 Abelian Model with a Bulk Higgs Boson

Here, we shall discuss an extension of the Abelian model outlined above by adding a bulk Higgs scalar. The 5D Lagrangian of this theory reads

\[ \mathcal{L}(x, y) = -\frac{1}{4} F^{MN} F_{MN} + (D_M \Phi)^* (D_M \Phi) - V(\Phi) + \mathcal{L}_{GF}(x, y), \quad (2.12) \]

where \( D_M = \partial_M + ie_5 A_M \) is the covariant derivative, \( e_5 \) denotes the 5-dimensional gauge coupling, and \( \Phi(x, y) \) is the 5-dimensional complex scalar field

\[ \Phi(x, y) = \frac{1}{\sqrt{2}} \left( h(x, y) + i \chi(x, y) \right) \quad (2.13) \]

that transforms under a U(1) gauge transformation as

\[ \Phi(x, y) \rightarrow \exp \left(-ie_5 \Theta(x, y)\right) \Phi(x, y). \quad (2.14) \]

In (2.12), the 5-dimensional Higgs potential is given by

\[ V(\Phi) = \mu_5^2 |\Phi|^2 + \lambda_5 |\Phi|^4, \quad (2.15) \]

with \( \lambda_5 > 0 \).

After imposing the \( S^1/Z_2 \) compactification conditions \( \Phi(x, y) = \Phi(x, y + 2\pi R) \) and \( \Phi(x, y) = \Phi(x, -y) \) on \( \Phi(x, y) \), we can perform a Fourier decomposition of the scalar fields \( h(x, y) \) and \( \chi(x, y) \) in terms of cosines

\[ h(x, y) = \frac{1}{\sqrt{2\pi R}} h_0(x) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} h_n(x) \cos \left( \frac{ny}{R} \right), \]
\[ \chi(x, y) = \frac{1}{\sqrt{2\pi R}} \chi_0(x) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} \chi_n(x) \cos \left( \frac{ny}{R} \right). \quad (2.16) \]

As we will see below, our choice of an even \( Z_2 \) parity for the bulk Higgs scalar \( \Phi \) ensures that the lowest lying KK modes describe a conventional 4-dimensional Abelian Higgs model. Instead, if \( \Phi \) were odd under \( Z_2 \), this would not allow Yukawa interactions of the Higgs scalars with fermions localized on a brane \( y = 0 \) and the generation of fermion masses through the Higgs mechanism would be impossible in this case.

Let us now turn our attention to the effective Higgs sector of our Abelian model. The effective 4-dimensional Lagrangian associated with the Higgs fields may conveniently be given by

\[ \mathcal{L}_{\text{Higgs}}(x) = \frac{1}{2} \sum_{n=0}^{\infty} \left[ (\partial_\mu h_n(x)) (\partial^\mu h_n(x)) - \frac{n^2}{R^2} h_n^2 - \mu^2 h_n^2 + (h \leftrightarrow \chi) \right] + \ldots, \quad (2.17) \]
where $\mu^2 = \mu_5^2$ and the ellipses denote quartic interactions which involve the Higgs fields $h_{(n)}$ and $\chi_{(n)}$ and which all depend on $\lambda = \lambda_5/(2\pi R) > 0$. In (2.17), the mass terms proportional to $n^2/R^2$ arise from compactifying the $y$-dimension. As in the usual 4-dimensional case, for $\mu^2 < 0$, the zero KK Higgs mode $\Phi_{(0)} = (h_{(0)} + i\chi_{(0)})/\sqrt{2}$ acquires a non-vanishing vacuum expectation value (VEV)

$$\langle \Phi_{(0)} \rangle = \frac{1}{\sqrt{2}} \langle h_{(0)} \rangle = \frac{v}{\sqrt{2}},$$

with $v = \sqrt{|\mu|^2/\lambda}$, which breaks the U(1) symmetry. Moreover, it can be shown that as long as the phenomenologically relevant condition $v < 1/R$ is met, $h_{(0)}$ will be the only mode to receive a non-zero VEV.

After spontaneous symmetry breaking, the effective kinetic Lagrangian of the theory for the $n$-KK mode may be cast into the form:

$$L_{\text{kin}}^{(n)}(x) = -\frac{1}{4} F_{\mu
u}^{(n)} F^{(n)\mu\nu} + \frac{1}{2} \left( \frac{n^2}{R^2} + e^2 v^2 \right) A_{(n)\mu} A^{\mu}_{(n)} + \frac{1}{2} (\partial_\mu A_{(n)5}) (\partial^\mu A_{(n)5})$$

$$+ \frac{1}{2} (\partial_\mu \chi_{(n)})(\partial^\mu \chi_{(n)}) - \frac{1}{2} \left( \frac{n}{R} \chi_{(n)} - ev A_{(n)5} \right)^2$$

$$+ A_{(n)\mu} \partial_\mu \left( \frac{n}{R} A_{(n)5} + ev \chi_{(n)} \right) + \ldots$$

(2.19)

where $e = e_5/\sqrt{2\pi R}$ and the dots indicate the omission of bilinear terms involving $h_{(n)}$. From (2.19), it is evident that the mass spectrum of the zero KK modes is identical to that of a conventional Abelian Higgs model, i.e. $m_{A_{(0)}} = ev$ and $m_{h_{(0)}} = \sqrt{2\lambda v}$. This is so, because $A_{(0)5}$ is absent and we are left with the standard 4-dimensional terms only. To determine the complete mass spectrum for the higher KK modes, we first introduce the (pseudo)-scalar KK modes $G_{(n)}$ and $a_{(n)}$ through the orthogonal linear transformations:

$$G_{(n)} = \left( \frac{n^2}{R^2} + e^2 v^2 \right)^{-1/2} \left( \frac{n}{R} A_{(n)5} + ev \chi_{(n)} \right),$$

$$a_{(n)} = \left( \frac{n^2}{R^2} + e^2 v^2 \right)^{-1/2} \left( ev A_{(n)5} - \frac{n}{R} \chi_{(n)} \right).$$

(2.20)

Then, with the aid of (2.20), $L_{\text{kin}}^{(n)}$ in (2.19) can be rewritten in the more compact form

$$L_{\text{kin}}^{(n)}(x) = -\frac{1}{4} F_{\mu
u}^{(n)} F^{(n)\mu\nu} + \frac{1}{2} (m_{A_{(n)}} A_{(n)\mu} + \partial_\mu G_{(n)}) \left( m_{A_{(n)}} A^{\mu}_{(n)} + \partial^\mu G_{(n)} \right)$$

$$+ \frac{1}{2} (\partial_\mu a_{(n)})(\partial^\mu a_{(n)}) - \frac{1}{2} m_{a_{(n)}}^2 a_{(n)}^2 + \ldots,$$

(2.21)

with $m_{A_{(n)}}^2 = m_{a_{(n)}}^2 = (n^2/R^2) + e^2 v^2$. From this last expression for $L_{\text{kin}}^{(n)}$, we readily see that $G_{(n)}$ plays the role of a Goldstone mode in an Abelian Higgs model, while the
pseudoscalar field $a_{(n)}$ describes a physical KK excitation degenerate in mass with the KK gauge mode $A_{(n)\mu}$. In particular, since the zero KK modes of the fields are expected to be much lighter than their first KK excitations, i.e. $ev \ll 1/R$, the masses of all higher $n$-KK gauge and Higgs modes are approximately $m_{(n)} = n/R$ and the Goldstone modes $G_{(n)}$ may almost be identified with $A_{(n)5}$, i.e. $G_{(n)} \approx A_{(n)5}$ as in 5D-QED.

From the above discussion, it becomes now clear that the appropriate gauge-fixing Lagrangian in (2.12) for a 5-dimensional generalized $R_\xi$-gauge should be

$$L_{\text{GF}}(x, y) = -\frac{1}{2\xi} \left[ \partial_\mu A_\mu - \xi (\partial_5 A_5 + ev \chi) \right]^2. \quad (2.22)$$

Taking (2.22) into account, we arrive at the total effective kinetic Lagrangian

$$L_{\text{kin}}^{(n)}(x) = -\frac{1}{4} F_{\mu\nu} F_{(n)\mu\nu} + \frac{1}{2} m_{A_{(n)}\mu} A_{(n)\mu}^\mu - \frac{1}{2\xi} (\partial_\mu A_\mu_{(n)})^2 + \frac{1}{2} (\partial_\mu G_{(n)}) (\partial^\mu G_{(n)}) - \frac{\xi}{2} m_{A_{(n)}}^2 G_{(n)}^2 + \frac{1}{2} (\partial_\mu a_{(n)}) (\partial^\mu a_{(n)}) - \frac{1}{2} m_{a_{(n)}}^2 a_{(n)}^2 + \frac{1}{2} (\partial_\mu h_{(n)}) (\partial^\mu h_{(n)}) - \frac{1}{2} m_{h_{(n)}}^2 h_{(n)}^2. \quad (2.23)$$

In the above, $m_{h_{(n)}} = \sqrt{(n^2/R^2) + 2\lambda v^2}$ are the KK Higgs masses and $m_{A_{(n)}}$ and $m_{a_{(n)}}$ are the KK masses of $A_{(n)}$ and $a_{(n)}$ given after (2.21). Observe finally that the limit $\xi \to \infty$ in (2.23) consistently corresponds to the unitary gauge.

### 2.2 Abelian Model with a Brane Higgs Boson

A qualitatively different way of implementing the Higgs sector in a higher-dimensional Abelian model is to localize the Higgs field at the $y = 0$ boundary of the $S^1/Z_2$ orbifold. The 5-dimensional Lagrangian of this theory reads

$$\mathcal{L}(x, y) = -\frac{1}{4} F^{MN} F_{MN} + \delta(y) \left[ (D_\mu \Phi^*) (D^\mu \Phi) - V(\Phi) \right] + L_{\text{GF}}(x, y). \quad (2.24)$$

Here, the covariant derivative $D_\mu = \partial_\mu + ie_5 A_\mu(x, y)$ and the Higgs potential $V(\Phi) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$ have their familiar 4-dimensional forms, and the $\delta$-function $\delta(y)$ confines the Higgs sector on the brane $y = 0$. Under a gauge transformation, the brane Higgs field $\Phi(x)$ transforms as

$$\Phi(x) \rightarrow \exp \left( -ie_5 \Theta(x, 0) \right) \Phi(x). \quad (2.25)$$
Under (2.25) and the local transformation (2.3) of the gauge field $A_M(x, y)$, the theory exhibits U(1) invariance. Notice that the bulk scalar field $A_5(x, y)$ vanishes on the brane $y = 0$ as a result of its odd $Z_2$-parity.

After compactification and integration over the $y$-dimension, the effective Lagrangian of the model under discussion will be the sum of two terms: the effective Lagrangian (2.7) of 5D-QED and the square bracket $[\ldots]_{y=0}$ in (2.24). Obviously, $\Phi = 1/\sqrt{2}(h + i\chi)$ being a brane field does not possess KK excitations and, for $\mu^2 < 0$ (with $\lambda > 0$), acquires a VEV: $\langle\Phi\rangle = (h)/\sqrt{2} = v/\sqrt{2}$. After spontaneous symmetry breaking, masses are generated for all the KK gauge modes $A_\mu^{(n)}$. However, unlike in the Abelian model with a bulk Higgs boson discussed in Section 2.1, the corresponding gauge-boson mass matrix here is no longer diagonal and has the form

$$M_A^2 = \begin{pmatrix}
m^2 & \sqrt{2}m^2 & \sqrt{2}m^2 & \cdots \\
\sqrt{2}m^2 & 2m^2 + (1/R)^2 & 2m^2 & \cdots \\
\sqrt{2}m^2 & 2m^2 & 2m^2 + (2/R)^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (2.26)$$

where $m^2 = e^2v^2$ denotes the mass generated by the Higgs mechanism. The eigenvalues of $M_A^2$ follow from

$$\det(M_A^2 - \lambda I) = \prod_{n=1}^{\infty} \left( n^2/R^2 - \lambda \right) \left( m^2 - \lambda - 2\lambda m^2 \sum_{n=1}^{\infty} \frac{1}{(n/R)^2 - \lambda} \right) = 0. \quad (2.27)$$

Since $\lambda = (n/R)^2$ is not a solution as can be easily seen, the mass eigenvalues are given by the zeros of the second big bracket in (2.27). This is equivalent to solving the transcendental equation

$$\sqrt{\lambda} = m^{(n)} = \pi m^2 R \cot(\pi m^{(n)} R). \quad (2.28)$$

The respective KK mass eigenstates $\hat{A}_\mu^{(n)}$ are given by

$$\hat{A}_\mu^{(n)} = -\left(1 + \pi^2 m^2 R^2 + \frac{m^{(n)}_2}{m^2} \right)^{-1/2} \sum_{j=0}^{\infty} \frac{2m^{(n)}_2 m}{m^{(n)}_2 - (j/R)^2} \left( \frac{1}{\sqrt{2}} \right)^{\delta_{j,0}} A_\mu^{(j)}. \quad (2.29)$$

