Kappa-symmetric higher derivative terms in brane actions

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Abstract
Using the superembedding formalism we construct supermembrane actions with higher derivative terms which can be viewed as possible higher order terms in effective actions. In particular, we provide the first example of an action for an extended supersymmetric object with fully $\kappa$-symmetric extrinsic curvature terms.
1 Introduction.

Most discussions of $p$-brane effective actions are confined to the lowest nontrivial order in the momentum expansion. Higher derivative terms will appear when the expansion is continued, however, and their supersymmetrisation is of interest either when considering the possible structure of the effective action or, taking a different attitude, as possible counterterms/regulators in a quantum theory of branes/supergravities. For $p$-branes the bosonic structure of such terms has been considered in some cases. In particular, the curvature square terms of effective $Dp$-brane actions that correspond to $O(\alpha')^2$ corrections to string scattering amplitudes were presented in [1], (see also [2], [3]), and higher $\partial F$ corrections to the open string effective action were given in [4], (and in, e.g., [5]). Typically, the higher derivative corrections will include extrinsic curvature contributions. When contemplating the full theory, this reopens the problem of finding a globally supersymmetric and locally $\kappa$-symmetric formulation of a $p$-brane with extrinsic curvature terms. This is the topic of the present note.

Partly motivated by Polyakov’s suggestion that the QCD string should be viewed as a string with extrinsic curvature terms [6], “rigid” strings and $p$-branes were quite extensively studied in the 80’s, [7, 8, 9] and early 90’s [10]. A $\kappa$-symmetric Green-Schwarz type “rigid” string was proposed in [11], and a generalization to higher $p$ was suggested in [12]. The $\kappa$-symmetry was only shown (for the string) to second order in the spinorial target space coordinate $\theta$, however, and a fully $\kappa$-symmetric formulation was only found for the “rigid” superparticle [13]. (For the 2-brane, spinning, i.e., locally world-volume supersymmetric, formulations with extrinsic curvature terms were found [14]).

In the $\kappa$-symmetric description of the “rigid” superparticle, the $\kappa$-symmetry is embedded in a local worldline superconformal symmetry, a fact which points to the way we understand $\kappa$-symmetry today, namely as defined in terms of the local supersymmetry of the worldsurface in the superembedding approach to $p$-branes. This latter view is our starting point in this paper.\footnote{The use of superspace methods to construct higher derivative counterterms in field theory was first used in [15] in the context of $D = 4$ supergravity theories}

The superembedding approach yields conditions for super $p$-branes to be embedded in a $D$-dimensional superspace. Depending on the values of $d = p + 1$ and $D$, the embedding conditions are more or less restrictive. In the most restrictive cases, the embedded surface is given on-shell, but for certain other cases, there is freedom left corresponding to an off-shell world surface multiplet. We focus on one of these latter cases ($d = 3$ in $D = 4$), and utilize the off-shell multiplet to construct our higher order actions. In constructing these we are to a certain extent guided by the bosonic higher derivative actions studied previously. The superspace actions contains an auxiliary scalar superfield, however, and eliminating this field turns out to be non-trivial and to lead to a (presumably infinite) series of higher derivative terms in general. A particularly simple bosonic action results if we choose a specific superspace action. An interesting feature of this action is that it only contains the trace-free part of the extrinsic curvatures.

The paper is organized as follows: in section 2 we give the necessary background, presenting a brief summary of the superembedding formalism. In section 3 we specialize to the case of a membrane in $D = 4$ and analyze the torsion equation in detail. Section 4 contains a discussion
of how to construct actions in the embedding context, and in section 5 this technique is utilized to find higher derivative actions. Section 6 contains our conclusions and some of our notation and conventions are collected in an appendix.

2 Superembeddings

The superembedding formalism was pioneered in the context of superparticles in three and four dimensions by Sorokin, Tkach, Volkov and Zheltukin [16], and has been applied by these and other authors to various other branes; for a review see [17]. In [18] it was shown that the formalism can be applied to arbitrary branes including those with various types of worldvolume gauge fields, and it was then used to construct the full non-linear equations of motion of the M-theory 5-brane in an arbitrary supergravity background [19].

