Free fermionic propagators on a lattice

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Abstract

A method used recently to obtain a discrete formalism for classical fields with nonlocal actions preserving chiral symmetry and uniqueness of fermion fields yields a discrete version of Huygens' principle with free discrete propagators that recover their continuum forms when the number of the non–equispaced lattice points tends to infinity.
1 Introduction

Lattice theory has been used since many years ago as one of the non-perturbative approaches to study physical effects that could occur in QCD or QED [1]. Nevertheless, the discrete formalism yielded by the Lattice theory does not preserve important properties such as chirality and uniqueness which are present in the standard field theory. Nielsen and Ninomiya [2] have shown that for a local and translationally invariant hermitian discretization of the fields it is not possible to have simultaneously chiral symmetry and uniqueness. Many attempts have been made to circumvent this restriction. The use of nonlocal operators can be found among others (see for example [3]–[5]) in spite that the locality of the Dirac operator is in general a desired property. In this context, a new approach to the problem of discretization of fields is presented in [6]. This approach is based on a non–equispaced lattice, performed through the zeros of Hermite polynomials, where both discrete derivatives and Fourier transform have support [7, 8]. The technique yields a projection of the quantum algebras on a finite linear space yielding matrix representations for the partial derivatives that produce discrete hermitian actions with nonlocal kinetic terms. The fermion doubling and the chiral symmetry breaking are absent in such a formulation. In this work we obtain a discrete version of Huygens’ principle for free spinor fields in 1 + 1 D. We show that whenever the number of the nodes in the non–equispaced lattice tends to infinity, the propagator we found approaches to their continuum form. We also show that the discrete propagator is indeed a discrete Green function by taking the inverse of the Dirac operator.

This work is organized as follows. In section 2 we review the essentials of our discretization method and introduce a discrete version of the sign and step functions. In section 3 we obtain the discrete Huygens’ principle on the non–equispaced lattice. Section 4 is devoted to study the discrete version of the free Dirac propagator in 1 + 1 D. Finally in section 5 we give our main conclusions.

2 Discrete technique

2.1 Review of the method

In this subsection we only present the main results of our discretization scheme; proofs and further applications can be found in [6, 7, 8, 9, 10, 11].

Let us consider the non–equispaced four dimensional lattice constructed with the set of nodal points $x^\mu_j$ ($\mu$ denotes the Lorentz index and $j = 1, 2, \ldots, N_\mu$, with $N_\mu$ being the number of nodes along the direction $\mu$) performed by the zeros of the Hermite polynomial $H_{N_\mu}(\xi)$. We denote $N = N_0N_1N_2N_3$ as the total lattice points. From this set of points we construct the four $N \times N$ matrices:

\[
\begin{align*}
D_0 &= D_0 \otimes 1_{N_3} \otimes 1_{N_2} \otimes 1_{N_1}, \\
D_3 &= 1_{N_0} \otimes D_3 \otimes 1_{N_2} \otimes 1_{N_1},
\end{align*}
\]
\[ D_2 = 1_{N_0} \otimes 1_{N_3} \otimes D_2 \otimes 1_{N_1}, \]
\[ D_1 = 1_{N_0} \otimes 1_{N_3} \otimes 1_{N_2} \otimes D_1, \]

where \( 1_{N_\mu} \) is the identity matrix of dimension \( N_\mu \) and \( D_\mu \) is the skew-symmetric matrix

\[
(D_\mu)_{jk} = \begin{cases} 
0, & i = j, \\
\frac{1}{x_j^{\mu} - x_k^{\mu}}, & i \neq j.
\end{cases}
\]

Let us define the diagonal matrix

\[ S = S_0 \otimes S_3 \otimes S_2 \otimes S_1 \]

where \((S_\mu)_{jk} = \delta_{jk} \exp[-(x_j^{\mu})^2/2] H'_{N_\mu}(x_j^{\mu})\). Then,

\[ SD_\mu S^{-1} \]

is a projection of the partial derivative \( \partial_\mu \) in the subspace of functions \( U \) generated by products of the form

\[ u_n(\xi) = \exp(-\xi^2/2) H_n(\xi), \quad n = 0, 1, \ldots, N_\mu - 1, \]

with \( \xi = x^\mu \) and \( \mu = 0, 1, 2, 3 \). This means that such matrices are exact nonlocal representations of the partial derivatives for functions in \( U \). Therefore, whenever \( N_\mu \to \infty \) we will get convergent approximations to the partial derivatives of a function \( \psi(x) \) spanned by the basis \( \{u_n(\xi)\}_0^\infty \). Thus, the discrete version of a dynamical differential variable operating on such functions is essentially the matrix obtained under the replacement

\[ \partial_\mu \to SD_\mu S^{-1}. \]