To find the appropriate form of the gauge-fixing term $\mathcal{L}_{GF}(x, y)$ in (2.24), we follow (2.22), but restrict the scalar field $\chi$ to the brane $y = 0$, viz.

$$\mathcal{L}_{GF}(x, y) = -\frac{1}{2\xi} \left[ \partial_\mu A_\mu^{(n)} - \xi \left( \partial_5 A_5 + e_5 v_5 \chi \delta(y) \right) \right]^2. \quad (2.30)$$
Then, the effective 4-dimensional gauge-fixing Lagrangian $\mathcal{L}_{\text{GF}}(x)$ is given by

$$
\mathcal{L}_{\text{GF}}(x) = -\frac{1}{2\xi} \left( \partial_\mu A_\mu^{(0)} \right)^2 - \frac{1}{2\xi} \sum_{n=1}^{\infty} \left( \partial_\mu A_\mu^{(n)} - \xi \frac{n}{R} A_{(n)5} \right)^2
+ e\nu \chi \left( \partial_\mu A_\mu^{(0)} \right) + \sqrt{2} e\nu \chi \sum_{n=1}^{\infty} \left( \partial_\mu A_\mu^{(n)} \right)
+ \xi \sqrt{2} e\nu \chi \sum_{n=1}^{\infty} \left( \frac{n}{R} A_{(n)5} - \frac{\xi}{2} e_5^2 \nu^2 \chi^2 \delta(n) \right).
$$

On the $S^1/Z_2$ orbifold, the $\delta$-function may be represented by

$$
\delta(y) = \frac{1}{2\pi R} + \sum_{n=1}^{\infty} \frac{1}{\pi R} \cos \left( \frac{ny}{R} \right),
$$

which implies

$$
\delta(0) = \frac{1}{2\pi R} + \sum_{n=1}^{\infty} \frac{1}{\pi R}.
$$

It is interesting to verify whether our 5-dimensional gauge-fixing term in (2.30) does consistently lead to the generalized $R_\xi$ gauge after integration over the extra dimension. In doing so, we apply the $R_\xi$-gauge-fixing prescription individually to each KK gauge mode in the effective Lagrangian, instead of using (2.30). It is then not difficult to obtain

$$
\mathcal{L}_{\text{GF}}^{(n)}(x) = -\frac{1}{2\xi} \left[ \partial_\mu A_\mu^{(n)} - \xi \left( \frac{n}{R} A_{(n)5} + \sqrt{2} \delta(n,0) e\nu \chi \right) \right]^2.
$$

This analytic result coincides with the one stated in (2.31), provided $e_5 = \sqrt{2\pi R} e$ and (2.33) are used. As is also expected from a generalized $R_\xi$ gauge, all mixing terms of the gauge modes $A_\mu^{(n)}$ with $A_{(n)5}$ and $\chi$ disappear up to total derivatives. Hence, the eigenvalues $m_{(n)}$ as derived from (2.28) do represent the physical masses.

The unphysical mass spectrum of the Goldstone modes may be determined by diagonalizing the following $\xi$-dependent mass matrix of the fields $\chi$ and $A_{(n)5}$:

$$
\mathcal{L}_{\text{mass}}^\xi(x) = -\frac{\xi}{2} \left( \chi, A_{(1)5}, A_{(2)5}, \ldots \right) M_\xi^2 \left( \begin{array}{c}
\chi \\
A_{(1)5} \\
A_{(2)5} \\
\vdots
\end{array} \right),
$$
with

\[
M_\xi^2 = \begin{pmatrix}
  e^2 v^2 (1 + \sum_{n=1}^\infty 2) & \sqrt{2} (1/R) e v & \sqrt{2} (2/R) e v & \cdots \\
  \sqrt{2} (1/R) e v & (1/R)^2 & 0 & \cdots \\
  \sqrt{2} (2/R) e v & 0 & (2/R)^2 & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]  

(2.36)

It can be shown that the characteristic polynomial of \(M_\xi^2\) is formally identical to the one of \(M_A^2\) given in (2.27):

\[
\det \left( M_\xi^2 - \lambda I \right) = \det \left( M_A^2 - \lambda I \right).
\]  

(2.37)

Consequently, the mass eigenvalues of \(M_\xi^2\) are given by \(m(n)\) in (2.28). Thus, as is expected from an \(R_\xi\) gauge, we find an one-to-one correspondence of each physical vector mode of mass \(m(n)\) to an unphysical Goldstone mode with gauge-dependent mass \(\sqrt{\xi} m(n)\).

Moreover, the Goldstone mass eigenstates are given by

\[
\hat{G}(n) = \left( 1 + \pi^2 m^2 R^2 + \frac{m^2(n)}{m^2} \right)^{-1/2} \left( \sqrt{2} \chi + \sum_{j=1}^\infty \frac{2 (j/R) m}{m^2(n) - (j/R)^2} A_{(j)5} \right).
\]  

(2.38)

In the unitary gauge \(\xi \to \infty\), the fields \(\hat{G}(n)\), or equivalently the fields \(A_{(n)5}\) and \(\chi\), are absent from the theory. Therefore, as opposed to the bulk-Higgs model of Section 2.1, the present brane-Higgs model does not predict other KK massive scalars apart from the physical Higgs boson \(h\).

### 2.3 Abelian 2-Higgs Model

It is now interesting to consider a model with two complex Higgs fields: one Higgs field \(\Phi_1(x, y)\) propagating in the bulk and the other field \(\Phi_2(x)\) localized on a brane at \(y = 0\). The 5-dimensional Lagrangian of this Abelian 2-Higgs model is given by

\[
\mathcal{L}(x, y) = -\frac{1}{4} F^{MN} F_{MN} + (D_M \Phi_1)^* (D^M \Phi_1) + \delta(y) (D_\mu \Phi_2)^* (D^\mu \Phi_2) - V(\Phi_1, \Phi_2) + \mathcal{L}_{GF}(x, y),
\]

(2.39)

where \(V\) is the most general Higgs potential allowed by gauge invariance

\[
V(\Phi_1, \Phi_2) = \mu_1^2 (\Phi_1^\dagger \Phi_1) + \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \delta(y) \left[ \frac{1}{2} \mu_2^2 (\Phi_2^\dagger \Phi_2) + m_{12}^2 (\Phi_1^\dagger \Phi_2) + \frac{1}{2} \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \frac{1}{2} \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \frac{1}{2} \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \lambda_6 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_7 (\Phi_2^\dagger \Phi_2)(\Phi_1^\dagger \Phi_2) + \text{h.c.} \right].
\]  

(2.40)
Note that all terms involving the brane field $\Phi_2$ are multiplied by a $\delta$-function. Here, we shall restrict ourselves to a CP-conserving Higgs sector, i.e. the parameters $m_{12}^2$, $\lambda_5$, $\lambda_6$ and $\lambda_7$ in (2.40) are real. Furthermore, we assume that both complex scalar fields acquire real VEV's. Thus, we may linearly expand $\Phi_1$ and $\Phi_2$ around their VEV's as follows:

$$\Phi_1(x, y) = \frac{1}{\sqrt{2}} \left( \frac{v_1}{\sqrt{2\pi R}} + h_1(x, y) + i\chi_1(x, y) \right), \quad (2.41)$$

$$\Phi_2(x) = \frac{1}{\sqrt{2}} \left( v_2 + h_2(x) + i\chi_2(x) \right). \quad (2.42)$$

Adopting the commonly used notation in 2-Higgs models, we define $v_1 = v \cos \beta$ and $v_2 = v \sin \beta$, i.e. $\tan \beta = v_2/v_1$.

In this 5-dimensional Abelian 2-Higgs model, the effective mass matrix $M_A^2$ of the Fourier modes $A_{(n)}^\mu$ is given by a sum of two matrices:

$$M_A^2 = M_{\text{brane}}^2 + M_{\text{bulk}}^2. \quad (2.43)$$

The first matrix $M_{\text{brane}}^2$, which includes the KK masses, may be obtained by (2.26) after replacing $m^2 = e^2v^2$ with $m^2 = e^2v^2 \sin^2 \beta$. The second matrix $M_{\text{bulk}}^2$ is proportional to unity, $M_{\text{bulk}}^2 = e^2v^2 \cos^2 \beta I$. Because of the particular structure of $M_A^2$ in this model, the mass eigenvalues of the KK gauge modes are given by

$$m_{A(n)}^2 = m_{(n)}^2 + \Delta m_{(n)}^2, \quad (2.44)$$

where $\Delta m_{(n)}^2 = e^2v^2 \cos^2 \beta$ and $m_{(n)}$ are the roots of the transcendental equation (2.28). The corresponding mass eigenstates $\hat{A}_{(n)}^\mu$ may in turn be determined by (2.29), after $m_{(n)}^2$ has been replaced with $m_{(n)}^2 - \Delta m_{(n)}^2$.

Following a similar $R_\xi$-gauge-fixing prescription as above, we may eliminate the mixing terms between $A_{(n)}^\mu$ and the fields $A_{(n)5}$, $\chi_{1(n)}$ and $\chi_2$ by choosing

$$\mathcal{L}_{\text{GF}}(x, y) = -\frac{1}{2\xi} \left[ \partial_\mu A^\mu - \xi \left( \partial_5 A_5 + e_5 \frac{v}{\sqrt{2\pi R}} \cos \beta \chi_1 + e_5 v \sin \beta \chi_2 \delta(y) \right) \right]^2. \quad (2.45)$$

In Appendix A, we show that the resulting Goldstone modes $\hat{G}_{(n)}$ in this model have masses $\sqrt{\xi m_{A(n)}}$. These Goldstone modes may be expressed in terms of the other pseudoscalar fields $A_{(n)5}$, $\chi_{1(n)}$ and $\chi_2$ as follows:

$$\hat{G}_{(n)} = E_{\chi_2}^{(n)} \chi_2 + \sum_{j=0}^{\infty} \left( E_{\chi_{1(j)}}^{(n)} \chi_{1(j)} + E_{A_{(j)5}}^{(n)} A_{(j)5} \right), \quad (2.46)$$
with

\[ E_{\chi_2}^{(n)} = \frac{1}{N}, \quad E_{A(j)}^{(n)} = -\frac{1}{N} \frac{\sqrt{2} e v \sin \beta(j/R)}{(j/R)^2 + e^2 v^2 \cos^2 \beta - m_{(n)}^2}, \]

\[ E_{\chi_1(j)}^{(n)} = -\frac{1}{N} \frac{\sqrt{2} e v \sin \beta(j/R)}{(j/R)^2 + e^2 v^2 \cos^2 \beta - m_{(n)}^2}, \]

and

\[ N^2 = \frac{1}{2} \frac{m_{(n)}^2}{m_{(n)}^2 - e^2 v^2 \cos^2 \beta} \left( 1 + \pi^2 e^2 v^2 \sin^2 \beta R^2 + \frac{m_{(n)}^2 - e^2 v^2 \cos^2 \beta}{e^2 v^2 \sin^2 \beta} \right). \]

The masses of the lowest-lying KK Higgs scalars strongly depend on the details of the Higgs potential, whereas the masses of the higher \( n \)-KK Higgs modes are approximately \( n/R \).

We conclude this section by remarking that even for the most general Abelian case, an appropriate higher-dimensional gauge-fixing condition analogous to (2.45) can always be found that leads after compactification to the usual \( R_\xi \) gauge as known from ordinary 4-dimensional theories. In the following, we shall see how the above gauge-fixing quantization procedure can be extended to non-Abelian models as well.

### 3 Higher-Dimensional Non-Abelian Theory

In this section, we shall consider a pure non-Abelian theory, such as 5-dimensional Quantum Chromodynamics (5D-QCD), without interactions to matter. The 5D-QCD Lagrangian takes on the simple general form

\[ \mathcal{L}(x, y) = -\frac{1}{4} F_{MN}^a F^{aMN} + \mathcal{L}_{GF} + \mathcal{L}_{FP}, \]  

where

\[ F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + g_5 f_{abc} A_M^b A_N^c \]

and \( f_{abc} \) are the structure constants of the gauge group \( SU(N) \), with \( N = 3 \) for 5D-QCD. In (3.1), the gauge-fixing term \( \mathcal{L}_{GF} \) and the induced Fadeev–Popov Lagrangian \( \mathcal{L}_{FP} \) will be determined later in this section.