We consider a superembedding $f : M \rightarrow \mathcal{M}$, where $M$ is the worldvolume of the brane and $\mathcal{M}$ is the target space. Our index conventions are as follows; coordinate indices are taken from the middle of the alphabet with capitals for all, Latin for bosonic and Greek for fermionic, $M = (m, \mu)$, tangent space indices are taken in a similar fashion from the beginning of the alphabet so that $\alpha = (a, \alpha)$. We denote the coordinates of $M$ by $X^M = (x^m, \xi^\mu)$. The distinguished tangent space bases are related to coordinate bases by means of the supervielbein, $E_M^A$, and its inverse $E_A^M$. We use exactly the same notation for the target space but with all of the indices underlined. Indices for the normal bundle are denoted by primes, so that $A' = (a', \alpha')$.

The embedding matrix is the derivative of $f$ referred to the preferred tangent frames, thus

$$E_A^A := E_A^M \partial_M X^M E_M^A$$

The basic embedding condition is

$$E_{\alpha a} = 0$$

Geometrically this states that the odd tangent space of the brane is a subspace of the odd tangent space of the target superspace at any point on the brane. To see the content of this constraint we can consider a linearised embedding in a flat target space in the static gauge. This gauge is specified by identifying the coordinates of the brane with a subset of the coordinates of the worldvolume, so that

$$x^a = \begin{cases} x^a \\ x'(x, \xi) \end{cases}$$

$$\xi^\alpha = \begin{cases} \xi^\alpha \\ \xi'(x, \xi) \end{cases}$$

Since
\[ E^a = da^a - \frac{i}{2} d\xi^a (\Gamma^a)_{\alpha\beta} \xi^\beta \]  

in flat space, it is easy to see, to first order in the transverse fields, that the embedding condition implies that

\[ D_{\alpha} \hat{x}^{a'} = i (\Gamma^{a'})_{\alpha\beta} \xi^{\beta'} \]  

where

\[ \hat{x}^{a'} = x^{a'} + \frac{i}{2} \xi^{\alpha} (\Gamma^{a'})_{\alpha\beta} \xi^{\beta'} \]  

From this equation the nature of the worldvolume multiplet specified by the embedding condition can be determined. Depending on the dimensions involved, this multiplet can be one of three types: (i) on-shell, i.e. the multiplet contains only physical fields and these fields satisfy equations of motion, (ii) Lagrangian off-shell, meaning that the multiplet also contains auxiliary fields (in most cases), and that the equations of motion of the physical fields are not satisfied, although they can be derived from a superfield action, and (iii) underconstrained, which means that further constraints are required to obtain multiplets of one of the first two types. For thirty-two target space supersymmetries, the multiplets are always of type (i) or of type (iii), whereas for sixteen or fewer supersymmetries all three types of multiplet can occur, with type (iii) arising typically for cases of low even codimension.

In this paper we shall be concerned with the membrane in \( N = 1, D = 4 \) superspace. The worldvolume is therefore \( N = 1, d = 3 \) superspace, and the worldvolume multiplet is an off-shell Lagrangian multiplet (type (ii)) which is simply a scalar multiplet. This can be seen from (7). Since the index \( a' \) only takes on one value we see that this equation determines the transverse fermionic superfield \( \xi^{a'} \) to be the fermionic derivative of the transverse bosonic superfield, but that there are no further constraints on this field. So the multiplet is a scalar multiplet as claimed. The components are a scalar, a spin one-half fermion and an auxiliary scalar.

In order to determine the consequences of the superembedding condition in the non-linear case, and to find the induced supergeometry on the brane one uses the torsion equation which is the pull-back of the equation defining the target space torsion two-form. This reads

\[ 2\nabla_{[A}E_{B]}^{\mathcal{C}} + T_{AB}^{\mathcal{C}} E_{C}^{\mathcal{C}} = (-1)^{A(B+E)} E_{B}^{\mathcal{B}} E_{A}^{\mathcal{A}} T_{AB}^{\mathcal{C}} \]  