The error arising from this procedure can be estimated in special cases [11] and can be related to the complement of \( \psi(x) \) with respect to \( U \) [6]. Thus, if \( \psi(x) \in U \) and \( \Psi \) denotes the \( N \times 1 \) vector of components

\[ \Psi_q = \psi(x_q) \equiv \psi(x_j^1, x_k^2, x_l^3, x_m^0), \]

ordered according to

\[ q = j + (k - 1)N_1 + (l - 1)N_1N_2 + (m - 1)N_1N_2N_3 \]

where first \( j \) runs over \( j = 1, \ldots, N_1 \), then \( k \) over \( k = 1, \ldots, N_2 \), then \( l \) over \( l = 1, \ldots, N_3 \), and finally we take \( m = 1, \ldots, N_0 \). In this form we have that the vector \( \Psi_{.\mu} \) constructed with the values of \( \psi_{.\mu} = \partial_\mu \psi \) at the site \( x_q = (x_m^0, x_l^1, x_k^2, x_j^3) \) and ordered as in (5), is given by

\[ \Psi_{.\mu} = SD_\mu S^{-1} \Psi. \]
The similarity transformation given by $S$ changes $\Psi$ into itself except for an alternating change of sign along each direction when $N_\mu \to \infty$ (see for example [6]) therefore, we may use $D_\mu$ instead (2) as a discrete representation of $\partial_\mu$.

Let $g(x^\mu)$ be a given function and $g$ and $g'$ the vectors of components $g(x_j^\mu)$ and $\partial g(x_j^\mu)/\partial x^\mu$, respectively. Let us denote by $G = \text{diag}(g)$ the diagonal matrix whose nonzero elements are the components of $g$. Then, we have that [6]

\[(S_\mu D_\mu S_\mu^{-1})G = G(S_\mu D_\mu S_\mu^{-1}) + G' + R_\mu,\]

where $G' = \text{diag}(g')$ and $R$ is the residual matrix which projects an arbitrary vector on the orthogonal subspace generated by the complement basis of (3), i.e., by $\{u_n(\xi)\}_{N_\mu}^{\infty}$. This equation, applied to the vector constructed with the values of the function $h(x^\mu)$ at the nodes, is the finite representation of the familiar formula

\[
\partial (gh)/\partial x^\mu = g\partial h/\partial x^\mu + h\partial g/\partial x^\mu.
\]

Since the complement basis becomes empty as the number of elements of (3) goes to infinite, we have that

\[D_\mu G = GD_\mu + G', \quad N_\mu \to \infty, \quad (7)\]

where $G'$ can be substituted by the diagonal matrix $\text{diag}(D_\mu g)$.

Now, let us define the symmetric function

\[F(\xi, \eta) = \sum_{l=0}^{N-1} (i)^l \varphi_l(\xi) \varphi_l(\eta), \quad (8)\]

where

\[\varphi_l(\xi) = \sqrt{(N-1)!2^{N-1-l}/N!} H_l(\xi)/H_{N-1}(\xi).\]

Since

\[iD_0 F = FP^0, \quad -iD_3 F = FP^3, \quad (9)\]

the commuting matrices (1) can be diagonalized simultaneously by the unitary and symmetric matrix

\[F = F_0^1 \otimes F_3 \otimes F_2 \otimes F_1, \quad (10)\]

where

\[(F_\mu)_{jk} = F(x_j^\mu, p_k^\mu) \quad (11)\]

is a discrete Fourier transform for one variable [8]. Here, $p_j^\mu$ is also a zero of $H_{N_\mu}(\xi)$ and it represents an eigenvalue of the discretized momentum. The elements of the four-dimensional discrete Fourier transform (10) satisfy the important asymptotic formula

\[\lim_{N \to \infty} F_{q'q} = C_N e^{-i p_0 x_q', \quad (12)\]

where $p_q \cdot x_{q'} = p_0 x_0 - p_3 x_3 - p_2 x_2 - p_1 x_1$ and $C_N$ is the product of the constants $C_{N_\mu} = 2^{N_\mu-3/2}(\Gamma[(N_\mu+1)/2])^2/N_\mu!$ except for an alternating change of sign [7, 6]. By using an asymptotic expression for the Hermite zeros we find that $C_{N_\mu}$ becomes proportional
to the standard measure of a Riemann integral in each variable (the difference between two consecutive lattice points):

$$C_{N_{\mu}} = \frac{1}{\sqrt{2\pi}} \Delta x^\mu = \frac{1}{\sqrt{2\pi}} \Delta p^\mu; \quad (13)$$

where $\Delta p^\mu = \Delta x^\mu = \pi/\sqrt{2N_{\mu}}$. The equations (12) and (13) are the backbone of the quadrature formula for the Fourier transform [8]. Additional and useful properties of $F_{\mu}$, are

$$F_{\mu}^\dagger = (-1)^{N_{\mu}+1}F_{\mu}U, \quad F_{\mu}U = UF_{\mu}; \quad (14)$$

where $U$ is the $N_{\mu} \times N_{\mu}$ matrix whose entries are given by $U_{jk} = \delta_{j,N_{\mu}-k+1}$. Denoting by $[\cdot]_k$ the $k$th column of a matrix, the first equality means that

$$[F_{\mu}]_k \rightarrow (-1)^{N_{\mu}+1}[F_{\mu}^\dagger]_k, \quad (15)$$

under $p^\mu_k \rightarrow -p^\mu_k, \mu = 0, 1, 2, 3$.