As we did for the Abelian case, we compactify each of the \( N \) gauge fields \( A_M^a(x, y) \) separately on \( S^1/Z_2 \) through the constraints (2.4). Moreover, under a \( SU(N) \) gauge transformation, \( A_M^a(x, y) \) transforms as

\[ A_M^a(x, y) \rightarrow A_M^a(x, y) + \partial_M \Theta^a(x, y) - g_5 f_{abc} \Theta^b(x, y) A_M^c(x, y). \]
After a Fourier expansion of $A^a_{\mu}(x, y)$, $A^5_{\mu}(x, y)$ and $\Theta^a(x, y)$ according to (2.5), one finds that the local SU($N$) transformation (3.3) amounts to [6]

$$
A^a_{(0)\mu} \rightarrow A^a_{(0)\mu} + \partial_\mu \Theta^a_{(0)} - \frac{1}{2} \frac{g_5}{\sqrt{2\pi R}} f^{abc} \sum_{m=0}^{\infty} \Theta^b_{(m)} \left( (1 + \delta_{m,0}) \sqrt{2^{1-\delta_{m,0}}} A^c_{(m)\mu} \right),
$$

$$
A^a_{(n)\mu} \rightarrow A^a_{(n)\mu} + \partial_\mu \Theta^a_{(n)} - \frac{1}{2} \frac{g_5}{\sqrt{2\pi R}} f^{abc} \sum_{m=0}^{\infty} \Theta^b_{(m)} \left( (1 + \delta_{m,n}) \sqrt{2^{-\delta_{m,n}}} A^c_{(m-n)\mu} + A^c_{(m+n)\mu} \right),
$$

$$
A^a_{(n)5} \rightarrow A^a_{(n)5} - \frac{n}{R} \Theta^a_{(n)} - \frac{1}{2} \frac{g_5}{\sqrt{2\pi R}} f^{abc} \sum_{m=0}^{\infty} \Theta^b_{(m)} \left( \text{sgn}(n-m) A^c_{(m-n)5} + A^c_{(m+n)5} \right),
$$

(3.4)

where $n \geq 1$. As opposed to the Abelian case, the new feature here is that the KK modes can now mix with each other under a gauge transformation. As a result of this mixing, any attempt to truncate the theory at a given KK mode $n = n_{\text{truc}}$ will explicitly break gauge invariance.†

It is straightforward to generalize the gauge-fixing term of 5D-QED given in (2.9) to the 5D-QCD case. The gauge-fixing term in 5D-QCD is given by

$$
L_{\text{GF}}(x, y) = -\frac{1}{2\xi} (F^a(A^a))^2,
$$

(3.5)

with

$$
F^a(A^a) = \partial^\mu A^a_\mu - \xi \partial_5 A^a_5.
$$

(3.6)

In this generalized $R_\xi$ gauge, all mixing terms $A^a_{(n)\mu} \partial_\mu A^a_{(n)5}$ disappear, so the Fourier modes represent mass eigenstates. As in the Abelian case, the latter is spoiled by a Higgs mechanism involving brane interactions.

In non-Abelian theories, the $R_\xi$ gauge induces an interacting ghost sector, which is described by the Fadeev–Popov Lagrangian

$$
L_{\text{FP}}(x, y) = c^a \frac{\delta F^a(A^a)}{\delta \Theta^a} c^b - c^a \left[ \partial^\mu \left( \partial_\mu \delta^{ab} - g_5 f^{abc} A^c_\mu \right) - \xi \partial_5 \left( \partial_\mu \delta^{ab} - g_5 f^{abc} A^c_5 \right) \right] c^b.
$$

(3.7)

†To overcome this difficulty, recent papers [20, 21] suggested to match the truncated theory with a manifestly gauge-invariant non-Abelian chiral-type Lagrangian. Although the two theories agree well for $n \ll n_{\text{truc}}$, they have a significantly different mass spectrum close to the truncation energy scale, i.e. for KK modes $n \approx n_{\text{truc}}$. 

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**Figure 1:** Feynman rules for the triple gauge boson coupling. \( \delta_{k,l,m} \) and \( \tilde{\delta}_{l,k,m} \) are defined in (3.8).

**Figure 2:** Feynman rules for the quartic gauge boson coupling. \( \delta_{k,l,m,n} \) and \( \tilde{\delta}_{k,n,l,m} \) are defined in (3.9).
In the above, $c^a(x, y)$ denote the higher-dimensional ghost fields, which are even under $Z_2$: $c^a(x, y) = c^a(x, -y)$, i.e. they share the same transformation properties with the group parameters $\Theta^a(x, y)$.

In Figs. 1 and 2, we exhibit the Feynman rules for the self-interactions of the KK modes $A^a_{(n)\mu}$ and $A^a_{(n)5}$ in the effective 4-dimensional theory in the $R_\xi$-gauge (3.5). In the unitary gauge, i.e. $\xi \to \infty$, the 5D-QCD Feynman rules reduce to those presented in [22].

The factors $\delta_{k,l,m}$, $\tilde{\delta}_{k,l,m}$, $\delta_{k,l,m,n}$ and $\tilde{\delta}_{k,n,l,m}$ imply selection rules for the triple and quartic coupling of the KK modes $A^a_{(n)\mu}$ and $A^a_{(n)5}$, which are typical for the interactions between bulk fields. These factors are given by

$$\delta_{k,l,m} = \delta_{k+l+m,0} + \delta_{k+l-m,0} + \delta_{k-l+m,0} + \delta_{k-l-m,0},$$

$$\tilde{\delta}_{k,l,m} = -\delta_{k+l+m,0} + \delta_{k+l-m,0} - \delta_{k-l+m,0} + \delta_{k-l-m,0},$$

for the triple gauge boson coupling and

$$\delta_{k,l,m,n} = +\delta_{k+l+m+n,0} + \delta_{k+l+m-n,0} + \delta_{k+l-m+n,0} + \delta_{k+l-m-n,0} + \delta_{k-l+m+n,0} + \delta_{k-l+m-n,0} + \delta_{k-l-m+n,0} + \delta_{k-l-m-n,0},$$

$$\tilde{\delta}_{k,l,m,n} = -\delta_{k+l+m+n,0} + \delta_{k+l+m-n,0} + \delta_{k+l-m+n,0} - \delta_{k+l-m-n,0} - \delta_{k-l+m+n,0} + \delta_{k-l+m-n,0} + \delta_{k-l-m+n,0} - \delta_{k-l-m-n,0},$$

for the quartic gauge boson coupling.

4 5-Dimensional Extensions of the Standard Model

In this section we shall study minimal 5-dimensional extensions of the SM compactified on an $S^1/Z_2$ orbifold, in which the SU(2)$_L$ and U(1)$_Y$ gauge bosons as well as the Higgs doublets may not all propagate in the bulk. In all these higher-dimensional scenarios, we shall assume that the chiral fermions are localized on a brane at the $y = 0$ fixed point of the $S^1/Z_2$ orbifold.

4.1 SU(2)$_L$⊗U(1)$_Y$-Bulk Model

To start with, we shall first consider the most frequently investigated model, where all electroweak gauge fields propagate in the bulk and couple to both a brane and a bulk Higgs doublets. The Lagrangian of the gauge-Higgs sector of this higher-dimensional standard
model (HDSM) is given by
\[ \mathcal{L}(x, y) = -\frac{1}{4} B_{MN} B^{MN} - \frac{1}{4} F^a_{MN} F^{aMN} + (D_M \Phi_1)^\dagger (D^M \Phi_1) \]
\[ + \delta(y)(D_\mu \Phi_2)^\dagger (D^\mu \Phi_2) - V(\Phi_1, \Phi_2) + \mathcal{L}_{GF}(x, y) + \mathcal{L}_{FP}(x, y), \] (4.1)
where \( B_{MN} \) and \( F^a_{MN} \) are the field strength tensors of the U(1) and SU(2) gauge fields, respectively. As usual, we define the covariant derivative \( D_M \) as
\[ D_M = \partial_M - ig_5 A_M^a \tau^a - ig'_5 2 \delta(y) \sqrt{2} \pi R \Phi_0^\dagger \Phi_1 \]
\[ - ig_5 \left( \Phi_1^\dagger \tau^a \Phi_0 - \Phi_0^\dagger \tau^a \Phi_1 \right) \sin \beta \delta(y). \] (4.2)
The Higgs potential \( V(\Phi_1, \Phi_2) \) of this SU(2)_L ⊗ U(1)_Y-bulk model has the very same analytic form as in (2.40), where \( \Phi_1(x, y) \) is a bulk Higgs doublet and \( \Phi_2(x) \) a brane one. After spontaneous symmetry breaking, the Higgs doublets will linearly be expanded about their VEVs, i.e.
\[ \Phi_1(x, y) = \begin{pmatrix} \chi_1^+ \\ \frac{1}{\sqrt{2}} \left( \frac{v_1}{\sqrt{2} \pi R} + h_1 + i \chi_1 \right) \end{pmatrix}, \quad \Phi_2(x) = \begin{pmatrix} \chi_2^+ \\ \frac{1}{\sqrt{2}} (v_2 + h_2 + i \chi_2) \end{pmatrix}. \] (4.3)
Here, we shall not repeat the calculational steps for determining the particle mass spectrum of the SU(2)_L ⊗ U(1)_Y-bulk model, as they are analogous to those of the Abelian model discussed in Section 2.3 (see also Appendix B). In fact, the above analogy in the derivation of the particle mass spectrum becomes rather explicit if the bulk gauge fields are written in terms of their higher-dimensional mass eigenstates:
\[ W_M^\pm = \frac{1}{\sqrt{2}} \left( A_M^1 + i A_M^2 \right), \]
\[ Z_M = \frac{1}{\sqrt{g_5^2 + g'_5^2}} \left( g_5 A_M^3 - g'_5 B_M \right), \]
\[ A_M = \frac{1}{\sqrt{g_5^2 + g'_5^2}} \left( g'_5 A_M^3 + g_5 B_M \right). \] (4.4)
Proceeding as in the Abelian case, we may easily determine the appropriate \( R_\xi \) gauge-fixing functions for the SU(2)_L and U(1)_Y gauge bosons:
\[ F^a(A^a) = \partial_\mu A^{a\mu} - \xi_A \left[ \partial_5 A_5^a - i \frac{g_5}{\sqrt{2} \pi R} \left( \Phi_1^\dagger \tau^a \Phi_0 - \Phi_0^\dagger \tau^a \Phi_1 \right) \cos \beta \right. \]
\[ - \left. i g_5 \left( \Phi_1^\dagger \tau^a \Phi_0 - \Phi_0^\dagger \tau^a \Phi_1 \right) \sin \beta \delta(y) \right], \] (4.5)
\[ F(B) = \partial_\mu B^\mu - \xi_B \left[ \partial_5 B_5 - i \frac{g'_5}{2 \sqrt{2} \pi R} \left( \Phi_1^\dagger \Phi_0 - \Phi_0^\dagger \Phi_1 \right) \cos \beta \right. \]
\[ - \left. i \frac{g'_5}{2} \left( \Phi_1^\dagger \Phi_0 - \Phi_0^\dagger \Phi_1 \right) \sin \beta \delta(y) \right], \] (4.6)
with
\[ \Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}; \quad v = \sqrt{v_1^2 + v_2^2}. \quad (4.7) \]

To avoid gauge-dependent photon-Z-mixing terms at the tree level, we will assume in the following that it is always \( \xi_A = \xi_B = \xi \). Under this assumption, the gauge-fixing Lagrangian \( \mathcal{L}_{GF}(x, y) \) in (4.1) may be expressed in terms of the real gauge-fixing functions \( F^a(A^a) \) and \( F(B) \) as follows:
\[ \mathcal{L}_{GF}(x, y) = -\frac{1}{2\xi} (F^a(A^a))^2 - \frac{1}{2\xi} (F(B))^2. \quad (4.8) \]
Furthermore, the Fadeev–Popov term \( \mathcal{L}_{FP}(x, y) \) in (4.1) is induced by the variations of \( F^a(A^a) \) and \( F(B) \) with respect to SU(2)_L and U(1)_Y gauge transformations. More explicitly, \( \mathcal{L}_{FP}(x, y) \) may be computed in the standard way by
\[ \mathcal{L}_{FP}(x, y) = c^a \frac{\delta F^a(A^a)}{\delta \Theta^b} c^b + \bar{c} \frac{\delta F(B)}{\delta \Theta} c, \quad (4.9) \]
where \( c^a(x, y) \) and \( c(x, y) \) are the 5-dimensional ghost fields associated with the SU(2)_L and U(1)_Y gauge groups, respectively. As in the 5D-QCD, the ghost fields are even under \( Z_2 \).

In the above \( R_\xi \)-gauge-fixing prescription, the complete kinetic Lagrangian of the gauge sector written in terms of the fields defined in (4.4) becomes rather analogous to the corresponding one of the Abelian model investigated in Section 2.3. In Appendix B, we give the propagators of the KK gauge and Goldstone modes in the \( R_\xi \) gauge, together with the exact analytic results for the couplings of the gauge bosons to fermions to be discussed in Section 4.4.