In this equation the covariant derivative acts on both worldvolume and target space tensor indices. The latter are taken care of by the pull-back of the target space connection, while the worldvolume connection can be chosen in a variety of different ways. One also has to parametrise the embedding matrix. Having done all this, one can work through the torsion equation starting at dimension zero. In this way the consequences of the embedding condition (2) can be worked out in a systematic and covariant fashion.
The equations of the component or Green-Schwarz formalism can be obtained by taking the leading \((\xi = 0)\) components of the various equations that describe the brane multiplet. A key feature of the formalism is that these component equations are guaranteed to be \(\kappa\)-symmetric because \(\kappa\)-symmetry can be identified with the leading term in a worldvolume local supersymmetry transformation. We recall briefly how this works [16]. Let \(v^M\) be a worldvolume vector field generating an infinitesimal diffeomorphism. If we write the superembedding in local coordinates as \(f^M = X^M(X)\), then the effect of such a transformation on \(X(X)\) is

\[
\delta X^M = v^M \partial_M X^M
\]  

If we express this in the preferred bases we find

\[
\delta X^A := \delta X^M E_M^A = v^A E_A^A
\]  

Now if we take an odd worldvolume diffeomorphism, \(v^a = 0\), and use the embedding condition (2) we get

\[
\begin{align*}
\delta X^a &= 0 \\
\delta X^\alpha &= v^\alpha E_\alpha^\alpha
\end{align*}
\]  

This can be brought to the more usual \(\kappa\)-symmetric form if we define

\[
\kappa^\alpha := v^\alpha E_\alpha^\alpha
\]

and note that it satisfies

\[
\kappa^\alpha = \kappa^\tilde{\alpha} P_{\tilde{\alpha}}^\alpha := \frac{1}{2} \kappa^\tilde{\alpha} (1 + \Gamma) \tilde{\alpha}^\alpha
\]

where \(P\) is the projection operator onto the worldvolume subspace of the odd tangent space of the target superspace. We can always write \(P = 1/2(1 + \Gamma)\), and so \(\Gamma\) is computable in terms of the embedding matrix. Substituting this into (12) we recover the normal form of \(\kappa\)-symmetry transformations. (Strictly, we should evaluate this equation at \(\xi = 0\) to get the correct component form.) The explicit form of \(P\) is

\[
P_{\tilde{\alpha}}^\alpha = (E^{-1})_{\tilde{\alpha}}^\gamma E_\gamma^\alpha
\]

where the inverse is taken in the fermionic tangent space (of \(M\)).

3 The membrane in \(D = 4\)

In this section we shall give the details of the non-linear embedding of the membrane in \(D = 4\).
To simplify the discussion we shall take the target space to be flat. For the embedding matrix we can take, as usual,

\begin{align}
E_\alpha^{\underline{a}} &= 0 \\
E_\alpha^{\underline{a}_1} &= u_\alpha^{\underline{a}_1} + h_\alpha^{\underline{b}_1} u_{\underline{b}_1}^{\alpha},
\end{align}

where both \(\alpha\) and \(\alpha'\) are \(d = 3\) spinor indices taking two values. We also choose

\[ E_\alpha^{\underline{a}} = m_a^{\underline{b}} u_\alpha^{\underline{b}}. \]

where

\[ \text{Spin}(1, 2) \ni u = \begin{pmatrix} u_\alpha^{\underline{a}} \\ u_{\alpha'}^{\underline{a}} \end{pmatrix} \]

and where the corresponding element of \(SO(1, 2)\) is made up by \(u_\alpha^{\underline{a}}\) and a normal component \(u_{\beta}^{\underline{a}}\). (Such variables have been referred to as "Lorentz harmonics" in the literature; see, for example, reference [20].)

The dimension zero component of the torsion equation (8) gives

\[ h_\alpha^{\underline{b}} = i\delta_\alpha^{\underline{b}} h \]

and we can choose

\[ T_{\alpha\beta}^{\underline{c}} = -i(\gamma^{\underline{c}})_{\alpha\beta}, \]

if

\[ E_\alpha^{\underline{a}} = f u_\alpha^{\underline{a}}, \quad f = 1 + h^2 \]

We could have chosen to include the scale factor in the dimension zero torsion instead, but we make the above choice in order to have the standard worldvolume dimension zero torsion. In fact, we shall continue with this policy at higher dimensions so that the torsion components will be the standard ones of off-shell \(N = 1, d = 3\) supergravity, that is

\begin{align}
T_{\alpha\beta}^{\gamma} &= 0 \\
T_{ab}^{\underline{c}} &= 0 \\
T_{ab}^{\underline{c}} &= 0 \\
T_{ab}^{\underline{c}} &= i(\gamma_a)_{\beta}^{\gamma} S
\end{align}

