O(a) improved QCD: The 3-loop beta-function, and the critical hopping parameter

A. Bodea, H. Panagopoulosb, Y. Proestosc

aCSIT, Tallahassee, USA
bDepartment of Physics, University of Cyprus
cPresent address: Department of Physics, Ohio State University, USA

We calculate the 3-loop bare $\beta$-function of QCD, formulated on the lattice with the clover fermionic action. The dependence of our result on the number of colors $N$, the number of fermionic flavors $N_f$, and the clover parameter $c_{SW}$, is shown explicitly. A direct outcome of our calculation is the two-loop relation between the bare coupling constant $g_0$ and the one renormalized in the $\overline{MS}$ scheme. Further, we can immediately derive the three-loop correction to the relation between the lattice $\Lambda$-parameter and $g_0$, which turns out to be very pronounced.

We also calculate the critical value of the hopping parameter, $\kappa_c$, in the clover action, to two loops in perturbation theory. This quantity is an additive renormalization; as such, it exhibits a linear divergence in the lattice spacing. We compare our results to non-perturbative evaluations of $\kappa_c$ coming from MC simulations.

1. INTRODUCTION

The clover action for lattice fermions was introduced a number of years ago [1], as a means of reducing finite lattice spacing effects. It is widely used nowadays in Monte Carlo simulations.

To monitor the onset of the continuum limit, tests of scaling must be performed on measured quantities. In particular, asymptotic scaling is governed by the bare $\beta$-function:

$$\beta_L(g_0) = -a \frac{d g_0}{d a} |_{g_0, \mu},$$

(1)

($a$ is the lattice spacing, $g$ ($g_0$) the renormalized (bare) coupling constant, $\mu$ the renormalization scale). For $g_0 \to 0$ one may write $\beta_L$ as:

$$\beta_L(g_0) = -b_0 g_0^3 - b_1 g_0^5 - b_2 g_0^7 + \ldots$$

(2)

The first two coefficients, $b_0$ and $b_1$, are universal and well known in $SU(N)$ gauge theory with $N_f$ fermion species; the 3-loop coefficient $b_2^L$, on the other hand, is regularization dependent. In the case at hand, $b_2^L$ is thus expected to depend not only on $N$ and $N_f$, but also on the parameter $c_{SW}$ of the clover action (see next Section).

We calculate $b_2^L$ for arbitrary $N$, $N_f$ and $c_{SW}$. The analogous calculation for pure gauge theory without fermions [2,3], as well as for Wilson fermions [4], was done a few years ago. We follow the general setup of those publications.

The $\beta$-function enters directly into the relation defining the parameter $\Lambda_L$:

$$a \Lambda_L = \exp \left( -\frac{1}{2 b_0 g_0^2} \right) \left( b_0 g_0^2 - b_1 / 2 b_0^2 \right),$$

$$\cdot \left[ 1 + q g_0^2 + \ldots \right], \quad q = (b_1^2 - b_0 b_2^L) / 2 b_0^3,$$

(3)

The “correction” factor $q$ turns out to be very pronounced for typical values of $c_{SW}$ and $g_0$.

A direct outcome of our calculation is the two-loop relation between the $\overline{MS}$ coupling $\alpha \equiv g^2 / (4 \pi)$ and $\alpha_0 \equiv g_0^2 / (4 \pi)$:

$$\alpha = \alpha_0 + d_1(a \mu) \alpha_0^2 + d_2(a \mu) \alpha_0^3 + O \left( \alpha_0^4 \right),$$

(4)

This relation is useful in studies involving running couplings or renormalized quark masses.

In Sec. 2 we present our results for $b_2^L$, $d_1(a \mu)$ and $d_2(a \mu)$, as functions of $N$, $N_f$ and $c_{SW}$. Further technical details and checks of our calculations are relegated to a longer write-up [5].

We have also calculated the critical value of the hopping parameter $\kappa_c$, to two loops, using the
clover action. Since Wilson fermions break chiral invariance explicitly, merely setting their bare mass to zero does not ensure chiral symmetry in the continuum limit; quantum corrections introduce an additive renormalization to the fermionic mass, which must then be fine tuned to a vanishing renormalized value. Consequently, the hopping parameter $\kappa$ is shifted from its naive value.