### 4.2 SU(2)_L-Brane, U(1)_Y-Bulk Model

Let us now consider a new minimal 5-dimensional alternative to the SM, in which only the U(1)_Y gauge boson propagates in the bulk, while the SU(2)_L gauge field lives on the \( y = 0 \) boundary of the \( S^1/Z_2 \) orbifold. The Lagrangian of this SU(2)_L-brane, U(1)_Y-bulk model is
\[ \mathcal{L}(x, y) = -\frac{1}{4} B_{MN} B^{MN} - \delta(y) \left[ \frac{1}{4} F_{MN} F^{MN} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) \right] + \mathcal{L}_{GF}(x, y) + \mathcal{L}_{FP}(x, y). \quad (4.10) \]
Observe that only a brane Higgs doublet
\[ \Phi(x) = \begin{pmatrix} \chi^\dagger \\ \frac{1}{\sqrt{2}} (v + h + i\chi) \end{pmatrix} \quad (4.11) \]
can be added in this model. The reason is that a bulk Higgs doublet would destroy the
gauge invariance of the theory in the bulk if one coupled it to the covariant derivative
\( D_\mu = \partial_\mu - ig A_\mu^a(x) \tau^a - \frac{ig'}{2} B_\mu(x, 0) \) on the \( y = 0 \) brane. As a consequence, the Higgs
potential of this model has the known SM form: \( V(\Phi) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4. \)

In the SU(2)_L-brane, U(1)_Y-bulk model, only the \( B^\mu(x, y) \) boson has to be expanded
in Fourier modes. Although the W-boson sector is completely standard, the neutral gauge
sector gets complicated by the brane-bulk mixing of \( B_\mu(x, y) \) with \( A_\mu^3(x) \) through the VEV
of the brane Higgs field \( \Phi(x) \). To be more precise, we find the effective mass-matrix
Lagrangian of the neutral gauge sector

\[
\mathcal{L}^N_{\text{mass}}(x) = \frac{1}{2} \begin{pmatrix} A_\mu^3, B_\mu^{(0)}, B_\mu^{(1)}, \ldots \end{pmatrix}^T M_N^2 \begin{pmatrix} A_\mu^3 \\ B_\mu^{(0)} \\ B_\mu^{(1)} \\ \vdots \end{pmatrix}
\]

(4.12)

with

\[
M_N^2 = \begin{pmatrix}
\frac{m^2}{g^2} & \frac{m^2}{g} & \sqrt{2} \frac{m^2}{g} & \cdots \\
\frac{m^2}{g} & m^2 & \sqrt{2} m^2 & \cdots \\
\sqrt{2} m^2 & \sqrt{2} m^2 & 2 m^2 + (1/R)^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(4.13)

and \( g'_5 = g' \sqrt{2\pi R}, \ m^2 = g'^2 v^2 / 4. \) The mass matrix \( M_N^2 \) contains a zero eigenvalue which

\[
\hat{A}_\mu = s_w A_\mu^3 + c_w B_\mu^{(0)},
\]

(4.14)

where \( s_w = \sqrt{1 - c_w^2} = g' / \sqrt{g^2 + g'^2} \) is the sine of the weak mixing angle. The other
non-zero mass eigenvalues \( m_{Z(n)} \) of \( M_N^2 \) in (4.13) may be determined by the roots of the transcendental equation

\[
m_{Z(n)} = \pi m^2 R \cot \left( \frac{\pi m_{Z(n)} R}{2} \right) + \frac{g^2}{g'^2} \frac{m^2}{m_{Z(n)}}.
\]

(4.15)

The respective mass eigenstates are given by

\[
\hat{Z}_\mu^{(n)} = \frac{1}{N} \left[ \frac{m_{Z}}{m_{Z(n)}} c_w A_\mu^3 - \sum_{j=0}^{\infty} \frac{\sqrt{2} m_{Z(n)} m_{Z}}{m_{Z(n)}^2 - (j/R)^2} \left( \frac{1}{\sqrt{2}} \right)^{\delta_{j,0}} s_w B_\mu^{(j)} \right],
\]

(4.16)

where \( m_{Z} = \sqrt{g^2 + g'^2} v / 2, \)

\[
N^2 = \frac{1}{2} \left[ \frac{c_w^2}{s_w^2} \left( \frac{m_{Z}^2}{m_{Z(n)}^2} - 2 \right) + s_w^2 \pi^2 m_{Z}^2 R^2 + \frac{m_{Z(n)}^2}{m_{Z}^2 s_w^2} + 1 \right].
\]

(4.17)
Notice that the KK mass eigenmode \( \tilde{Z}_{(0)} \) has to be identified with the observable \( Z \) boson.

In analogy to the SM-bulk model, the appropriate \( R_\xi \) gauge-fixing functions for this brane-bulk model are written

\[
F^a(A^a) = \partial_\mu A^{a\mu} + \xi i g \left( \Phi_1^\dagger \tau^a \Phi_0 - \Phi_0^\dagger \tau^a \Phi_1 \right) \quad (4.18)
\]

\[
F(B) = \partial_\mu B^\mu - \xi \left[ \partial_5 B_5 - i \frac{g'_5}{2} \left( \Phi_2^\dagger \Phi_0 - \Phi_0^\dagger \Phi_2 \right) \delta(y) \right] , \quad (4.19)
\]

with

\[
\Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \quad (4.20)
\]

Nevertheless, because of the specific brane-bulk structure of the higher-dimensional model, the corresponding gauge-fixing Lagrangian has now the form

\[
\mathcal{L}_{GF}(x, y) = -\frac{1}{2\xi} (F^a(A^a))^2 \delta(y) - \frac{1}{2\xi} (F(B))^2 . \quad (4.21)
\]

Like the charged gauge sector, the charged scalar sector is completely standard in this model. The neutral scalar sector, however, has a structure very similar to the one of the Abelian model discussed in Section 2.2. Again, one can show the existence of an one-to-one correspondence between the KK gauge modes with mass \( \tilde{m}_{Z(n)} \) and their associate would-be Goldstone modes with mass \( \sqrt{\xi} \tilde{m}_{Z(n)} \). The latter KK modes are given by

\[
\tilde{G}^0_{(n)} = \frac{1}{N} \left( \chi + \frac{g'v}{\sqrt{2}} \sum_{j=1}^{\infty} \frac{j/R}{m^2_{Z(n)} - (j/R)^2} B_{(j)5} \right) , \quad (4.22)
\]

where the normalization factor \( N \) is defined in (4.17).

The Fadeev-Popov Lagrangian \( \mathcal{L}_{\text{FP}} \) can also be obtained in the standard fashion. Taking the brane-bulk structure of the model into account, we may determine \( \mathcal{L}_{\text{FP}} \) by

\[
\mathcal{L}_{\text{FP}}(x, y) = \bar{c}^a(x) \frac{\delta F^a(A^a(x))}{\delta \Theta^b(x)} c^b(x) \delta(y) + \bar{c}(x, y) \frac{\delta F(B(x, y))}{\delta \Theta(x, y)} c(x, y) , \quad (4.23)
\]

where the \((x, y)\)-dependence of the different quantities involved is explicitly indicated.

### 4.3 SU(2)\(_L\)-Bulk, U(1)\(_Y\)-Brane Model

Another minimal 5-dimensional extension of the SM, complementary to the one discussed in Section 4.2, emerges if the SU(2)\(_L\) gauge boson is the only field that feels the presence
of the fifth compact dimension. By analogy, the Lagrangian of this model reads

\[
\mathcal{L}(x, y) = -\frac{1}{4} F_{MN}^a F^{aMN} - \delta(y) \left[ \frac{1}{4} B_{MN} B^{MN} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) \right] \\
+ \mathcal{L}_{GF}(x, y) + \mathcal{L}_{FP}(x, y),
\]

with \( D_\mu = \partial_\mu - i g_5 A_\mu^a(x, 0) \tau^a - i g' 2 B_\mu(x) \). As in the model discussed in the previous section, there is only one Higgs field on the brane \( y = 0 \) and the Higgs potential is of the SM form. Because only the SU(2)_L gauge boson lives in the bulk, the charged gauge sector of this higher-dimensional standard model is equivalent to that of the SM-bulk model discussed in Section 4.1 in the limit \( \sin \beta \to 0 \), i.e. only the Higgs field restricted to the brane \( y = 0 \) acquires a non-vanishing VEV. Thus, the SU(2)_L-bulk, U(1)_Y-brane model predicts a KK tower of \( W \)-boson excitations, while the neutral gauge sector is quite analogous to the one discussed in the previous section. Specifically, the effective mass-matrix Lagrangian of the neutral gauge sector is given by

\[
\mathcal{L}_{\text{mass}}^N(x) = \frac{1}{2} \left( B^\mu, A_\mu^{3(0)}, A_\mu^{3(1)}, \ldots \right) M_N^2 \left( \begin{array}{c} B_\mu \\
A_\mu^{3(0)} \\
A_\mu^{3(1)} \\
\vdots \end{array} \right),
\]

with

\[
M_N^2 = \begin{pmatrix}
m^2 g'^2 / g^2 & m^2 g' & 2 m^2 & \cdots \\
m^2 g' & m^2 & \sqrt{2} m^2 & \cdots \\
\sqrt{2} m^2 g' / g & \sqrt{2} m^2 & (1/R)^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots \end{pmatrix},
\]

\( g_5 = g \sqrt{2\pi R} \) and \( m^2 = g^2 v^2 / 4 \). Again, we find that the zero KK mode given by the linear combination \( \hat{A}_\mu = s_w A_\mu^{3(0)} + c_w B_\mu \) represents a massless vector field, the photon. The higher KK modes are massive and their masses may be obtained by the solutions of the transcendental equation

\[
m_{Z(n)} = \pi m^2 R \cot \left( \pi m_{Z(n)} R \right) + \frac{g'^2}{g^2} \frac{m^2}{m_{Z(n)}}.
\]

The Z boson, denoted as \( Z_{(0)} \), and its heavier KK mass eigenmodes may be conveniently expressed in terms of the gauge eigenstates as

\[
\hat{Z}_\mu^{(n)} = \frac{1}{N} \left[ \sum_{j=0}^{\infty} \frac{\sqrt{2} m_{Z(n)} m_Z}{m^2 Z(n) - (j/R)^2} \left( \frac{1}{\sqrt{2}} \right)^{\delta_{j,0}} c_w A_\mu^{3(j)} - \frac{m_Z}{m_{Z(n)}} s_w B_\mu \right],
\]

23
where
\[ N^2 = \frac{1}{2} \left[ \frac{s_w^2}{c_w^2} \left( \frac{m_Z^2}{m_{Z_{(n)}}^2} - 2 \right) + c_w^2 \pi^2 m_Z^2 R^2 + \frac{m_{Z_{(n)}}^2}{m_Z^2 c_w^2} + 1 \right]. \] (4.29)

In close analogy to the previous section, the higher-dimensional gauge-fixing functions leading to the generalized \( R_\xi \)-gauge are given by
\[
F^a (A^\mu) = \partial_\mu A^{\mu a} - \xi \left[ \partial_5 B_5 - ig_5 \left( \Phi_1^\dagger \tau^a \Phi_0 - \Phi_0^\dagger \tau^a \Phi_1 \right) \delta(y) \right],
\] (4.30)
\[
F (B) = \partial_\mu B^\mu + \xi i \frac{g'}{2} \left( \Phi_2^\dagger \Phi_0 - \Phi_0^\dagger \Phi_2 \right),
\] (4.31)
giving rise to the gauge-fixing Lagrangian
\[
\mathcal{L}_{GF}(x, y) = -\frac{1}{2\xi} (F^a (A^\mu))^2 - \frac{1}{2\xi} (F (B))^2 \delta(y).
\] (4.32)

The charged scalar sector of this model is identical to that of the SM-bulk model of Section 4.1, without the presence of a Higgs field on the \( y = 0 \) boundary. On the other hand, the neutral scalar sector predicts a KK tower of would-be Goldstone modes associated with the longitudinal polarization degrees of the massive KK gauge modes \( Z_{(n)} \). The would-be KK Goldstone modes are determined by
\[
\hat{G}^0_{(n)} = \frac{1}{N} \left( \chi + \frac{g v}{\sqrt{2}} \sum_{j=1}^{\infty} \frac{j/R}{m_{Z_{(n)}}^2 - (j/R)^2} A^3_{j(5)} \right),
\] (4.33)
with \( N \) as defined in (4.29). The Fadeev-Popov Lagrangian can be calculated as in the model described earlier in Section 4.2 (cf. (4.23)), by considering the obvious modifications that take account of the complementary brane-bulk structure of the model.

### 4.4 Localization of Fermions on the Brane

In the minimal 5-dimensional extensions of the SM we have been studying, we have assumed that all the SM fermions are localized at the \( y = 0 \) fixed point of the \( S^1/\mathbb{Z}_2 \) orbifold. Therefore, upon integrating out the \( y \) dimension, both the effective kinetic terms of fermions and the effective Yukawa sector take on the usual 4-dimensional SM structure. Clearly, the SM fermions do not have KK modes. Under a gauge transformation, the left- and right-handed fermions transform according to
\[
\Psi_L(x) \rightarrow \exp \left( ig_5 \Theta^a(x, 0) \tau^a + ig'_5 Y^L \Theta(x, 0) \right) \Psi_L(x),
\]
\[
\Psi_R(x) \rightarrow \exp \left( ig'_5 Y^R \Theta(x, 0) \right) \Psi_R(x).
\] (4.34)
The corresponding covariant derivatives that couple the chiral fermions to the gauge fields are given by

\[ D^L_\mu = \partial_\mu - i g_5 A^a_\mu(x, y) \tau^a - i g'_5 Y^L B_\mu(x, y), \]
\[ D^R_\mu = \partial_\mu - i g'_5 Y^R B_\mu(x, y). \]  

(4.35)

It is obvious that the effective coupling of a fermion to a gauge boson restricted to the same brane \( y = 0 \) has its SM value. On the other hand, the effective interaction Lagrangian describing the coupling of a fermion to a gauge boson living in the bulk has the generic form

\[ L_{\text{int}}(x) = -\overline{\Psi} \gamma^\mu (g_V + g_A \gamma^5) \Psi \left( A_{(0)\mu} + \sqrt{2} \sum_{n=1}^{\infty} A_{(n)\mu} \right). \]  

(4.36)

Again, the coupling parameters \( g_V \) and \( g_A \) are set by the quantum numbers of the zero KK gauge mode and receive their SM values. Because the KK mass eigenmodes generally differ from the Fourier modes, their couplings to fermions \( \hat{g}_{V(n)} \) and \( \hat{g}_{A(n)} \) have to be calculated for each model individually by taking into account the appropriate weak-basis transformations. The precise values of \( \hat{g}_{V(n)} \) and \( \hat{g}_{A(n)} \) will be very important for our phenomenological discussion in the next section. The Feynman rules for the interactions of the KK gauge mass eigenmodes to fermions are exhibited in Appendix B.