It is always possible to bring the components of the torsion tensor to this form by making appropriate choices for the connection and the even tangent space. The field \(S\) will be determined later in terms of the worldvolume multiplet so that the geometry is indeed induced.
To analyse the higher-dimensional components of the torsion equation it is convenient to introduce the \( \text{spin}(1, 2) \) valued one-form \( X = duu^{-1} \). We have, by group theory,

\[
X_{\beta\gamma} = \frac{1}{4} (\gamma^{bc})_{\beta}^{\gamma} X_{bc}
\]

\[
X_{\beta}^{\prime\gamma} = \frac{1}{4} (\gamma^{bc})_{\beta'}^{\gamma'} X_{bc}
\]

\[
X_{\beta}^{\gamma'} = \frac{1}{2} (\gamma^b)_{\beta}^{\gamma} X_{b3}
\]

\[
X_{\beta'}^{\gamma} = \frac{i}{2} (\gamma^b)_{\beta'}^{\gamma} X_{b3}
\]

(24)

where \( X_{bc} = -X_{cb} \). Since we have specified the worldvolume connection already we have to use the torsion equation to solve for the components of \( X \).

It is now a matter of straightforward algebra to go through the torsion equation and determine the various unknown quantities in terms of the worldvolume multiplet. If we write \( E_a^\alpha \) in the form

\[
E_a^\alpha = \Lambda_a^\alpha u_{a,\alpha} + \psi_a^{\alpha'} u_{a',\alpha}
\]

(25)

and put

\[
X_{\alpha,b3} = \chi_{ab3}
\]

\[
X_{\alpha,bc} = \epsilon_{cda} \chi_{d}
\]

(26)

we find that all of the dimension one-half fields can be expressed in terms of \( \Lambda_a \) which should be thought of as the derivative of the transverse fermionic field specifying the embedding. For any vector-spinor \( \zeta_a \) we put

\[
\zeta_a = \hat{\zeta}_a + \gamma_a \zeta
\]

(27)

where the hatted quantity is gamma-traceless. The results are

\[
\hat{\psi}_a = ih \hat{\Lambda}_a
\]

\[
\hat{\chi}_a = 0
\]

\[
\hat{\chi}_{a3} = -\hat{\Lambda}_a
\]

(28)

while for the spin one-half components we have

\[
\psi = -\frac{2ih}{1 + 3h^2} \Lambda
\]
\[ \chi = \frac{3h}{1 + 3h^2} \Lambda \]
\[ \chi_3 = -\frac{1}{1 + 3h^2} \Lambda \]  
\[ (29) \]

and

\[ \nabla_a h = -\frac{3(1 + h^2)}{2(1 + 3h^2)} \Lambda \alpha \]  
\[ (30) \]

At dimension one we find

\[ -i\nabla_\alpha \Lambda_{\alpha\beta} = (\gamma^h)_{\alpha\beta} \left( -\frac{1}{2} X_{ab3} - \frac{h}{2} X_{ab} + \eta_{ab} h S \right) + \epsilon_{\alpha\beta} \nabla_a h \]  
\[ (31) \]

and

\[ X_{a,bc} := \epsilon_{bcd} X_a^d; \quad X_{ab} = -\frac{2h}{f} \epsilon_{abc} \nabla^c h \]
\[ X_{a,b3} = X_{ab3} = X_{ba3} \]  
\[ (32) \]

as well as

\[ S = -\frac{h}{2(1 + 3h^2)} X_3; \quad (X_3 := \eta^{ab} X_{ab3}) \]  
\[ (33) \]

In equations (31), (32), (33) we have omitted fermion bilinear terms (\( \Lambda^2 \) terms) as we shall not need them later, but they can easily be computed. The leading component of the quantity \( X_{a,b3} \), which is symmetric up to fermion bilinears, can be thought as the bosonic extrinsic curvature (second fundamental form) associated with the superembedding.

We note also that

\[ -i\nabla_\alpha \Lambda_\beta = \epsilon_{\alpha\beta} \left( \frac{X_3}{6} - h S \right) - \frac{(1 + 3h^2)}{3f} (\gamma^a)_{\alpha\beta} \nabla_a h \]  
\[ (34) \]

again up to fermion bilinears. One can of course also compute the dimension three-halves torsion and curvature components and the dimension two curvature in terms of the brane multiplet but we shall not need these results in the rest of this paper.