### 2.2 Discrete distributions

In section 4 a representation of the step function $\theta(t-t')$ on the lattice will be needed to construct the discrete propagators. To obtain it, we begin by giving a discrete form of the sign function $\epsilon(t-t')$. Is it well known that

$$\epsilon(t-t') = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{e^{it-t'E}dE}{E} = \frac{i}{\pi} \int_{-\infty}^{\infty} e^{itE} \frac{1}{E} e^{-it'E}dE.$$

According to (12) and (13) the integral with the integrand $e^{itE}$ becomes the matrix $\sqrt{2\pi}F_0$ [defined through (11)] applied to the remaining part of the integrand. Thus, the discrete form of $\epsilon(t-t')$ is the skew-symmetric matrix

$$\Xi = -\frac{i}{\pi} (\sqrt{2\pi}F_0)(P^0)^{-1} (\sqrt{2\pi}F_0^\dagger) = -\frac{2i}{\Delta E} F_0(P^0)^{-1} F_0^\dagger,$$

where $P^0$ is the diagonal matrix whose nonzero elements are $E_k$. Of course, $\Xi$ has to be a nonsingular matrix, restricting us to consider $N_0$ even. The fact that the entries of $\Xi$ are real is guaranteed by (14), and (12)–(13) yields a finite asymptotic value for any fixed element\(^1\) of $\Xi$. Since $D_0F_0^* = -iF_0^*P^0$, the discrete derivative of $\Xi$ is two times the identity matrix divided by the measure, i.e.,

$$D_0\Xi = 2\left(\frac{1}{\Delta E}\right)1_{N_0 \times N_0}. \quad (16)$$

\(^1\)By a fixed element of a $N \times N$ matrix $A$, we mean the entry $A_{N/2+j,N/2+k}$, with $j$ and $k$ fixed. Here $N$ is even and $j, k = \pm 1, \pm 2, \ldots, \pm N/2$. 
The measure goes to zero as \( N_0 \to \infty \), therefore, this result is the discrete version of
\[
\frac{d}{dt} \epsilon(t - t') = 2\delta(t - t').
\]
From (16) we see that \(-iF_0(P^0)^{-1}F_0^\dagger\) is the inverse of \( D_0 \) and that \( D_0 \Xi - \Xi D_0 = 0 \). On the other hand, the step function accepts the Fourier representation
\[
\theta(t - t') = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(t-t')E}dE}{E - i\epsilon}.
\]
Expanding the integrand up to the first order in \( \epsilon \) this equation can be rewritten as
\[
\theta(t - t') = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(t-t')E}dE}{E} + \frac{\epsilon}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(t-t')E}dE}{E^2}. \tag{18}
\]
The first integral is \( \epsilon(t - t') \) and the second integral is equal to
\[
\frac{\epsilon}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(t-t')E}dE}{(E^2 + \epsilon^2)} = 1/2. \tag{19}
\]
Therefore, (18) is just the simple relation \( \theta(t) = [\epsilon(t) + 1]/2 \). Since the first positive zero of \( H_{N_0}(E) \) (the first positive value of \( E_k \), i.e., \( E_{N_0/2+1} \)) goes to zero as \( N_0 \to \infty \), we may give a discrete representation of \( \theta(t - t') \) through (18). The identification of \( \epsilon \) with \( E_{N_0/2+1} \) gives the matrix
\[
\Theta = \frac{1}{2}(\Xi + \Sigma), \tag{20}
\]
as the representation of \( \theta(t - t') \) on the lattice, where we have defined
\[
\Sigma = \frac{2E_{N_0/2+1}}{\Delta E} F_0(P^0)^{-1}F_0^\dagger.
\]
Due to (19), a fixed element of this symmetric matrix becomes asymptotically equal to 1 divided by the measure, i.e.,
\[
(\Sigma)_{jk} = \frac{1}{\Delta E}, \quad N_0 \to \infty. \tag{21}
\]
Note that the discretization of (17):
\[
-\frac{i}{\Delta E} F_0(P^0 - iE_{N_0/2+1}1_{N_0\times N_0})^{-1}F_0^\dagger
\]
and (20) are equal up to the first order in \( E_{N_0/2+1} \). The application of \( D_0 \) to \( \Sigma \) yields
\(-E_{N_0/2+1}\Xi \). Since a fixed element of \( \Xi \) is bounded as \( N_0 \to \infty \) and \( E_{N_0/2+1} \to 0 \), we get the expected property
\[
D_0 \Theta = \delta, \tag{22}
\]
where $\delta$ is the identity matrix divided by the measure, i.e., $\delta = 1_{N_0 \times N_0}/\Delta E$. In order to normalize the right-hand side of (22) is convenient to our purpose define

$$\tilde{\Theta} = -iF_0(P^0 - iE_{\frac{N_0}{2}+1}1_{N_0 \times N_0})^{-1}F_0^\dagger = \frac{1}{2}(\tilde{\Xi} + \tilde{\Sigma})$$