Likewise, the Feynman rules for the interaction of the Goldstone modes to fermions can also be obtained from the SM Yukawa sector by relating the KK weak modes to the respective KK mass eigenmodes. It is worth remarking here that although the \( Z_2 \)-odd fifth component of a bulk gauge boson \( A_M, A_5 \), does not couple directly to the brane fermions, \( A_5 \) is involved in fermionic couplings due to its mixing with the CP-odd Higgs fields which are even under \( Z_2 \). In particular, one can show that the resulting Goldstone couplings to fermions have the proper analytic structure to assure gauge invariance in the computation of S-matrix elements.

5 Global-Fit Analysis

In this section, we shall evaluate the bounds on the compactification scale \( 1/R \) of minimal higher-dimensional extensions of the SM by analyzing a large number of high precision electroweak observables. To be specific, we proceed as follows. We relate the SM prediction \( O^{\text{SM}} \) for an electroweak observable to the prediction \( O^{\text{HDSM}} \) for the same observable
obtained in the higher-dimensional SM under investigation through

\[ \mathcal{O}^{\text{HDSM}} = \mathcal{O}^{\text{SM}} (1 + \Delta_{\mathcal{O}}^{\text{HDSM}}). \]  

(5.1)

Here, \( \Delta_{\mathcal{O}}^{\text{HDSM}} \) is the tree-level modification of a given observable \( \mathcal{O} \) from its SM value due to the presence of one extra dimension. To enable a direct comparison of our predictions with the electroweak precision data [16], we include SM radiative corrections to \( \mathcal{O}^{\text{SM}} \). However, we neglect SM- as well as KK- loop contributions to \( \Delta_{\mathcal{O}}^{\text{HDSM}} \) as higher order effects.

As input SM parameters for our theoretical predictions, we choose the most accurately measured ones, namely the Z-boson mass \( M_Z \), the electromagnetic fine structure constant \( \alpha \) and the Fermi constant \( G_F \). In all the 5-dimensional models under study, the tree-level Z-boson mass \( m_{Z(0)} = \hat{m}_Z \) generally deviates from its SM form \( m_Z = \sqrt{g^2 + g'^2} v / 2 \). Therefore, we parameterize this deviation as follows:

\[ \hat{m}_Z^2 = m_Z^2 (1 + \Delta_Z X), \]  

(5.2)

where

\[ X = \frac{\pi^2 m_Z^2}{3 M^2} \]  

(5.3)

(with \( M = 1/R \)) represents the typical scale quantifying the higher-dimensional effect and \( \Delta_Z \) is a model-dependent parameter of order unity. Since the massless photon retains its SM properties through the entire process of compactification, the electromagnetic fine structure constant is still given by its SM value

\[ \alpha = \frac{e^2}{4\pi}. \]  

(5.4)

Instead, the Fermi constant \( G_F \) as determined by the muon lifetime may receive direct as well as indirect contributions due to KK states. We may account for this modification of \( G_F \) by writing

\[ G_F = \frac{\pi \alpha}{\sqrt{2} s_w c_w \hat{m}_Z^2} (1 + \Delta_G X), \]  

(5.5)

where the order unity parameter \( \Delta_G \) strongly depends on the details of the 5-dimensional model under consideration.

In the computation of the electroweak precision observables, it will be necessary to express the weak mixing angle \( \theta_w \) in terms of the input parameters \( \alpha, \hat{m}_Z \) and \( G_F \), by
means of (5.5). In this respect, it is useful to define an effective weak mixing angle \( \hat{\theta}_w \) using the tree-level SM relation

\[
G_F = \frac{\pi \alpha}{\sqrt{2} s_w^2 \hat{c}_w^2 \hat{m}_Z^2}.
\]  

(5.6)

With the above definition for \( \hat{\theta}_w \) and (5.5), we may relate the squared sines of the two weak mixing angles by

\[
s_w^2 = s_w^2(1 + \Delta_\theta X).
\]  

(5.7)

Again, \( \Delta_\theta \) in (5.7) is a model-dependent parameter of order unity to be determined below.

### 5.1 SU(2)_L \otimes U(1)_Y-Bulk Model

Before we present predictions for the electroweak observables in the SU(2)_L \otimes U(1)_Y-bulk model, let us first give the KK modifications for some of the fundamental parameters of the theory. The KK modifications of the \( Z \)- and \( W \)-boson masses are found to be

\[
\Delta_Z = -s_\beta^4,
\]

\[
\Delta_W = -s_\beta^4 \hat{c}_w^2,
\]

(5.8)

where \( \Delta_W \) is defined in analogy to (5.2). In (5.8) and in the following, we will often use the following short-hand notations for trigonometric functions: \( s_x = \sin x \), \( c_x = \cos x \), \( s_{2x} = \sin 2x \), \( c_{2x} = \cos 2x \).

KK effects also cause tree-level shifts to the \( W \)- and \( Z \)-boson gauge couplings. The physical gauge-boson couplings are given by

\[
\hat{g}_W = \hat{g}_W(0) = g \left( 1 - \frac{\pi^2}{3} s_\beta^2 \hat{c}_w^2 \frac{\hat{m}_Z^2}{M^2} \right),
\]

\[
\hat{g}_Z = \hat{g}_Z(0) = g \left( 1 - \frac{\pi^2}{3} s_\beta^2 \frac{\hat{m}_Z^2}{M^2} \right).
\]  

(5.9)

These last two relations are approximate, i.e. they are obtained by expanding the exact analytic results for the masses and couplings, stated in Appendix B, to leading order in the parameter \( X \) defined in (5.3). Finally, the KK tree-level shift \( \Delta_G \) of the Fermi constant \( G_F \) is

\[
\Delta_G = \hat{c}_w^2 \left( 1 - 2s_\beta^2 - \frac{s_w^2}{\hat{c}_w^2} s_\beta^4 \right),
\]  

(5.10)
which implies

\[
\Delta \theta = -\frac{e^4}{c_{2w}} \left(1 - 2s^2 - \frac{s^2}{c^2 s_w} s^4 \right).
\]  (5.11)

Notice that \(\Delta \theta\) determining the difference between \(s_w\) and \(\hat{s}_w\) is a key parameter in the computation of many precision observables, as it additionally enters via the vector coupling of the \(Z\) boson.

In our calculations of the electroweak observables to leading order in \(X\), we consistently use \(\hat{m}_{Z(n)} \approx m_{W(n)} \approx n/R\) and \(\hat{g}_{Z(n)} \approx g_{W(n)} \approx \sqrt{2}g\) for \(n \geq 1\). Within this approximative framework, we compute the following high precision observables: the \(W\)-boson mass \(M_{W,\text{SM}}\), the invisible width \(\Gamma_{Z(\nu \nu)}\), the leptonic widths \(\Gamma_{Z(l^+l^-)}\), the \(Z\)-boson hadronic width \(\Gamma_{Z(\text{had})}\), the weak charge of cesium \(Q_w\) measuring atomic parity violation, various ratios \(R_l\) and \(R_q\) involving partial \(Z\)-boson widths and the \(Z\)-boson hadronic width, fermionic asymmetries \(A_f\) at the \(Z\) pole, and various fermionic forward-backward asymmetries \(A_{FB}^{(0,f)}\) at vanishing polarization. A complete list of the considered observables along with the SM predictions and their experimental values is given in Appendix C.

In Table 1, we present predictions for the parameter \(\Delta_{\text{HDSM}}^{\text{HDSM}}/X\) in the SM-bulk model, where \(\Delta_{\text{HDSM}}^{\text{HDSM}}\) and \(X\) are defined by (5.1) and (5.3), respectively. Moreover, the auxiliary parameters that occur in Table 1 are given by

\[
\Delta_V = \frac{4}{2T_{3f} - 4Q_f s_w^2} \Delta \theta,
\]

\[
\Delta_f = \frac{8 s_w^2 Q_f (2T_{3f} - 4Q_f s_w^2)}{(2T_{3f} - 4Q_f s_w^2)^2 + (2T_{3f})^2} \Delta \theta,
\]

\[
\Delta_h = \frac{8 s_w^2 \sum_{q=u,d,c,s,b} Q_q (2T_{3q} - 4Q_q s_w^2)}{\sum_{q=u,d,c,s,b} [(2T_{3q} - 4Q_q s_w^2)^2 + (2T_{3q})^2]} \Delta \theta,
\]

\[
Q_w^{\text{SM}} = (Z - N) - 4Z \hat{s}_w^2,
\]

where \(Q_f\) and \(T_{3f}\) are the electric charge and the third component of the weak isospin of a fermion \(f\), respectively, and \(N = 78\) is the number of neutrons and \(Z = 55\) the number of protons in the cesium nucleus. In (5.12), the parameters \(\Delta_V\), \(\Delta_f\) and \(\Delta_h\) are all proportional to \(\Delta \theta\), since they arise from substituting \(s_w^2\) by \(s_w^2\) into the different electroweak observables. In detail, \(\Delta_V\) parameterizes the KK shift in the vector coupling of the \(Z\) boson to fermions. \(\Delta_f\) results from an analogous KK shift in the sum of the squared vector and axial vector couplings for a given fermion \(f\). Similarly, \(\Delta_h\) gives the KK shift in the total hadronic width of the \(Z\) boson.
<table>
<thead>
<tr>
<th>Observable</th>
<th>$\Delta_O^{HDSM}/X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_W$</td>
<td>$\frac{1}{2}(s_\beta^4 \hat{s}_w^2 + \hat{s}<em>c^2 \Delta</em>\theta)$</td>
</tr>
<tr>
<td>$\Gamma_Z(\nu\nu)$</td>
<td>$\hat{s}<em>w^2 (s</em>\beta^2 - 1)^2 - 1$</td>
</tr>
<tr>
<td>$\Gamma_Z(l^+l^-)$</td>
<td>$\hat{s}<em>w^2 (s</em>\beta^2 - 1)^2 - 1 + \Delta_l$</td>
</tr>
<tr>
<td>$\Gamma_Z$(had)</td>
<td>$\hat{s}<em>w^2 (s</em>\beta^2 - 1)^2 - 1 + \Delta_h$</td>
</tr>
<tr>
<td>$Q_W(Cs)$</td>
<td>$\left[(1 - s_\beta^2)^2 + 4 Z (Q_{w\nu}^{SM})^{-1} \Delta_\theta \right] \hat{s}_w^2$</td>
</tr>
<tr>
<td>$R_l$</td>
<td>$- \Delta_l + \Delta_h$</td>
</tr>
<tr>
<td>$R_q$</td>
<td>$\Delta_q - \Delta_h$</td>
</tr>
<tr>
<td>$A_f$</td>
<td>$\Delta_V - \Delta_f$</td>
</tr>
<tr>
<td>$A_{FB}^{(0,f)}$</td>
<td>$\Delta_V - \Delta_f + f \leftrightarrow e$</td>
</tr>
</tbody>
</table>

**Table 1:** Predictions for $\Delta_O^{HDSM}/X$ in the SU(2)$_L \otimes$ U(1)$_Y$-bulk model. The auxiliary parameters $\Delta_V$, $\Delta_f$ and $\Delta_h$ are defined in (5.12).

Employing the results of $\Delta_O^{HDSM}$ in Table 1, we can compute the predictions for all the electroweak observables listed in Appendix C, by virtue of (5.1). We will confront these predictions with the respective experimental values, which are also listed in Appendix C. To do so, we perform a $\chi^2$ test to obtain bounds on the compactification scale $M = 1/R$ as a function of the bulk-brane angle sin $\beta$. Thus, in our global-fit analysis (ignoring correlation effects between the observables to first approximation), a compactification radius is considered to be compatible at the $n\sigma$ confidence level (CL), if $\chi^2(R) - \chi^2_{\text{min}} < n^2$, where

$$\chi^2(R) = \sum_i \frac{(O_i^{\text{exp}} - O_i^{HDSM})^2}{(\Delta O_i)^2}$$

and $\chi^2_{\text{min}}$ is the minimum of $\chi^2$ for a compactification radius in the physical region, i.e. for $R^2 > 0$. In (5.13), $i$ runs over all the observables listed in Table 6 in Appendix C. From this table, one easily sees that the total experimental and theoretical uncertainty $(\Delta O_i)^2$ of an observable $O_i$ is dominated by its experimental uncertainty.