4 Actions

In [21] a method of computing the Green-Schwarz action of any brane starting from the superembedding formalism was given. It is similar in some respects to the generalised action principle of [22]. This method works even for on-shell brane multiplets (with the exception of those that
have constrained self-dual tensor fields), but in cases where the brane multiplet is off-shell one can find the superspace Lagrangian as well. Some examples of this in the context of codimension zero superembeddings were given in a recent paper [23]. This sort of construction has also been discussed in the context of general supersymmetric theories in [24].

We recall briefly the construction of brane actions in the context of the \(D = 4\) membrane [21, 25]. There is a closed worldvolume four-form \(W_4\) which in this case is the pull-back of a target space four-form \(G_4\),

\[
W_4 = f^*G_4 = d(f^*C_3)
\]  
(35)

where \(C_3\) is the target space potential three-form. As the bosonic worldvolume is three-dimensional, and since the de Rham cohomology of a real supermanifold is the same as that of its body, it follows that \(W_4\) is exact and can be written as \(W_4 = dK_3\) for some globally defined worldvolume three-form \(K_3\). The three-form \(\mathcal{L}_3 := K_3 - f^*C_3\) is closed, and one can construct the GS action in terms of \(\mathcal{L}_3\) as follows:

\[
S_{GS} = \int_{M_0} dx^m dx^n dx^p \mathcal{L}_{mnp}(x, 0)
\]  
(36)

where \(M_0\) is the body of \(M\). This action is guaranteed to be invariant under reparametrisations of \(M_0\) and \(\kappa\)-symmetry because these transformations are the leading components of even and odd worldvolume diffeomorphisms respectively and because \(\mathcal{L}_3\) is closed.

In fact we can also construct a superfield action from \(\mathcal{L}_3\), or indeed from any closed three-form \(L_3\), \(dL_3 = 0\). Since any action will be unaffected by a shift in \(L_3\) of the form \(L_3 \rightarrow L_3 + dY_2\), it follows that we can use this freedom to choose \(L_{\alpha\beta\gamma} = 0\) and \(L_{\alpha\beta\gamma} = i(\gamma_\alpha)_{\alpha\beta}L\), where \(L\) is an unconstrained scalar superfield. Working through the component equations of \(dL_3 = 0\) one then finds that

\[
\begin{align*}
L_{\alpha\beta\gamma} &= 0 \\
L_{\alpha\beta\gamma} &= i(\gamma_\gamma)_{\alpha\beta}L \\
L_{\alpha\beta\gamma} &= (\gamma_{ab})_{\delta} \nabla^\delta L \\
L_{abc} &= \epsilon_{abc}L_\alpha
\end{align*}
\]  
(37)

where

\[
L_\alpha = 2i \nabla^2 L - 4\sigma L
\]  
(38)

with

\[
\nabla^2 := -\frac{1}{2} \nabla^\alpha \nabla_\alpha
\]  
(39)
so that

\[ \nabla_\alpha \nabla_\beta L = \frac{i}{2} (\gamma^a)_{\alpha\beta} \nabla_a L + \epsilon_{\alpha\beta} \nabla^2 L \]  

(40)

We can interpret \( L \) as the superfield Lagrangian while \( L_o \) is related to the component Lagrangian which can be obtained from (36). Equivalently, given a scalar superfield \( L \) of the appropriate dimension, we can construct a closed three-form \( L_3 \) which has components given by (37). We can then compute the component Lagrangian using (36).

The Green-Schwarz action for the membrane was discussed from this point of view in [21]. One finds that the only non-vanishing component of \( K_3 \) is \( K_{abc} := \epsilon_{abc} K \), where

\[ K = \frac{1 - h^2}{1 + h^2} \]  

(41)

The contribution this term makes to the component GS action can be found using

\[ K_{mnp} = E_m^a E_n^b E_p^c \epsilon_{abc} K \]  

(42)

so if we define the GS Lagrangian density \( \mathcal{L}(x) \) by

\[ \mathcal{L}(x) := \epsilon^{mnp} \mathcal{L}_{mnp}(x, 0) \]  

(43)

we find, for the membrane

\[ \mathcal{L}(x) = (\det (E_m^a) K)(x, 0) + \mathcal{L}_{WZ} \]  

(44)

where \( \mathcal{L}_{WZ} \) denotes the Wess-Zumino term which is given by the pull-back of \( C_3 \) onto \( M_o \).