(23)
as the discrete form of $\theta(t - t')$, where $\tilde{\Xi} = (\Delta E)\Xi$ and $\tilde{\Sigma} = (\Delta E)\Sigma$. In this way we have that $D_0\tilde{\Theta} = 1_{N_0 \times N_0}$. Another important relation between these matrices follows from (7). If we choose the vector $g$ as the $j$th column of $\tilde{\Theta}$, i.e. $[\tilde{\Theta}]_j$, Eq. (7) becomes

$$D_0 \text{diag}([\tilde{\Theta}]_j) = \text{diag}([\tilde{\Theta}]_j)D_0 + [1_{N_0 \times N_0}]_j, \quad N_0 \to \infty,$$

(24)

### 3 Huygens’ Principle

The above formalism is applied in Ref. [6] to find discrete spinor fields and some of their properties. In particular, the discrete version of the free Dirac equation is found to be

$$i\gamma^\mu \otimes D_\mu \Psi = m 1_{4N}\Psi,$$

(25)

where $\gamma^\mu$ are Dirac matrices, $1_{4N}$ is the identity matrix of dimension $4N$ and $\Psi$ is the discretized field whose four spinorial components $\Psi_a$ have spatial–temporal indexes $(\Psi_a)_q = \Psi_a(x_q)$ ordered according to (4–5). The square of a mass eigenvalue of (25), say $m_r$, is given by the discrete form of the energy–momentum relation

$$m_r^2 = (p_0^0)^2 - (p_1^1)^2 - (p_2^2)^2 - (p_3^3)^2,$$

(26)

where $p_i^\mu$ is a nonzero component of the diagonal matrix $P^\mu$ representing the discretized momentum $p^\mu$. Such values are zeros of Hermite polynomials and therefore, $m_r$ is degenerate. The indexes in (26) are ordered according to

$$r = j + (k - 1)N_1 + (l - 1)N_1N_2 + (m - 1)N_1N_2N_3 + (i - 1)N_1N_2N_3N_0$$

where the slowest index is $i = 1, 2, 3, 4$. The degeneracy of $m_r$ does not depend on $N_\mu$ and the discrete mass–shell consists only in few points even when $N_\mu \to \infty$. However, since the Hermite zeros become dense on the axes, $m_r^2$ approaches to any real number as $N_\mu \to \infty$ and its degeneracy is given by the number of points sufficiently close to the mass–shell defined by $m_r$. This behavior is illustrated in Fig. 1.
Figure 1: The degeneracy of \( m_r \) is approximated by the number of points \( p_i^\mu \) of (26) located between the hyperbolas defined by \( |m - m_r| = \epsilon / 2 \) corresponding to the dashed lines in this figure for the case \( m_r = 7.3 \). The points satisfying this condition are only displayed and the distance between the hyperbolas has been exaggerated. Since we can always find points \( p_i^\mu \) close enough to any point laying on the lower or the upper hyperbola, both surfaces are completely covered when \( N_\mu \to \infty \).

Thus, the sum over the discrete mass–shell can be substituted asymptotically by a sum over all the values \( p_i^k, k = 1, 2, 3, i = 1, 2, \ldots, N_k \), and \( p_i^0 \) given by

\[
(p_m^0)^2 - (p_j^1)^2 - (p_k^2)^2 - (p_l^3)^2 = m^2,
\]

where \( m \) is now a parameter whose numerical value is close to \( m_r \). This approximation will be used later.

On the other hand, a general discrete solution of (25) can be written as a linear combination in the degenerate subspace corresponding to the eigenvalue \( m_r \). By using (15), this combination can be written as a sum over positive values of the energy and other sum over the negative values. Since the spinor space is not affected by the discretization procedure we can obtain the positive and negative energy bispinors following the standard procedure. Thus, a plane–wave solution of (25) corresponding to the mass eigenvalue \( m_r \) is the vector

\[
\Psi_{m_r} = \sum_{p_i^k = m_r^2}^+ \left( (a_q^1 \mathbf{u}_q + a_q^2 \mathbf{u}_q^a) \otimes [\mathbf{F}]_q \right) + \sum_{p_i^k = m_r^2}^- \left( (b_q^1 \mathbf{v}_q + b_q^2 \mathbf{v}_q^a) \otimes [\mathbf{F}^\dagger]_q \right),
\]

where \( a_q^l \) and \( b_q^l, l = 1, 2, \) are complex numbers and

\[
\mathbf{u}_q^1 = C_{mr}[1, 0, p_i^3/(p_m^0 + m_r), p_{jk}^i/(p_m^0 + m_r)]^T,
\]

\[
\mathbf{u}_q^2 = C_{mr}[0, 1, p_{jk}^3/(p_m^0 + m_r), -p_i^3/(p_m^0 + m_r)]^T,
\]

\[
\mathbf{v}_q^1 = C_{mr}[p_i^3/(p_m^0 + m_r), p_{jk}^i/(p_m^0 + m_r), 1, 0]^T,
\]