Figure 3 shows lower bounds on the compactification scale $1/R$ coming from different
Figure 3: Lower bounds on the compactification scale $M = 1/R$ at the 3σ level from precision observables and a χ² analysis in the SM-bulk model.

types of observables as functions of $\sin^2 \beta$, where we take into account only one observable at a time. In addition, Fig. 3 displays the result obtained by a global χ² fit. For a model dominated by a brane Higgs field ($\sin \beta = 1$), the most stringent bound on $1/R$ is set by the forward-backward asymmetry involving $b$-quarks, while for a bulk-Higgs dominated model (with $\sin \beta = 0$) $1/R$ is most severely constrained by the hadronic $Z$-boson width. A global χ² analysis yields a lower bound on $1/R$ of about 4 TeV at the 3σ CL, for the two limiting cases for which only one Higgs field that either lives in the bulk or on the $y = 0$ brane has a non-vanishing VEV. The lower bound on $1/R$ may decrease to $\sim 3$ TeV, for a mixed brane-bulk Higgs scenario with $\sin^2 \beta \sim 0.5$. This is so, because all the observables but the various $Z$-boson widths do not lead to useful lower limits on $1/R$ in the region of $\sin^2 \beta \sim 0.5$.

5.2 SU(2)$_L$-Brane, U(1)$_Y$-Bulk Model

Next we shall investigate the model, in which only the U(1)$_Y$ gauge boson feels the presence of the extra dimension, whereas the SU(2)$_L$ gauge boson is confined on the $y = 0$ brane.
In this case, we have
\[ \Delta_Z = -s_w^2, \]
\[ \Delta_W = 0. \tag{5.14} \]
Obviously, the $W$-boson mass does not change by KK effects. However, the modification of the $Z$-boson coupling to fermions becomes more involved in this model. Specifically, KK effects induce non-factorizable shifts both in the vector and axial part of the $Z\bar{f}f$-coupling, when the result is expressed in terms of the $Z$-boson mass-eigenstate. To leading order in $X$, we can account for these new non-factorizable modifications by parameterizing the $Z\bar{f}f$-coupling in terms of an effective electric charge $\hat{Q}_f$ and an effective third component of the weak isospin $\hat{T}_3$:
\[ \hat{Q}_f = \hat{Q}_{f(0)} = Q_f \left( 1 - s_w^2 X \right), \]
\[ \hat{T}_{3f} = \hat{T}_{3f(0)} = T_{3f} \left( 1 - X \right), \tag{5.15} \]
with $Q_f = T_{3f} + Y_f$. The exact relations between $\hat{Q}_f$ and $Q_f$ and between $\hat{T}_{3f}$ and $T_{3f}$ are given in Appendix B.

Taking the above results into account, we find
\[ \Delta_G = -s_w^2 \]
and, thereby,
\[ \Delta_\theta = \frac{s_w^2 c_w^2}{c^2_w}. \tag{5.17} \]
The simplicity of the above results is a consequence of the fact that the charged gauge sector lives on the brane and hence is not affected by KK effects.

With the help of the new auxiliary parameters, we exhibit in Table 2 the tree-level KK shifts $\Delta_{O}^{HDSM}$ to the different electroweak observables. The parameters $\delta_V$ and $\delta_A$ give the KK modifications in the vector and axial-vector part of the $Z\bar{f}f$ coupling, except of the modifications which are purely due to the difference between $\theta_w$ and $\hat{\theta}_w$, i.e.
\[ \delta_V = \frac{-2T_{3f} s_w^2 + 4Q_f s_w^2}{2T_{3f} - 4Q_f s_w^2}, \]
\[ \delta_A = -s_w^2. \tag{5.18} \]
The parameter $\delta_f$ quantifies the KK shift in the sum of the squared vector and axial vector couplings of a given fermion $f$ to the $Z$ boson in this SU(2)$_L$-brane, U(1)$_Y$-bulk model.
The parameter $\delta_f$ is given by

$$
\delta_f = \frac{\left( -16T_{3f}^2 + 16T_{3f}Q_f \right) s_w^2 + \left( 16T_{3f}Q_f - 32Q_f^2 \right)s_w^4}{(2T_{3f} - 4Q_f s_w^2)^2 + (2T_{3f})^2}.
$$

(5.19)

Moreover, the parameters $\Delta_V$, $\Delta_f$ and $\Delta_h$ are defined in (5.12) with $\Delta_\theta$ given by (5.17).

Following the procedure outlined in the previous section, we can now evaluate the lower bounds on the compactification scale $M = 1/R$ in the SU(2)$_L$-brane, U(1)$_Y$-bulk model. In Table 3, we display the lower limits on $1/R$ for each observable separately, together with that found by a global analysis. The most restrictive bound is obtained by the $b$-quark forward-backward asymmetry, giving rise to a lower limit on $1/R$ of $\sim 4.4$ TeV at the $3\sigma$ CL. Finally, our global-fit analysis leads to the slightly less restrictive lower bound: $1/R \gtrsim 3.5$ TeV.

Table 2: Predictions for $\Delta_O^{\text{HDSM}}/X$ in the SU(2)$_L$-brane, U(1)$_Y$-bulk model. See text for the definition of the delta parameters.

<table>
<thead>
<tr>
<th>Observable</th>
<th>$\Delta_O^{\text{HDSM}}/X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_W$</td>
<td>$\frac{1}{2}(s_w^2c_w^2/c_{2w})$</td>
</tr>
<tr>
<td>$\Gamma_Z(\nu\nu)$</td>
<td>$s_w^2$</td>
</tr>
<tr>
<td>$\Gamma_Z(l^+l^-)$</td>
<td>$s_w^2 + \Delta_l + \delta_l$</td>
</tr>
<tr>
<td>$\Gamma_Z(\text{had})$</td>
<td>$s_w^2 + \Delta_h + \delta_h$</td>
</tr>
<tr>
<td>$Q_W(\text{Cs})$</td>
<td>$4Z \left( Q_{w}^{\text{SM}} \right)^{-1} s_w^2 \Delta_\theta$</td>
</tr>
<tr>
<td>$R_l$</td>
<td>$-\Delta_l + \Delta_h - \delta_l + \delta_h$</td>
</tr>
<tr>
<td>$R_q$</td>
<td>$\Delta_q - \Delta_h + \delta_q - \delta_h$</td>
</tr>
<tr>
<td>$A_f$</td>
<td>$\Delta_V - \Delta_f - \delta_f + \delta_V + \delta_A$</td>
</tr>
<tr>
<td>$A_{FB}^{(0,f)}$</td>
<td>$\Delta_V - \Delta_f - \delta_f + \delta_V + \delta_A + f \leftrightarrow e$</td>
</tr>
</tbody>
</table>
Let us finally consider the complementary scenario, in which only the SU(2)\(_L\) gauge boson propagates in the higher-dimensional space. In this case, the KK-mass shifts for the \(Z\) and \(W^\pm\) bosons are computed to be

\[ \Delta Z = \Delta W = -c_w^2. \quad (5.20) \]

By analogy, the KK effects on the \(Z\bar{f}f\)-coupling can also be taken into account by introducing an effective third component of the weak isospin:

\[ \hat{T}_{3f} = \hat{T}_{3f(0)} = T_{3f} \left( 1 - \hat{c}_w^2 X \right). \quad (5.21) \]

Unlike in the model discussed in the previous section, the electric-charge term in the \(Z\bar{f}f\)-coupling remains unaffected by KK effects, i.e. \(\hat{Q}_f = \hat{Q}_{f(0)} = Q_f\). Thus, from the muon decay, we calculate

\[ \Delta G = -\hat{c}_w^2, \quad (5.22) \]

---

**Table 3:** Lower bounds (in TeV) on the compactification scale \(1/R\) at the 3σ CL in models where either only the U(1)\(_Y\) or only the SU(2)\(_L\) gauge boson propagates in the higher-dimensional space.

<table>
<thead>
<tr>
<th>Observable</th>
<th>U(1)(_Y) in bulk</th>
<th>SU(2)(_L) in bulk</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_W)</td>
<td>1.2</td>
<td>1.2</td>
</tr>
<tr>
<td>(\Gamma_Z(\text{had}))</td>
<td>0.8</td>
<td>2.3</td>
</tr>
<tr>
<td>(Q_W(\text{Cs}))</td>
<td>0.4</td>
<td>0.9</td>
</tr>
<tr>
<td>(A_{FB}^{(0,b)})</td>
<td>4.4</td>
<td>2.4</td>
</tr>
<tr>
<td>(A_{\tau})</td>
<td>2.5</td>
<td>1.4</td>
</tr>
<tr>
<td>(R_{\tau})</td>
<td>1.0</td>
<td>0.5</td>
</tr>
<tr>
<td>global analysis</td>
<td>3.5</td>
<td>2.6</td>
</tr>
</tbody>
</table>
which leads to

$$\Delta \theta = \frac{c_w^4}{c_w^2}. \quad (5.23)$$

As in the previous section, we introduce the auxiliary parameters $\Delta V, \Delta f, \Delta h, \delta V, \delta_A$ and $\delta_f$, which enables us to cast the tree-level KK shifts $\Delta \Omega^{\text{HDSM}}$ to the electroweak observables in Table 4. The meaning of these auxiliary parameters are the same as in Sections 5.1 and 5.2. In particular, $\Delta V, \Delta f, \Delta h$ are given by (5.12), with $\Delta \theta$ in (5.23), while $\delta V, \delta_A$ and $\delta_f$ are respectively found to be

$$\delta_V = -\frac{2T_{3f}c_w^2}{2T_{3f} - 4Q_f s_w^2},$$

$$\delta_A = -\frac{c_w^2}{c_w^2}, \quad (5.24)$$

$$\delta_f = \frac{(-16T_{3f}^2 + 16T_{3f}Q_f s_w^2)c_w^2 + (16T_{3f}Q_f - 32Q_f^2)s_w^4}{(2T_{3f} - 4Q_f s_w^2)^2 + (2T_{3f})^2}. $$

Table 4: Predictions for $\Delta \Omega^{\text{HDSM}}/X$ in the SU(2)$_L$-bulk, U(1)$_Y$-brane model. See text for the definition of the auxiliary parameters.
Table 5: Lower bounds (in TeV) on the compactification scale $1/R$ at $2\sigma$, $3\sigma$ and $5\sigma$ CLs.

<table>
<thead>
<tr>
<th>model</th>
<th>$2\sigma$</th>
<th>$3\sigma$</th>
<th>$5\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U(1)$_Y$ bulk SU(2)$_L$ brane</td>
<td>4.3</td>
<td>3.5</td>
<td>2.7</td>
</tr>
<tr>
<td>U(1)$_Y$ brane SU(2)$_L$ bulk</td>
<td>3.0</td>
<td>2.6</td>
<td>2.1</td>
</tr>
<tr>
<td>U(1)$_Y$ bulk SU(2)$_L$ bulk (brane Higgs)</td>
<td>4.7</td>
<td>4.0</td>
<td>3.1</td>
</tr>
<tr>
<td>U(1)$_Y$ bulk SU(2)$_L$ bulk (bulk Higgs)</td>
<td>4.6</td>
<td>3.8</td>
<td>3.0</td>
</tr>
</tbody>
</table>

In Table 3, we also present the lower bounds on $1/R$ for the different type of observables. In the present model, the $b$-quark forward-backward asymmetry offers the most stringent lower bound on the compactification scale as well: $1/R \gtrsim 2.4$ TeV at the $3\sigma$ CL. Most interestingly, we observe that this lower bound on $1/R$ is much more relaxed than the one found in the previous models. The same observation applies to our global fit as well, i.e. a $\chi^2$ analysis constrains the compactification scale $M = 1/R$ to be higher than about 2.6 TeV at the $3\sigma$ CL.

In Table 5, we summarize the lower bounds on $1/R$ obtained by our global fits in the minimal higher-dimensional extensions of the SM under discussion. We find that the $\chi^2$ values increase rapidly as the compactification scale decreases, such that the lower bounds on $1/R$ at higher confidence levels are relatively stable. Thus, from Table 5, we see again that the lower bound on the compactification scale is the smallest in the SU(2)$_L$-bulk, U(1)$_Y$-brane model.
6 Conclusions

We have studied new possible 5-dimensional extensions of the SM compactified on an $S^1/Z_2$ orbifold, in which the SU(2)$_L$ and U(1)$_Y$ gauge fields and Higgs bosons may or may not all experience the presence of the fifth dimension. Moreover, the fermions in these models are considered to be confined to the one of the two boundaries of the $S^1/Z_2$ orbifold. We have paid special attention to consistently quantize the higher-dimensional models in the generalized $R_\xi$ gauges. Specifically, we have been able to identify the appropriate higher-dimensional gauge-fixing conditions which should be imposed on the theories so as to yield the known $R_\xi$ gauge after the fifth dimension has been integrated out. Based on the so-quantized effective Lagrangians, we have derived analytic expressions for the KK-mass spectrum of the gauge bosons and for their interactions to the fermionic matter.