Now the leading term of \( E_m^a = e_m^a + O(\xi) \) is not quite the usual GS dreibein due to the factor of \( f \) included in \( E_a^a \), instead we find

\[ \bar{e}_m^a = f e_m^a \]  

(45)

where \( \bar{e}_m^a \) is the GS dreibein. It is the dreibein for the GS metric \( \bar{g}_{mn} \) defined by

\[ \bar{g}_{mn} = \mathcal{E}_m^a \mathcal{E}_n^b \eta_{ab} \]  

(46)

where

\[ \mathcal{E}_m^a := \partial_M X^M E_M^a \]  

(47)

the vertical bar denoting evaluation at \( \xi = 0 \). (This object is often called II in GS notation).

We therefore find
The equation of motion for the auxiliary field $h$ is simply $h = 0$, and substituting this back in (48) one finds the usual GS Lagrangian.

This formula can be adapted for a general Lagrangian defined by a scalar field $L$ as described above. The bosonic terms in the component Lagrangian are found from

$$L(x) = \sqrt{-\det g} f^{-3} K + L_{WZ}$$

where the dots indicate terms involving contributions involving fermions.

5 Higher derivative Lagrangians

In this section we shall apply the above method to construct higher derivative Lagrangians for the membrane. The GS Lagrangian is of standard two-derivative type in the bosonic fields so we shall look at terms in the bosonic part of the Lagrangian which have four derivatives. These terms are of extrinsic curvature squared type. To include them we shall need to introduce a parameter, $\beta$ say, which has dimension $-2$.

The idea is then to write down all possible scalar superfields constructed from $h$ and $\Lambda_a$ which have the correct dimension, i.e. 1. There are two possible quadratic terms one can construct from $\Lambda$, 

$$L^{(1)} = -\frac{i}{2} \Lambda^{\alpha \alpha} \Lambda_{\alpha \alpha}$$

and

$$L^{(2)} = -\frac{i}{2} (\gamma^{ab})_{\alpha \beta} \Lambda_{\alpha \alpha} \Lambda_{\beta \beta}$$

Since $h$ has dimension zero one can multiply each of these by arbitrary functions of $h$. It is straightforward to calculate the contributions to $L_{\alpha}$ that these Lagrangians make. Neglecting fermion terms we find

$$-i \nabla^\alpha \nabla_\alpha L^{(1)} = \frac{1}{2} X_{ab3} X^{ab3} + \frac{h^2(2 + 9 h^2)}{2(1 + 3 h^2)^2} - \frac{2(1 + 2 h^2 + 3 h^4)}{f^2}(\nabla h)^2$$

$$-i \nabla^\alpha \nabla_\alpha L^{(2)} = \frac{1}{2} X_{ab3} X^{ab3} - \left(1 + \frac{h^2(2 + 9 h^2)}{(1 + 3 h^2)^2}\right) X^3 + \frac{4 h^2(2 + 3 h^2)}{f^2}(\nabla h)^2$$

A general linear combination of these two terms with $h$-dependent coefficients will lead to a rather complicated bosonic Lagrangian. However, there are two particular combinations which generate simpler results. Clearly we can find functions of $h$ such that when the two terms multiplied by
these functions are added the terms involving derivatives of \( h \) will drop out. For these choices the \( h \) equation of motion would remain algebraic although it would be very complicated and difficult to solve. The second possibility is to take the linear combination \( L^{(1)} + \frac{1}{2}L^{(2)} \). In this case there are still derivative \( h \) terms but all the coefficient functions simplify dramatically. We find

\[
L^{(1)}_a + \frac{1}{2}L^{(2)}_a = \frac{3}{4} \hat{X}_{ab3} \hat{X}^{ab3} - \frac{2}{f^2} (\nabla h)^2 \tag{54}
\]

where \( \hat{X}_{ab3} := X_{ab3} - 1/3 \eta_{ab} X_3 \) is traceless.