\[
\mathbf{v}_q^2 = C_{mr}[p_{jk}^3/(p_m^0 + m_r), -p_i^3/(p_m^0 + m_r), 0, 1]^T.
\]
Here, $C_{mr} = (m_r/p_m^0) [(p_m^0 + m_r)/2m_r]^{1/2}$ (the $u$ spinors are normalized to one), $p_{jk}^\pm = p_j^1 \pm ip_k^2$ and $j, k, l, m$ are the indexes of the components of $p_q^2$ satisfying (26) for a given eigenvalue $m_r$. Eq. (26) is the condition that defines the sums given in (28): positive energies on the discrete mass–shell correspond to the first sum and negative energies to the second one. Since $C_{mr}$ depends on the inverse of $p_m^0$ (eigenvalue of $P^0$) the number of temporal nodes must be an even integer to have $p_m^0 \neq 0$.

To obtain a relation between the temporal components of $\Psi_{m_r}$ we need to write down the explicit dependence on their spatial and temporal indexes maintaining the spinor structure; thus, the order of the tensor product of (28) should be taken according to this. We begin by taking only 1 + 1 variables to illustrate a standard procedure that can be generalized to more variables. The representation of the Dirac matrices employed here is $\gamma^0 = \sigma_z$ and $\gamma^1 = i\sigma_y$. By using the notation of (11) and taking into account (15), Eq. (28) can be written as

$$
\begin{align*}
(\Psi_{m_r})_q &= \Psi_{m_r}(x_j, t_k) = \sum_{p_j^2 = m_r^2}^+ \left[ a(p_j, E_{k'}) u(p_j, E_{k'}) F^\ast(t_k, E_{k'}) F(x_j, p_{j'}) \right. \\
&\quad + b(-p_{j'}, -E_{k'}) v(-p_{j'}, -E_{k'}) F^\ast(t_k, -E_{k'}) F(x_j, -p_{j'}),
\end{align*}
$$

(29)

where the sign appearing in (15) has been included in the complex constant $b_{q'} = b(-p_{j'}, -E_{k'})$ and the relations between indexes are

$$
q = j + (k - 1)N_1, \quad q' = j' + (k' - 1)N_1,
$$

(30)

where $j, j' = 1, \ldots, N_1$ and $k, k' = 1, \ldots, N_0$. Here, the spinors are given by

$$
u(p, E) = \frac{1}{2E} \left( \frac{m_r/\sqrt{E + p}}{\sqrt{E + p}} \right), \quad v(-p, -E) = \frac{1}{2E} \left( \frac{\sqrt{E - p}}{-m_r/\sqrt{E - p}} \right).
$$

If we multiply (29) by $F^\ast(x_j, p_{jn})$ and sum over $x_j$ we obtain

$$
\begin{align*}
\sum_{x_j} F^\ast(x_j, p_{jn}) \Psi_{m_r}(x_j, t_k) &= \sum_{p_j^2 = m_r^2}^+ \left[ a(p_j, E_{k'}) u(p_j, E_{k'}) F^\ast(t_k, E_{k'}) \delta_{p_{jn}, p_{j'}} \right. \\
&\quad + b(-p_{j'}, -E_{k'}) v(-p_{j'}, -E_{k'}) F^\ast(t_k, -E_{k'}) \delta_{p_{jn}, -p_{j'}},
\end{align*}
$$

(31)

where we have made use of the orthogonality relations

$$
\begin{align*}
\sum_{x_j} F^\ast(x_j, p_{jn}) F(x_j, p_{j'}) &= \delta_{jnj'} = \delta_{p_{jn}, p_{j'}}, \\
\sum_{x_j} F^\ast(x_j, p_{jn}) F(x_j, -p_{j'}) &= \delta_{jn, N_1} = \delta_{p_{jn}, -p_{j'}}.
\end{align*}
$$

The right-hand side of (31) is different from zero if $p_{jn}$ is on the discrete mass–shell. Thus, multiplying (31) by $u^\dagger(p_{jn}, E_{k'})$, using the relations

$$
u^\dagger(p_{jn}, E_{k'}) u(p_{jn}, E_{k'}) = 1, \quad u^\dagger(p_{jn}, E_{k'}) v(p_{jn}, -E_{k'}) = 0,
$$

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and rearranging indexes we obtain that

$$a(p_j, E_k) = u^\dagger(p_j, E_k) \sum_{x_j'} F^*(x_j', p_j) \Psi_m(x_j', t_{k'})$$

(32)

for any index $1 \leq k' \leq N_0$. Similarly, multiplying (29) by $v^\dagger(p_j, -E_{k'})F^*(x_j, p_j)$ we obtain

$$b(p_j, -E_k) = v^\dagger(p_j, -E_k) \sum_{x_j'} \frac{F^*(x_j', p_j)}{F^*(t_{k'}, -E_k)} \Psi_m(x_j', t_{k'})$$