The additive mass renormalization is linearly divergent with the lattice spacing. This adverse feature of Wilson fermions poses an additional problem to a perturbative treatment. Indeed, our calculation serves as a check on the limits of applicability of perturbation theory, by comparison with non perturbative Monte Carlo results.

In the present work we follow the procedure of Ref. [6], in which $\kappa_c$ was computed using Wilson fermions without $\mathcal{O}(a)$ improvement. In Sec. 3 we present our results on $\kappa_c$, showing explicitly the dependence on $N, N_f$ and $c_{SW}$. Details on our calculation can be found in our publication [7].

2. THE $\beta$ FUNCTION

Our starting point is the Wilson formulation of the QCD action on the lattice, with the addition of the clover [1] fermion term, $S_{SW}$, which reads in standard notation:

$$S_{SW} = \frac{i a^5}{4} c_{SW} \sum_{x, \mu, \nu, f} \tilde{\psi}_f(x) \sigma_{\mu \nu} \tilde{F}_{\mu \nu}(x) \psi_f(x),$$

$$[\tilde{F}_{\mu \nu} = \frac{1}{8} (Q_{\mu \nu} - Q_{\nu \mu}),$$

$$Q_{\mu \nu} = U_{\mu, \nu} + U_{\nu, -\mu} + U_{-\mu, -\nu} + U_{-\nu, \mu}$$

Here $U_{\mu, \nu}(x)$ is the usual product of link variables $U_{\mu}(x)$ along the perimeter of a plaquette in the $\mu-\nu$ directions, originating at $x$; $f$ is a flavor index; $\sigma_{\mu \nu} = (i/2)[\gamma_\mu, \gamma_\nu]$. The value of the parameter $c_{SW}$ can be chosen arbitrarily; it is normally tuned in a way as to minimize $\mathcal{O}(a)$ effects. The lattice $\beta$-function is independent of the renormalized fermionic masses, which may be set to zero.

We compute the relation between $g_0$ and $g$, defined in the $\overline{MS}$ renormalization scheme:

$$g_0 = Z_g(g_0, a\mu) g.$$  \hspace{1cm} (6)

The one- and two-loop terms of $Z_g^2$ have the form:

$$Z_g(g_0, a\mu)^2 = 1 + L_0(a\mu) g_0^2 + L_1(a\mu) g_0^4 + \ldots$$

$$L_0(x) = 2b_0 \ln x + l_0, \quad L_1(x) = 2b_1 \ln x + l_1.$$  \hspace{1cm} (7)

The constant $l_0$ is related to the ratio of the associated $\Lambda$ parameters:

$$l_0 = 2b_0 \ln (\Lambda_\Lambda/\Lambda_{SW}).$$  \hspace{1cm} (8)

Its value is known (see e.g. Ref. [8] and references therein) and is presented here with increased accuracy for the $c_{SW}$-dependent coefficients:

$$l_0 = \frac{1}{8N} - 0.169955991998031(2) N + N_f l_{01}$$

$$l_{01} = 0.006696001(5) - c_{SW} 0.00504671402(1) + c_{SW}^2 0.02984346720(1)$$

The dependence on $c_{SW}$ is quite pronounced, leading to changes in $\Lambda_\Lambda$ of up to a factor of 2.

The quantity $b_2^L$ can be obtained from $l_0, l_1$:

$$b_2^L = b_2 - b_1 l_0 + b_0 l_1.$$  \hspace{1cm} (10)

where $b_2$ is known from the continuum. Computing $l_1$ amounts to a two-loop calculation of the one-particle irreducible two-point function of a background gauge field. We find:

$$l_1 = \frac{3}{128N^2} + N_f [l_{11}/N + l_{12} N]$$

$$+ 0.018127763034(4) - N^2 0.0079101185(2)$$

$$+ c_{SW}^2 0.0052931(2) + c_{SW}^3 0.0005624(3)$$

$$+ c_{SW}^4 0.0008199(1),$$

$$l_{12} = 0.0009998(16) + c_{SW} 0.000342(4)$$

$$- c_{SW}^2 0.0048660(6) - c_{SW}^3 0.00021431(3)$$

$$- c_{SW}^4 0.0004382(1).$$  \hspace{1cm} (11)

Substitution in Eq. (10) now yields $b_2^L$, for any value of $N, N_f, c_{SW}$. The correction coefficient $q$ of Eq. (3) also follows immediately; it brings about a substantial correction to asymptotic scaling, with a pronounced $c_{SW}$ dependence. Finally, the coefficients $d_1(a\mu)$ and $d_2(a\mu)$ can be read off Eqs. (11), (4), (6), (7).