To compute the contribution of this term to the bosonic action one has to change to a coordinate basis and then remember that the bosonic metric in (54) is related by a scale factor to the GS metric. If one then converts to this metric, and drops all fermion terms, one finds the following Lagrangian for the sum of the GS and higher derivative Lagrangian:

\[
L(x) = \sqrt{-\det g} \left( \frac{1 - h^2}{1 + h^2} + \frac{\beta}{2} \hat{K}^2 - \frac{4\beta}{3f^2} (\nabla h)^2 \right) \tag{55}
\]

where

\[
K_{mn}^3 = (\nabla_m \partial_n x^c)(u^{-1})_c^3 \tag{56}
\]

with \( \hat{K} \) being the corresponding traceless tensor. The metric here is

\[
g_{mn} = \partial_m x_a \partial_n x_a \tag{57}
\]

i.e. the standard induced metric for the associated bosonic embedding. The tensor \( \hat{K}_{mn}^3 \) is then the traceless extrinsic curvature for this embedding.

If we now set \( h = \tan \phi \) the Lagrangian becomes

\[
L(x) = \sqrt{-\det g} \left( \frac{\beta}{2} \hat{K}^2 - \frac{4\beta}{3} (\nabla \phi)^2 + \cos 2\phi \right) \tag{58}
\]

Hence the equation of motion for the auxiliary field in this case is simply the sine-Gordon equation,

\[
\nabla^2 \phi - \frac{3}{4\beta} \sin 2\phi = 0 \tag{59}
\]

6 Conclusions

We have discussed a particular case, the membrane in \( N = 1, D = 4 \) flat superspace, where the embedding formalism allows us to construct \( \kappa \)-symmetric extensions with higher derivative terms rather straightforwardly. Note that the superembedding matrix \( E_{4\Delta} \) is invariant under rigid
target space supersymmetry transformations because it is constructed using the invariant target space supervielbein, and so the invariance of the actions we have discussed under this symmetry is manifest, provided that we include the fermion terms, of course. We have also seen that the component actions we have exhibited can be viewed as superspace actions which are manifestly invariant under local worldvolume supersymmetry and this guarantees the invariance of the component actions under reparametrisations of the bosonic worldvolume and local $\kappa$-symmetry. We could, of course, fix the worldvolume gauge to the physical (static) gauge, say, in which case our actions would be invariant under non-linearly realised target space supersymmetry with three-dimensional $N = 1$ supersymmetry as the linearly realised subgroup.

Above we gave an action with extrinsic curvature terms. What about intrinsic curvatures? In fact, at least at the bosonic level, this is to a certain extent a matter of how one chooses to express the action. The Gauss-Codazzi equations relate the intrinsic and extrinsic curvatures on the world volume to each other and to the pull-back of the ambient space time curvature. In our case the ambient (super-) space-time is flat and we may therefore trade extrinsic and intrinsic curvatures according to

$$R_{klmn} = K_{k[mn}^3 K_{n]l}^3,$$

which may be used, e.g., to rewrite the extrinsic curvature term in (58) as

$$\tilde{K}^2 = -R + \frac{2}{3}(K_3)^2,$$

where $R$ is the curvature scalar and $K_3$ is the trace of the extrinsic curvature. We can see from the results of section 3 that a similar situation holds in superspace. The worldvolume intrinsic curvature superfield, $S$, has dimension one and could thus appear in the superspace Lagrangian $L$ multiplied by another function of $h$. However, from (34) we see that $S$ can be expressed as a function $F$, say, of $h$ times $\nabla^\alpha \Lambda_\alpha$ up to fermion bilinears. So we could integrate such a term by parts to get $F^i(h)\nabla^\alpha h \Lambda_\alpha$, whereupon we can use (30) to express it as another function of $h$ times a fermion bilinear. Hence the possible intrinsic curvature term that could be included in the action does not give anything different and the actions we have constructed are the most general at this dimension.

In (58) the higher order term is fourth order in derivatives. In the effective action there will generically be terms of arbitrary order. The restriction on the order of derivatives may seem to be built into our choice of quadratic Lagrangians (50) and (51) in Sec. 5. Note however that a solution to the $h$ field equations for a generic linear combination of (52) and (53) will generate many terms of order higher than four when substituted back into the Lagrangian.

Ideally, when considering higher order terms in the effective actions, one would like to have access to predictions from string theory for comparison. In [1] such results may be found for $D$-branes to order $\alpha'^2$. This calls for extending our results to $R^2$ terms and $D$-branes. Quite generally, it would an interesting task to extend our results to non-flat backgrounds and, in a first attempt, to those other cases where the superembedding formalism yields off-shell multiplets, typically models which have eight or fewer supersymmetries on the brane.