(33)

for $1 \leq k' \leq N_0$. Therefore, Eq. (29) can be written as

$$\Psi_m(x_j, t_k) = \Psi_m^+(x_j, t_k) + \Psi_m^-(x_j, t_k)$$

(34)

where

$$\Psi_m^+(x_j, t_k) = \sum_{p_{j'}} \sum_{x_{j'}} \left[ u(p_{j'}, E_{k'})u^\dagger(p_{j'}, E_{k'}) \frac{F^*(x_{j'}, p_{j'})}{F^*(t_{k'}, E_{k'})} \right] F^*(t_{k'}, E_{k'}) \Psi_m(x_{j'}, t_{k'})$$

(35)

and

$$\Psi_m^-(x_j, t_k) = \sum_{p_{j'}} \sum_{x_{j'}} \left[ v(-p_{j'}, -E_{k'})v^\dagger(-p_{j'}, -E_{k'}) \frac{F^*(x_{j'}, -p_{j'})}{F^*(t_{k'}, -E_{k'})} \right] F^*(t_{k'}, -E_{k'}) \Psi_m(x_{j'}, t_{k'})$$

(36)

Eqs. (35) and (36) can be written in terms of

$$\gamma = E_{k'}\gamma^0 - p_{j'}\gamma^1 \pm m_r = \left( \pm m_r \begin{array}{c} E_{k'} \pm p_{j'} \\ \pm m_r \end{array} \right)$$

(37)

by using

$$u(p_{j'}, E_{k'})u^\dagger(p_{j'}, E_{k'}) = \frac{\gamma + m_r}{2E_{k'}} \gamma^0 \quad v(-p_{j'}, -E_{k'})v^\dagger(-p_{j'}, -E_{k'}) = \frac{\gamma - m_r}{2E_{k'}} \gamma^0$$

so that

$$\Psi_m(x_j, t_k) = \sum_{x_{j'}} G(x_j, t_k; x_{j'}, t_{k'}) \gamma^0 \Psi_m(x_{j'}, t_{k'})$$

(38)

where

$$G(x_j, t_k; x_{j'}, t_{k'}) = \sum_{p_{j'n}=m_r} \frac{1}{2E_{kn}} \left[ \left( \gamma + m_r \right) \frac{F^*(x_{j'}, p_{jn})}{F^*(t_{k'}, E_{kn})} F^*(t_{k}, E_{kn}) F(x_j, p_{jn}) \right]$$

$$+ \left( \gamma - m_r \right) \frac{F^*(x_{j'}, -p_{jn})}{F^*(t_{k'}, -E_{kn})} F^*(t_{k}, -E_{kn}) F(x_j, -p_{jn}) \right]$$

(39)
Eq. (38) seems to be a discrete version of Huygens’ principle but a simple calculation shows that it is just a resemblance.

It is true that such an equation yields an identity for $\Psi_m(x_j, t_k)$ as given by (29), when $k \rightarrow k'$ (interchange the sums of (38) and sum first over $x_j'$), but the function given by (39) does not satisfy $G(x_j, t_k'; x_j', t_k') = \gamma^0 \delta_{j,j'}$ and can not be considered as a discrete Green function.

However, according to the argument given at the beginning of this section, we can substitute asymptotically the sum over the discrete mass–shell given in (39) by the sum over all the values of the discrete spatial momentum satisfying a energy–momentum relation according to (27), i.e,

$$\sum_{p_j^2=m_r^2} \rightarrow \sum_{\sqrt{p_j^2+m_r^2}=E_k} \quad (40)$$

as $N_\mu \rightarrow \infty$ for $j = 1,\ldots,N_1$, $k = 1,\ldots,N_0$. Here, $m$ is a parameter close to $m_r$.

Therefore, in this limit (39) can be substituted by

$$\tilde{G}(x_j, t_k; x_j', t_k') = \sum_{\sqrt{p_j^2+m_r^2}=E_k} \frac{1}{2E_{kn}} \left[ (q_{jn} + m_r) F^*(x_j', p_{jn}) F^*(t_k', E_{kn}) F(x_j, p_{jn}) 
- (q_{jn} - m_r) F^*(x_j', -p_{jn}) F^*(t_k', -E_{kn}) \right], \quad (41)$$

It is not difficult to show that

$$\tilde{G}(x_j, t_k'; x_j', t_k') = \gamma^0 \delta_{j,j'}, \quad (41)$$

so that we can interpret (41) as the discrete Green function and

$$\Psi_m(x_j, t_k) = \sum_{x_j'} \tilde{G}(x_j, t_k; x_j', t_k') \gamma^0 \Psi_m(x_j', t_k'), \quad t_k' < t_k \quad (42)$$

as the discrete version of the Huygens’ principle. The condition $t_k' \leq t_k$ has been added to define the retarded discrete propagator.