There exist several constraints on the algebraic and numerical values of individual diagrams. A particularly strong check is provided by Ref. [8]. Eq. (5.6) in that reference reads:

$$d_2(N=3, c_{SW}^3=1) = 1.685(9) - 8.6286(2) c_{SW}^3.$$  \hspace{1cm} (12)

For the same quantity, our results lead to:

$$1.6828(8) - 8.62843775(1) c_{SW}^3.$$  \hspace{1cm} (13)

Both sets of numbers are clearly in very good agreement.
3. THE HOPPING PARAMETER

We set out to calculate the hopping parameter,
\[ \kappa \equiv \frac{1}{(2m_B a + 8r)} \]
which is an adjustable quantity in numerical simulations; \( m_B \) is the bare fermionic mass, and \( r \) is the Wilson parameter appearing in the fermionic action, usually set to 1. The critical value of \( \kappa \), at which chiral symmetry is restored, is thus \( 1/8r \) classically, but gets shifted by quantum effects.

For a vanishing renormalized mass we require:
\[ m_B = \Sigma^L(0, m_B, g_0) \]
where \( \Sigma^L(p, m_B, g_0) \) is the truncated, 1PI fermionic two-point function. We solve this recursive equation for \( m_B \) perturbatively. We write:
\[ \Sigma^L(0, m_B, g_0) = g_0^2 \Sigma^{(1)} + g_0^4 \Sigma^{(2)} + \cdots \]

Two diagrams contribute to \( \Sigma^{(1)} \), and 26 to \( \Sigma^{(2)} \). Certain sets of diagrams must be evaluated together for infrared convergence. The dependence on \( c_{SW} \) is polynomial. For \( \Sigma^{(1)} \) we find:
\[ \Sigma^{(1)} = \frac{N^2 - 1}{N} \left( -0.16285705871085(1) \right) \]
\[ + c_{SW} \left( 0.04348303388205(1) \right) \]
\[ + c_{SW}^2 \left( 0.01809576878142(1) \right) \]

One- and two-loop results pertaining to \( c_{SW} = 0 \) are as in Ref. [6], and can be found with greater accuracy in a subsequent work [9].

For \( c_{SW} \neq 0 \), only one-loop results exist so far in the literature; a recent presentation for \( N=3 \), \( c_{SW}=1 \) [8] agrees with our Eq. (15):
\[ \Sigma^{(1)} \rightarrow -0.2700753495(2), \quad \text{Ref. [8]} \]
\[ -0.270075349597(5), \quad \text{Eq. (15).} \]

We now turn to the much more cumbersome evaluation of \( \Sigma^{(2)} \). Our result is:
\[ \Sigma^{(2)}/(N^2-1) = \]
\[ \left[ (-0.017537(3) + 1/N^2 0.016567(2) \right. \]
\[ + N_f/N 0.00118618(8)) \]
\[ + (0.002601(2) - 1/N^2 0.000597(7) \right. \]
\[ - N_f/N 0.000545(2)) \]
\[ + (-0.0001556(3) + 1/N^2 0.0026226(2) \right. \]
\[ + N_f/N 0.0013652(1)) \]
\[ + (-0.000163(6) + 1/N^2 0.00015803(6) \right. \]
\[ - N_f/N 0.00069225(3)) \]
\[ + (-0.00017219(2) + 1/N^2 0.000042829(3) \right. \]
\[ - N_f/N 0.000198100(7)) \]

Several important consistency checks can be performed on the values of individual diagrams; our results satisfy these checks [7]. Eqs. (15, 16) lead immediately to the critical mass and hopping parameter, making use of Eqs. (13, 12).

A number of non-perturbative determinations of \( \kappa_c \) exist for particular values of \( g_0, c_{SW}, \) and \( N_f \). We present these in [7], along with our results and with “dressed” results obtained by a resummation of cactus diagrams [10]. Comparing with the Monte Carlo estimates, dressed results show a definite improvement over non-dressed values.

REFERENCES

5. A. Bode, H. Panagopoulos, e-print hep-lat/0110211.