Finally, a surprising feature of our Lagrangian (58) is the appearance of the Sine-Gordon equation
(59). One would like to know if this is just a coincidence or if, e.g., topological solutions to this equation have any particular significance.

Appendix

In this appendix we gather together some conventions and useful formulae which were used in the derivation of the equations presented in the text.

In $D = 4$ the metric is $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and the $\Gamma$-matrices in a split appropriate to the membrane are

\[
\begin{align*}
\Gamma^a &= \gamma^a \otimes \sigma_3 \\
\Gamma^3 &= 1 \otimes \sigma_2
\end{align*}
\]  

while the charge conjugation matrix $C$ and $\Gamma_5$ are given by

\[
\begin{align*}
C &= \epsilon \otimes 1 \\
\Gamma_5 &= -i(1 \otimes \sigma_1)
\end{align*}
\]  

In these equations $\gamma^a$ denotes the $d = 3$ gamma-matrices while the $\sigma$-matrices are the Pauli matrices. In $d = 3$ the metric is $\eta_{ab} = \text{diag}(-1, 1, 1)$ and we take $\epsilon^{012} = +1$. The Dirac matrices obey

\[\gamma^a \gamma^b = \epsilon^{abc} \gamma^c, \quad (a \neq b).\]  

The gamma matrices can be chosen to be real, a permissible explicit choice being $\gamma^0 = i\sigma_2, \gamma^1 = \sigma_1, \gamma^2 = \sigma_3$. The charge conjugation matrix is $C_3 = \epsilon = i\sigma_2$ and Majorana spinors a real. A four-dimensional Majorana spinor can be written as

\[\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}\]  

where the two-component spinors on the right are $d = 3$ spinors, $\phi$ being Majorana and $\chi$ being anti-Majorana. (This accounts for some of the factors of $i$ which appear in the dimension-half equations of section 3.)

The above formulae for gamma-matrices are valid when the indices are in the standard position for matrix multiplication, i.e. the first one down and the second one up. If the second index is lowered, by means of the four-dimensional charge conjugation matrix, one finds
\[
(\Gamma^a_{\alpha \beta}) = \begin{cases} 
(\gamma^a)_{\alpha \beta} = \begin{pmatrix} 0 & (\gamma^a)_{\alpha' \beta'} \\
0 & -((\gamma^a)_{\alpha' \beta'}) \end{pmatrix} \\
(\gamma^a)_{\alpha' \beta} = \begin{pmatrix} 0 & -i\epsilon_{\alpha \beta'} \\
0 & 0 \end{pmatrix}
\end{cases}
\] (66)

For the gamma-matrices with two vector indices we have, with the indices in normal matrix position,

\[
\Gamma^{ab} = \begin{cases} 
\Gamma^{ab} = \gamma^{ab} \otimes 1 \\
\Gamma^{a3} = -i\gamma^a \otimes \sigma_1
\end{cases}
\] (67)

With both indices down these become

\[
(\Gamma^{ab})_{\alpha \beta} = \begin{cases} 
(\gamma^{ab})_{\alpha \beta} = \begin{pmatrix} 0 & (\gamma^{ab})_{\alpha' \beta'} \\
0 & (\gamma^{ab})_{\alpha' \beta'} \end{pmatrix} \\
(\gamma^{a3})_{\alpha \beta} = \begin{pmatrix} 0 & -i(\gamma^a)_{\alpha' \beta'} \\
-i(\gamma^a)_{\alpha' \beta'} & 0 \end{pmatrix}
\end{cases}
\] (68)

\(\Gamma_5\) with indices down is

\[
(\Gamma_5)_{\alpha \beta} = -i \begin{pmatrix} 0 & \epsilon_{\alpha \beta'} \\
\epsilon_{\alpha' \beta} & 0 \end{pmatrix}
\] (69)

while the charge conjugation matrix, which naturally has its indices down, is

\[
C^{\alpha \beta} = \begin{pmatrix} \epsilon_{\alpha \beta} & 0 \\
0 & \epsilon_{\alpha' \beta'} \end{pmatrix}
\] (70)

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References