### 4 Propagators on the lattice

Let us denote by $K(x_j, t_k; x_j', t_k')$ the 1 + 1 retarded propagator. Thus

$$K(x_j, t_k; x_j', t_k') = \tilde{G}(x_j, t_k; x_j', t_k'), \quad t_k' < t_k \quad (43)$$

In this section we will obtain some properties of this function. First, we will show the equivalence between (43) and the discrete propagator given in [7]. By using the asymptotic formula (12) we have that

$$\lim_{N_\mu \rightarrow \infty} \frac{F^*(0, p_{jn})}{F^*(0, E_{kn})} = \lim_{N_\mu \rightarrow \infty} \frac{F^*(0, -p_{jn})}{F^*(0, -E_{kn})} = 1.$$
Therefore,

\[ K(x, t; 0, 0) = \sum_{\sqrt{p^2 + m^2} = E} \frac{1}{2E} \left[ (p + m)F^*(t, E)F(x, p) + (p - m)F^*(t, -E)F(x, -p) \right], \quad (44) \]

where \( t > 0 \) and the indexes have been suppressed to clarify the notation. By the aid of (37), taking the real and imaginary part of (11) defined through \( F(\xi, \eta) = C(\xi, \eta) + iS(\xi, \eta) \) and using the parity of these functions with respect to their arguments, we get the components

\[
K_{11}(x, t; 0, 0) = K_{22}(x, t; 0, 0) = \sum_{\sqrt{p^2 + m^2} = E} \left[ -\frac{im}{E}S(t, E)C(x, p) \right],
\]

\[
K_{12}(x, t; 0, 0) = \sum_{\sqrt{p^2 + m^2} = E} \left[ C(t, E)C(x, p) - \frac{p}{E}S(t, E)S(x, p) \right],
\]

\[
K_{21}(x, t; 0, 0) = \sum_{\sqrt{p^2 + m^2} = E} \left[ C(t, E)C(x, p) + \frac{p}{E}S(t, E)S(x, p) \right],
\]

which are the same that those given in [7], obtained here under a more general scheme. As it is shown in [7], this discrete Green function converges to the correct continuum propagator and can be rewritten as the weighted sum over lattice paths of the checkerboard model of Feynman.

As usual, the condition \( t_{k'} \leq t_k \) in (43) can be included as the product of the step and Green functions

\[
K(x_j, t_k; x_{j'}, t_{k'}) = \tilde{\Theta}(t_k, t_{k'}) \tilde{G}(x_j, t_k; x_{j'}, t_{k'}), \quad (45)
\]

where \( \tilde{\Theta} \) is defined by (23). The Feynman propagator can be defined along the same lines. The Hadamard product of two matrices \( A \) and \( B \) [defined as \( (A \circ B)_{jk} = A_{jk}B_{jk} \)] can be used to write (45) in matrix form. Let us define the \((2N) \times (2N)\) matrix

\[
\tilde{\Theta} = 1_{2 \times 2} \otimes \tilde{\Theta} \otimes O,
\]

where \( N = N_0N_1 \) and \( O \) is the identity matrix for the Hadamard product \((O_{jk} = 1, j, k = 1, 2, \ldots, N_1)\). Thus, taking the order for the tensor product used to obtain (41), we have that

\[
(\tilde{\Theta})_{qq'} = 1_{2 \times 2} \tilde{\Theta}_{kk'}.
\]

Now, if \( \tilde{F} \) denotes the matrix of components \((\tilde{F})_{qq'} = F(x_j, p_{j'})/F(t_k, E_{k'})\), the matrix block \( \tilde{G}(x_j, t_k; x_{j'}, t_{k'}) \) given by (41) takes the compact form

\[
(\tilde{G})_{qq'} = \sum_{\sqrt{p_{jq}^2 + m^2} = E_{kq}} \frac{1}{2} \left[ (\tilde{F})_{qq'} \left( \frac{\mathbf{p}_{jq} + m_{x}}{E_{kq}} \right)(\tilde{F})_{q'q} + (\tilde{F})_{qq'}^\dagger \left( \frac{\mathbf{p}_{jq} - m_{x}}{E_{kq}} \right)(\tilde{F})_{q'q}^\dagger \right], \quad (46)
\]

and (45) becomes the product \((\tilde{\Theta})_{qq'}(\tilde{G})_{qq'}\). This means that the retarded propagator is the \((2N) \times (2N)\) matrix

\[
K = \tilde{\Theta} \circ \tilde{G}. \quad (47)
\]
By using the relation
\[(A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D),\]
it is not difficult to see that the components of \(K\) can be written as
\[
(K)_{qq'} = \sum_{\sqrt{p_{j\mu} + m^2} = E_{kn}} \frac{1}{2i}\left[\left((\bar{\Theta} \circ F_0^i) \otimes F_1\right)_{qq'}\left(\mp \frac{p_{j\mu} + m_r}{E_{kl}}\right)(\bar{F}^i)_{q'q} + \left((\bar{\Theta} \circ F_0) \otimes F_1^i\right)_{qq'}\left(\mp \frac{p_{j\mu} - m_r}{E_{kl}}\right)(\bar{F})_{qq'}\right].
\]

Applying \((i\gamma^\mu \otimes D_\mu - m_r \mathbf{1}_{2N})\) to (47) and using (48), (24) and (9) we get
\[
\sum_s (i\gamma^\mu \otimes D_\mu - m_r \mathbf{1}_{2N})_{qs}(K)_{sq'} = i\delta_{qq'}\mathbf{1}_{2 \times 2}
\]
as \(N_\mu \to \infty\). Thus, the retarded propagator defined by (47) becomes asymptotically equal to \(-i\) times the inverse of the Dirac matrix:
\[
K = -i(i\gamma^\mu \otimes D_\mu - m_r \mathbf{1}_{2N})^{-1}.
\]