The aforementioned analytic expressions have proven very essential to obtain accurate predictions for low-energy as well as high-energy electroweak observables measured at LEP and SLC. In particular, we have performed an extensive global-fit analysis of recent high-precision electroweak data to three different 5-dimensional extensions of the SM: (i) the SU(2)$_L \otimes$U(1)$_Y$-bulk model, where all SM gauge bosons are bulk fields; (ii) the SU(2)$_L$-brane, U(1)$_Y$-bulk model, where only the $W^\pm$ bosons are restricted to the brane, and (iii) the SU(2)$_L$-bulk, U(1)$_Y$-brane model, where only the U(1)$_Y$ gauge field is confined to the brane. After carrying out a $\chi^2$-test, we obtain different sensitivities to the compactification radius $R$ for the above three models. For the often-discussed first model, we find the 2$\sigma$ (3$\sigma$) lower bounds on $1/R$: $1/R \gtrsim 4.6$ (3.6) and 4.7 (4.0) TeV, for a Higgs boson living in the bulk and on the brane, respectively. For the second and third models, the corresponding 2$\sigma$ (3$\sigma$) lower limits are 4.3 (3.5) and 3.0 (2.6) TeV. Consequently, we observe that the bounds on $1/R$ may be reduced even up to 1 TeV, if the $W^\pm$ bosons are the only fields that propagate in the bulk.

The analysis presented here involves a number of assumptions which are inherent in any non-stringy field-theoretic treatment of higher-dimensional theories. Although the results obtained in the higher-dimensional models with one compact dimension are convergent at the tree level, they become divergent if more than one extra dimensions are considered. Also, the analytic results are ultra-violet (UV) divergent at the quantum level, since the higher-dimensional theories are not renormalizable. Within a string-theoretic framework, the above UV divergences are expected to be regularized by the string mass scale $M_s$. Therefore, from an effective field-theory point of view, the phenomenological predictions will depend to some extend on the UV cut-off procedure [23] related to the string
scale $M_s$. Nevertheless, assuming validity of perturbation theory, we expect that quantum corrections due to extra dimensions will not exceed the 10% level of the tree-level effects we have been studying here. Finally, we have ignored possible model-dependent winding-number contributions, which become relevant when the compactification scale $1/R$ and $M_s$ turn out to be of comparable size [24].

The lower limits on the compactification scale derived by the present global analysis indicate that resonant production of the first KK state may only be accessed at the LHC, at which heavy KK masses up to 6–7 TeV [9, 14] might be explored. In particular, if the $W^\pm$ bosons propagate in the bulk with a compactification radius $R \sim 3$ TeV$^{-1}$, one may still be able to probe resonant effects originating from the second KK state, and so differentiate the model from other 4-dimensional new-physics scenarios.

Acknowledgements

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Note Added

Shortly after completion of our paper, we became aware of [25] and [26]. The focus of these papers is the SM-bulk model, in which KK effects on high-energy scattering processes at LEP2 and other colliders were analyzed. In addition to being complementary by concentrating on high-precision electroweak observables, we have investigated new minimal higher-dimensional extensions of the SM, where the SU(2)$_L$ and U(1)$_Y$ gauge bosons may not both propagate in the higher-dimensional space. In particular, we find that the lower limits on $1/R$ may be substantially relaxed in these scenarios. Finally, we address the issue of a consistent quantization of the higher-dimensional field theory in the generalized $R_\xi$ gauge.
In this appendix, we wish to show that the KK Goldstone modes given in (2.46) have the properties of true Goldstone particles as these are known from spontaneous symmetry breaking models. The higher-dimensional gauge-fixing Lagrangian in (2.45) induces at each KK level \( n \) the gauge fixing terms

\[
\mathcal{L}_{\text{GF}}^{(n)} = -\frac{1}{2\xi} \left[ \partial_\mu A^{(n)}_\mu - \xi \left( \sqrt{\frac{n^2}{R^2} + \epsilon^2 v^2 \cos^2 \beta} \, G^{(n)} + \sqrt{2^{1-\delta_{n,0}} \epsilon v \sin \beta} \right) \right]^2, \tag{A.1}
\]

where the factor of \( \sqrt{2} \) stems from the \( \delta \)-function (cf. (2.32)). In the Abelian 2-Higgs model, the fields \( G^{(n)} \) are defined analogously with (2.20) as

\[
G^{(n)} = \left( \frac{n^2}{R^2} + \epsilon^2 v^2 \cos^2 \beta \right)^{-1/2} \left( \frac{n}{R} A^{(n)5} + \epsilon v \cos \beta \chi^{(1)} \right). \tag{A.2}
\]

Thus, the \( \xi \)-dependent mass terms of the scalar modes in the \( \chi_2 G^{(n)} \)-basis are given by

\[
\mathcal{L}^{\xi}_{\text{mass}}(x) = -\frac{\xi}{2} \left( \chi_2, G^{(0)}, G^{(1)}, \ldots \right) M^2_\xi \left( \begin{array}{c} \chi_2 \\ G^{(0)} \\ G^{(1)} \\ \vdots \end{array} \right), \tag{A.3}
\]

with

\[
M^2_\xi = \begin{pmatrix}
\epsilon^2 v^2 (1 + \sum_{n=1}^{\infty} \frac{2}{2}) \sin^2 \beta & \epsilon^2 v^2 \sin \beta \cos \beta & \sqrt{2} \epsilon v c_1 \sin \beta & \cdots \\
\epsilon^2 v^2 \sin \beta \cos \beta & \epsilon^2 v^2 \cos^2 \beta & 0 & \cdots \\
\sqrt{2} \epsilon v c_1 \sin \beta & 0 & (1/R)^2 + \epsilon^2 v^2 \cos^2 \beta & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \tag{A.4}
\]

and \( c_n = \sqrt{(n/R)^2 + (\epsilon v \cos \beta)^2} \). The infinite sum in the upper left entry of \( M^2_\xi \) is due to \( \delta(0) \) according to (2.33). We expect that only the Goldstone modes of the theory acquire gauge-dependent masses coming from the gauge-fixing terms. Computing the characteristic polynomial of \( M^2_\xi \), we find

\[
\det(M^2_\xi - \lambda I) = -\lambda \det(M^2_A - \lambda I), \tag{A.5}
\]

where \( M^2_A \) is the gauge-boson mass matrix given in (2.43). As a consequence, we may assign a Goldstone mass eigenstate \( \hat{G}^{(n)} \) with mass \( \sqrt{\xi \hat{m}^{(n)}} \) for each KK gauge eigenmode.
with mass $\hat{m}_{(n)}$. This constitutes a necessary condition in order to obtain gauge-invariant S-matrix elements within the $R_\xi$ class of gauges. From (A.5), we observe the existence of an additional degree of freedom which does not acquire a $\xi$-dependent mass with no correspondence to a KK gauge mode. This additional CP-odd scalar field will generally receive a gauge-independent mass that will entirely depend on the parameters of the Higgs potential. Additionally, it may mix with the other physical CP-odd states to form mass eigenstates (see discussion below).

On the other hand, in a consistent theory, the KK Goldstone modes should not acquire any gauge-independent mass term apart from their $\xi$-dependent mass mentioned above. In addition to the KK mass terms, the physical mass matrix of the KK scalar modes is determined by the Higgs kinetic terms in (2.39) and the Higgs potential (2.40). Since the CP-even Higgs modes do not mix with $A_{(n)5}$ in the CP-conserving case, the scalar mass matrix is block diagonal and we can concentrate on the CP-odd mass matrix $M_{\text{CP-odd}}^2$, as it appears in the original Lagrangian

$$\mathcal{L}_{\text{mass}}^{\text{CP-odd}}(x, y) = -\frac{1}{2} (\chi_1, \chi_2) M_{\text{CP-odd}}^2 \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

(A.6)

before integrating out the $y$ dimension. After a straightforward computation from (2.40), this CP-odd mass matrix may be cast into the form

$$M_{\text{CP-odd}}^2 = \delta(y) \begin{pmatrix} m_{\chi_{11}}^2 & m_{\chi_{12}}^2 \\ m_{\chi_{12}}^2 & m_{\chi_{22}}^2 \end{pmatrix},$$

(A.7)

where

$$m_{\chi_{11}}^2 = -\tan \beta m_{A_{12}}^2 + 2v^2 \sin^2 \beta \lambda_5 + \frac{1}{2} v^2 \sin \beta \cos \beta \lambda_6 + \frac{1}{2} v^2 \sin^2 \beta \tan \beta \lambda_7. \quad (A.8)$$

The other entries of the CP-odd mass matrix $M_{\text{CP-odd}}^2$ can be related to $m_{\chi_{11}}^2$ via

$$m_{\chi_{22}}^2 = m_{\chi_{11}}^2 / \tan^2 \beta \quad \text{and} \quad m_{\chi_{12}}^2 = m_{\chi_{21}}^2 = -m_{\chi_{11}}^2 / \tan \beta. \quad (A.9)$$

In deriving (A.7), (A.8) and (A.9), we have made use of the minimization conditions on the Higgs potential, i.e. $\langle \partial V / \partial \Phi_i \rangle = 0$, with $i = 1, 2$. In particular, the latter enabled us to cast the CP-odd mass matrix into the simple form of (A.7), where all entries are proportional to an overall $\delta$-function. Note that the absence of bulk mass terms originating from the Higgs potential is a characteristic of the CP-odd scalar sector of the model under consideration.

After integrating out the $y$ dimension in (A.7), we obtain the effective mass matrix for all the CP-odd KK modes $\chi_{1(n)}$, $\chi_2$ and $A_{(n)5}$. From this effective CP-odd mass matrix
\[ \gamma^{(n)} \text{ propagator:} \quad \mu \rightarrow \nu = \frac{i}{k^2 - m_{\gamma}^2} \left( -g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m_{\gamma}^2} \right) \]

\[ \hat{W}^{\pm}_{(n)} \text{-boson propagator:} \quad \mu \rightarrow \nu = \frac{i}{k^2 - m_{\hat{W}}^2} \left( -g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m_{\hat{W}}^2} \right) \]

\[ \hat{Z}_{(n)} \text{-boson propagator:} \quad \mu \rightarrow \nu = \frac{i}{k^2 - m_{\hat{Z}}^2} \left( -g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m_{\hat{Z}}^2} \right) \]

\[ A_{(n)5} \text{ propagator:} \quad \mu \rightarrow \nu = \frac{i}{k^2 - \xi m_{\gamma}^2} \]

\[ \hat{G}^0_{(n)} \text{ propagator:} \quad \mu \rightarrow \nu = \frac{i}{k^2 - \xi m_{\gamma}^2} \]

\[ \hat{G}^\pm_{(n)} \text{ propagator:} \quad \mu \rightarrow \nu = \frac{i}{k^2 - \xi m_{\hat{W}}^2} \]

**Figure 4:** KK gauge- and Goldstone-boson propagators in the 5-dimensional extensions of the SM in the generalized \( R_\xi \)-gauge.

including the KK mass terms, it is straightforward, although somehow tedious, to show that the would-be Goldstone modes (2.46) do not receive indeed any gauge-independent mass from the Higgs potential, whereas all physical CP-odd mass eigenstates should acquire high enough mass eigenvalues to avoid conflict with experimental data.

**B** **Masses, Couplings and Feynman rules**

Here, we shall present exact analytic results for the masses and the couplings of the KK gauge modes to fermions in the minimal 5-dimensional extensions of the SM discussed in Section 4.

To start with, we display in Fig. 4 the propagators for the KK gauge and Goldstone modes in the \( R_\xi \) gauge. In addition, the masses of the KK gauge bosons may be determined as follows:
\( \hat{m}_{\gamma(n)} = \frac{n}{R}, \)  
\( \sqrt{\hat{m}_{W(n)}^2 - m_W^2 \cos^2 \beta} = \pi m_W^2 \sin^2 \beta R \cot \left( \pi R \sqrt{\hat{m}_{W(n)}^2 - m_W^2 \cos^2 \beta} \right), \)  
\( \sqrt{\hat{m}_{Z(n)}^2 - m_Z^2 \cos^2 \beta} = \pi m_Z^2 \sin^2 \beta R \cot \left( \pi R \sqrt{\hat{m}_{Z(n)}^2 - m_Z^2 \cos^2 \beta} \right), \)

where \( n = 0, 1, 2, \ldots, m_W = g v / 2 \) and \( m_Z = \sqrt{g^2 + g'^2} v / 2. \)

**SU(2)_L-Brane, U(1)_Y-Bulk Model:**

\( \hat{m}_{\gamma(0)} = 0, \quad \hat{m}_{W(0)} = m_W = g v / 2, \)  
\( \hat{m}_{Z(n)} = \pi m_Z^2 \sin^2 \theta_W R \cot \left( \pi R \hat{m}_{Z(n)} \right) + \frac{m_W^2}{\hat{m}_{Z(n)}}. \)

Note that there are no KK excitations for the photon and \( W \) boson in this model.