Similarly, the Feynman propagator \(S\) which propagate the positive frequencies toward positive times and the negative ones backward in time, is defined by
\[
(S)_{qq'} = \sum_{\sqrt{p_{j\mu} + m^2} = E_{kn}} \frac{1}{2i}\left[\left((\bar{\Theta} \circ F_0^i) \otimes F_1\right)_{qq'}\left(\mp \frac{p_{j\mu} + m_r}{E_{kl}}\right)(\bar{F}^i)_{q'q} - \left((\bar{\Theta}^T \circ F_0) \otimes F_1^i\right)_{qq'}\left(\mp \frac{p_{j\mu} - m_r}{E_{kl}}\right)(\bar{F})_{qq'}\right].
\]

Both \(S\) and \(K\) are related as in the continuum case. By using (23) and taking into account that \(\bar{\Xi}\) and \(\bar{\Sigma}\) are skew–symmetric and symmetric matrices respectively, we have that \(\bar{\Theta}^T = -\bar{\Theta} + \bar{\Sigma}\). On the other hand, (21) means that \(\bar{\Sigma} = O\) as \(N_0 \to \infty\), therefore
\[
(S)_{qq'} = -i(K)_{qq'} + \sum_{\sqrt{p_{j\mu} + m^2} = E_{kn}} \frac{i}{2}\left[\left(F^i\right)_{qq'}\left(\mp \frac{p_{j\mu} - m_r}{E_{kl}}\right)(\bar{F})_{qq'}\right].
\]

The second term of the right–hand side of this equation is a solution of \((i\gamma^\mu \otimes D_\mu - m_r \mathbf{1}_{2N})\Psi = 0\). Since \(K\) satisfies (49), \(S\) becomes the inverse matrix of the Dirac operator. We end this section by discussing the form that such a matrix should have for a finite \(N_\mu\). Strictly speaking, there is no inverse for \((i\gamma^\mu \otimes D_\mu - m_r \mathbf{1}_{2N})\) since \(m_r\) is an eigenvalue of \(i\gamma^\mu \otimes D_\mu\). However, according to the approximation procedure given at the beginning of section 3, we can substitute \(m_r\) by a value \(m \neq m_r\) laying close enough to \(m_r\) to yield a nonsingular Dirac operator. Proceeding in this way and taking the representation for the \(\gamma\)–matrices used before, we obtain
\[
S = (i\gamma^\mu \otimes D_\mu - m \mathbf{1}_{2N})^{-1} = \begin{pmatrix} m \Delta & \Delta D_- \\ D_+ \Delta & m D_-^{-1} \Delta D_- \end{pmatrix},
\]

13
where the $N \times N$ matrices $\Delta$ and $D_{\pm}$ are defined as follows

$$
\Delta = (D_{-}D_{+} - m^2 1_N)^{-1}, \quad D_{\pm} = i(D_{0} \pm D_{1}).
$$

(53)

Since $D_{0}$ and $D_{1}$ commutes,

$$
\Delta = (-D_{0}^2 + D_{1}^2 - m^2 1_N)^{-1},
$$

so that $\Delta$ is the inverse of the discrete representation of the Klein–Gordon operator $-\partial_{0}^2 + \partial_{1}^2 - m^2$. It is not difficult to see that $\Delta$, $D_{\mu}$ and $D_{\pm}$ also commute. Thus, (52) takes the more compact form

$$
S = (i\gamma^\mu \otimes D_{\mu} + m 1_N)(1_{2 \times 2} \otimes \Delta),
$$

(54)

which is the discrete form of the well-known relationship between fermionic and scalar propagators. Note that this result does not depend on any asymptotic limit; we only need to take $N_{1} \neq N_{0}$ and $m \neq m_{r}$ to have nonsingular finite matrices.

5 Final remarks

We end this work by pointing out the main differences between the discrete technique for a finite $N_{\mu}$ and its asymptotic formulation given in section 3. First of all, we would like to remark the similarity and convergence of the results of this method to the ones of standard field theory. As it was shown in sections 3 and 4, the convergence of the discrete Fourier transform it is not enough to obtain a propagator as a sum of plane–wave solutions of the equations; it is needed to include other terms laying in the neighborhood of the mass–shell. This is also a requirement in the standard field theory where the propagators are assumed off–shield and Eq. (26) is no longer satisfied. On the other hand, in a physical process such as dispersion of particles, the energy and momentum conservation are consequence of translational invariance which is accounted by Huygens’ principle. In our discretization scheme the translational invariance is lost; however later is recovered when the number of the lattice points goes to infinity, since the Hermite zeros become dense in the real axis.

Finally, we have shown at the end of the previous section how to get a finite matrix form of the propagators without reference to any asymptotic limit. This matrices can be computed easily in numerical calculations of this discrete field theory.

References