**SU(2)_L-Bulk, U(1)_Y-Brane Model:**

\( \sqrt{\hat{m}_{W(n)}^2 - m_W^2 \cos^2 \beta} = \pi m_W^2 \sin^2 \beta R \cot \left( \pi R \sqrt{\hat{m}_{W(n)}^2 - m_W^2 \cos^2 \beta} \right), \)  
\( \hat{m}_{Z(n)} = \pi m_Z^2 \cos^2 \theta_W R \cot \left( \pi R \hat{m}_{Z(n)} \right) + \frac{m_Z^2}{\hat{m}_{Z(n)}}. \)

There are no KK excitations for the photon field in this model.

In the following, we will give the exact analytic expressions for the couplings of KK gauge bosons to fermions. To this end, we first define the following generic interaction Lagrangian:

\[ \mathcal{L}_{\text{int}} = \sum_n \hat{g}_W(n) \left( \hat{W}^+_{(n)\mu} J^+_{W \mu} + \hat{W}^-_{(n)\mu} J^-_{W \mu} \right) + \sum_n \hat{g}_Z(n) \hat{Z}_{(n)\mu} J^\mu_Z + \sum_n \hat{e}_{(n)} \hat{A}_{(n)\mu} J^\mu_{EM}, \]

with

\[ J^+_{W \mu} = \frac{1}{2\sqrt{2}} \left[ \bar{\nu}_i \gamma^\mu \left( 1 - \gamma^5 \right) e_i + \bar{u}_i \gamma^\mu \left( 1 - \gamma^5 \right) d_j V_{ij} \right], \]

\[ J^\mu_Z = \frac{1}{4 \cos \theta_W} \bar{f} \gamma^\mu \left[ \left( 2 \hat{T}_{3f(n)} - 4 \hat{Q}_f \sin^2 \theta_w \right) - 2 \hat{T}_{3f(n)} \gamma^5 \right] f, \]

\[ J^\mu_{EM} = \bar{f} Q_f \gamma^\mu f \]

and \( \nu_i = (\nu_e, \nu_\mu, \nu_\tau), e_i = (e, \mu, \tau), u_i = (u, c, t) \) and \( d_i = (d, s, b) \). In addition, \( f \) denotes all the 12 SM fermions. After a basis transformation from the weak to the mass eigenstates,
we obtain the following effective gauge and quantum couplings related to the three different higher-dimensional models \((n = 0, 1, 2, \ldots)\):

**SU(2)\(_L\)⊗U(1)\(_Y\)-Bulk Model:**

\[
\hat{e}(0) = e, \quad \hat{e}(n \geq 1) = \sqrt{2} e,
\]

\[
\hat{g}_Z(n) = \sqrt{2} g \left( 1 + \frac{m_Z^2 \sin^2 \beta}{m_Z^2 - m_Z^2 \cos^2 \beta} + \frac{\alpha^2 m_W^4 \sin^2 \beta}{M^2 (m_Z^2 - m_Z^2 \cos^2 \beta)} \right)^{-1/2},
\]

\[
\hat{g}_W(n) = \sqrt{2} g \left( 1 + \frac{m_W^2 \sin^2 \beta}{m_W^2 - m_W^2 \cos^2 \beta} + \frac{\alpha^2 m_W^4 \sin^2 \beta}{M^2 (m_W^2 - m_W^2 \cos^2 \beta)} \right)^{-1/2},
\]

\[
\hat{T}_{3f(n)} = T_{3f}, \quad \hat{Q}_{f(n)} = Q_f,
\]

with \(M = 1/R\).

**SU(2)\(_L\)-Brane, U(1)\(_Y\)-Bulk Model:**

\[
\hat{e}(0) = e, \quad \hat{g}_Z(n) = g, \quad \hat{g}_W(0) = g,
\]

\[
\hat{T}_{3f(n)} = \frac{T_{3f} m_Z^2}{c_w m_Z^2} \left[ \frac{1}{2} \left( 1 - \frac{m_Z^2}{m_Z^2} \right) + \frac{s_w^2}{2 c_w^2} \left( \frac{\alpha^2 m_Z^2}{M^2} + \frac{m_Z^2}{m_Z^2 + \frac{4 m_Z^2}{m_Z^2}} \right) \right]^{-1/2},
\]

\[
\hat{Q}_{f(n)} = \frac{Q_f m_Z^2}{c_w m_Z^2} \left[ \frac{1}{2} \left( 1 - \frac{m_Z^2}{m_Z^2} \right) + \frac{s_w^2}{2 c_w^2} \left( \frac{\alpha^2 m_Z^2}{M^2} + \frac{m_Z^2}{m_Z^2 + \frac{4 m_Z^2}{m_Z^2}} \right) \right]^{-1/2}.
\]

**SU(2)\(_L\)-Bulk, U(1)\(_Y\)-Brane Model:**

\[
\hat{e}(0) = e, \quad \hat{g}_Z(n) = g, \quad \hat{g}_W(n) = \sqrt{2} g \left( 1 + \frac{m_W^2}{m_W^2} + \frac{\alpha^2 m_W^4}{M^2 m_W^2} \right)^{-1/2},
\]

\[
\hat{T}_{3f(n)} = \frac{T_{3f} m_Z^2}{s_w m_Z^2} \left[ \frac{1}{2} \left( 1 - \frac{m_Z^2}{m_Z^2} \right) + \frac{c_w^2}{2 s_w^2} \left( \frac{\alpha^2 m_Z^2}{M^2} + \frac{m_Z^2}{m_Z^2 + \frac{4 m_Z^2}{m_Z^2}} \right) \right]^{-1/2},
\]

\[
\hat{Q}_{f(n)} = \frac{Q_f m_Z^2}{s_w m_Z^2} \left[ \frac{1}{2} \left( 1 - \frac{m_Z^2}{m_Z^2} \right) + \frac{c_w^2}{2 s_w^2} \left( \frac{\alpha^2 m_Z^2}{M^2} + \frac{m_Z^2}{m_Z^2 + \frac{4 m_Z^2}{m_Z^2}} \right) \right]^{-1/2}.
\]

In Fig. 5 we display the Feynman rules for the couplings of the KK gauge bosons to fermions that pertain to the above minimal 5-dimensional extensions of the SM.

### C  Input Parameters, Observables and SM Predictions

In this Appendix, we list the numerical values of the input parameters and electroweak observables, along with their SM predictions. These numerical values were used in Section 5.
\[ f \rightarrow \hat{A}_\mu^{(n)} = i \hat{e}_f \gamma_\mu \]

\[ f \rightarrow \hat{W}_\mu^{(n)} = \frac{i}{2 \sqrt{2}} \hat{g}_{W(n)} \gamma_\mu (1 - \gamma^5) \]

\[ \nu_i \rightarrow \hat{W}_\mu^{(n)} = \frac{i}{2 \sqrt{2}} \hat{g}_{W(n)} \gamma_\mu (1 - \gamma^5) V_{ij} \]

\[ e_i \rightarrow \hat{W}_\mu^{(n)} = \frac{i}{2 \sqrt{2}} \hat{g}_{W(n)} \gamma_\mu (1 - \gamma^5) \gamma \mu \]

\[ u_j \rightarrow \hat{Z}_\mu^{(n)} = \frac{i}{4 \cos \theta_w} \hat{g}_{Z(n)} \gamma_\mu \left[ \left( 2 \hat{T}_{3f(n)} - 4 \hat{Q}_f(n) \sin^2 \theta_w \right) - 2 \hat{T}_{3f(n)} \gamma^5 \right] \]

\[ d_i \rightarrow \hat{Z}_\mu^{(n)} = \frac{i}{4 \cos \theta_w} \hat{g}_{Z(n)} \gamma_\mu \left[ \left( 2 \hat{T}_{3f(n)} - 4 \hat{Q}_f(n) \sin^2 \theta_w \right) - 2 \hat{T}_{3f(n)} \gamma^5 \right] \]

\[ f \rightarrow \hat{Z}_\mu^{(n)} = \frac{i}{4 \cos \theta_w} \hat{g}_{Z(n)} \gamma_\mu \left[ \left( 2 \hat{T}_{3f(n)} - 4 \hat{Q}_f(n) \sin^2 \theta_w \right) - 2 \hat{T}_{3f(n)} \gamma^5 \right] \]

**Figure 5:** Feynman rules for couplings of the KK gauge bosons to fermions in the minimal 5-dimensional extensions of the SM.

to constrain the parameters of the 5-dimensional models.

As input parameters for our theoretical predictions, we use the most accurately determined ones, namely the Fermi constant \( G_F \) measured in muon decay, the fine structure constant \( \alpha \) determined by the quantum Hall effect and the \( Z \)-boson mass \( M_Z \) [16]:

\[ G_F = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}, \]
\[ \alpha = 1/137.0359895(61), \]
\[ M_Z = 91.1872(21) \text{ GeV}, \] (C.1)

where the numbers in parentheses indicate the 1\( \sigma \) uncertainties.

Given the above input parameters, predictions can be made for a number of high-precision observables within the SM framework. The results of these predictions may be found in [16], together with experimental values of the observables. For reader’s convenience, the actual values taken into account in our analysis are also listed in Table 6. The theoretical values in this table are obtained by assuming a light SM Higgs boson.
As was already discussed in Section 5, we introduce an effective weak mixing angle \( \hat{\theta}_w \) by enforcing the tree-level SM relation

\[
G_F = \frac{\pi \alpha}{\sqrt{2} \sin^2 \theta_w \cos^2 \theta_w M_Z^2} .
\]  

(C.2)

If renormalization-group running of the parameters is included, e.g. \( \alpha(M_Z) = 1/128.92(3) \), we find

\[
\sin^2 \hat{\theta}_w = 0.23105(8) ,
\]

(C.3)

which is the value used for the Z-pole observables in Section 5.
<table>
<thead>
<tr>
<th>Observable</th>
<th>Exp. Value ($O^{\text{EXP}}$)</th>
<th>SM Prediction ($O^{\text{SM}}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_W$</td>
<td>80.448(62) GeV</td>
<td>80.378(20) GeV</td>
</tr>
<tr>
<td>$\Gamma_Z(\text{had})$</td>
<td>1.7439(20) GeV</td>
<td>1.7422(15) GeV</td>
</tr>
<tr>
<td>$\Gamma_Z(l^+l^-)$</td>
<td>83.96(9) MeV</td>
<td>84.00(3) MeV</td>
</tr>
<tr>
<td>$\Gamma_Z(\nu\bar{\nu})$</td>
<td>498.8(15) MeV</td>
<td>501.65(15) MeV</td>
</tr>
<tr>
<td>$Q_W(\text{Cs})$</td>
<td>-72.06(44)</td>
<td>-73.09(03)</td>
</tr>
<tr>
<td>$R_e$</td>
<td>20.803(49)</td>
<td>20.740(18)</td>
</tr>
<tr>
<td>$R_\mu$</td>
<td>20.786(33)</td>
<td>20.741(18)</td>
</tr>
<tr>
<td>$R_\tau$</td>
<td>20.764(45)</td>
<td>20.786(18)</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.2164(73)</td>
<td>0.2158(2)</td>
</tr>
<tr>
<td>$R_c$</td>
<td>0.1674(38)</td>
<td>0.1723(1)</td>
</tr>
<tr>
<td>$A_e$</td>
<td>0.15108(218)</td>
<td>0.1475(13)</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>0.137(16)</td>
<td>0.1475(13)</td>
</tr>
<tr>
<td>$A_\tau$</td>
<td>0.1425(44)</td>
<td>0.1475(13)</td>
</tr>
<tr>
<td>$A_b$</td>
<td>0.911(25)</td>
<td>0.9348(1)</td>
</tr>
<tr>
<td>$A_c$</td>
<td>0.630(26)</td>
<td>0.6679(6)</td>
</tr>
<tr>
<td>$A_s$</td>
<td>0.85(9)</td>
<td>0.9357(1)</td>
</tr>
<tr>
<td>$A_{FB}^{(0,e)}$</td>
<td>0.0145(24)</td>
<td>0.0163(3)</td>
</tr>
<tr>
<td>$A_{FB}^{(0,\mu)}$</td>
<td>0.0167(13)</td>
<td>0.0163(3)</td>
</tr>
<tr>
<td>$A_{FB}^{(0,\tau)}$</td>
<td>0.0188(17)</td>
<td>0.0163(3)</td>
</tr>
<tr>
<td>$A_{FB}^{(0,b)}$</td>
<td>0.0988(20)</td>
<td>0.1034(9)</td>
</tr>
<tr>
<td>$A_{FB}^{(0,c)}$</td>
<td>0.0692(37)</td>
<td>0.0739(7)</td>
</tr>
<tr>
<td>$A_{FB}^{(0,s)}$</td>
<td>0.0976(114)</td>
<td>0.1035(9)</td>
</tr>
</tbody>
</table>

**Table 6:** Precision measurements and the corresponding SM predictions for all observables considered in our analysis [16] (notation as in [16]).
References


