Functional Forms for Lattice Correlators at Small Times

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The analytic form of the lattice quark propagator is used to derive the functional form for short distance mesonic correlators. These are then used to calculate “Continuum Model” Ansätze which comprise of a pole, representing the ground state, plus a contribution for the excited states, coming from the short distance behaviour. These are compared to Monte Carlo data.

1. Introduction

On the lattice, the usual method for studying hadronic physics is to determine the two-point correlation functions of hadronic operators.

\[ G_2^\Gamma(t) = \sum_x \langle 0 | T \{ J_\Gamma(x,t) J_\Gamma(0,0) \} | 0 \rangle \rightarrow e^{-Mt} \quad \text{as} \quad t \rightarrow \infty, \quad (1) \]

where \( M \) is the mass of the ground state, and \( \Gamma \) is channel dependent. The presence of the decaying exponential has led to investigations being focused on the large Euclidean time “tails” of the two-point functions, and data at small times being discarded.

The question was asked, “Is there a way of obtaining information from short times?” The answer was “Yes” - a QCD Sum Rules approach. Lattice QCD allows the determination of correlation functions deep within the non-perturbative region. QCD Sum Rules (QCDSR) can be utilised in the near-perturbative regime where the excited states cannot be ignored. Both are techniques which have been widely employed to deepen the understanding of strongly interacting systems, and have been combined in the work of Leinweber [1,2], and Allton and Capitani [3].

In this study, we extend the work of [1–3] by applying the analysis with a continuum limit expansion of the lattice quark propagator, derived in [4].

The lattice data used is UKQCD generated, and is (quenched) Wilson at \( \beta = 6.0 \) with a volume of \( 16^3 \times 48 \).

2. QCDSR Continuum Model

In this section we introduce the QCDSR-CM and apply it to the continuum quark propagator. The QCDSR-CM replaces a discrete spectrum of excited states with a continuous spectrum of spectral density \( \rho \), and introduces a sharp threshold in the energy scale, \( s_0 \), marking the onset of the continuum.

The basic object studied is the continuum OPE for QCD quark propagator:

\[ S_\text{OPE}_q(x) = \frac{\gamma \cdot x}{2\pi^2 x^4} + \frac{m}{(2\pi x)^2} - \langle \overline{q} q \rangle \frac{2^3}{3} + \cdots, \quad (2) \]

where \( m \) is the quark mass. This is substituted into the (Wick-contacted) 2-point function at zero-momentum:

\[ G_2^\text{OPE}(t) = \int d^3x T \{ S_\text{q}(x)^\text{OPE} \Gamma S_\text{q}(-x)^\text{OPE} \Gamma \}. \quad (3) \]

It is useful to express the time-sliced correlation function, \( G_2(t)^\text{OPE} \), in the spectral representation:

\[ G_2(t)^\text{OPE} = \int_0^\infty \rho(s)^\text{OPE} e^{-st} ds, \quad (4) \]

and the OPE spectral density is calculated by means of an inverse Laplace transform of \( G_2^\text{OPE} \).

We invoke the QCDSR-CM by setting the threshold \( s_0 \) in the energy scale, so that the excited states’ contributions to \( G_2(t) \) is given by only the energies above that scale,

\[ \rho(s) = Z \delta(s - M) + \xi \theta(s - s_0) \rho(s)^\text{OPE}. \quad (5) \]

We have to include the ground state in order to obtain the full correlation function. This is in-
cluded as a δ function in ρ(s) in Eq.(5). So:

\[ G_2(t) = Ze^{-Mt} + \xi \int_0^\infty \rho(s) OPE e^{-st} \mathrm{d}s. \] (6)

There are four parameters in the fitting ansatz:
- \( Z \rightarrow \) normalisation of ground state
- \( M \rightarrow \) ground state mass
- \( \xi \rightarrow \) normalisation of excited states, introduced to allow for lattice distortions [1]
- \( s_o \rightarrow \) continuum threshold.

The OPE expansions for the degenerate mesons and baryons using the continuum quark propagator are listed in [1,3]. The analysis was extended to non-degenerate mesons in [5].

3. Wilson Quark Propagator

In terms of loop-divergent behaviours, a coordinate space formulation of a lattice theory, has the benefit over its momentum analogue of highlighting the short-distance phenomena which are the origin of the poles. In the continuum, the analytic expression for a fermion propagator in position space is well known, Eq.(2). On the lattice however, the standard representation involves integrals over Bessel functions, and proved difficult to analyse in the continuum limit. Lüscher and Weisz successfully analysed the massless propagator in 4-dimensions [6]. While in [7] the mass dependence of the propagator at \( x = 0 \) was derived. However, in [4], the asymptotic expansion for the modified Bessel function of the first kind, \( I_n \), which appears in the expression for the lattice propagator, (using Schwinger’s integral representation),

\[ S_q(x)_{WF} = \sum_{i=even} \left\{ m - ia^{-1} \sum_{\mu=1}^4 \gamma_\mu \sin \left[ a(-i\partial_\mu + k_\mu^i) \right] \right\} \]

was calculated, leading to the following expression for the Wilson lattice propagator:

\[ (4\pi)^2 S_q(x)_{WF} = \frac{4m^2}{(x^2)^2} \left[ 1 + \frac{ram}{2} K_1 \left[ m(x^2)^{\frac{1}{2}} \right] \right] \]

where \( K_1 \) and \( K_2 \) are modified Bessel functions of the second kind. This expression was then used in a QCDSR-CM style approach following the procedures outlined in section 2.

![Figure 1. Effective mass plot for the 2-exponential and \( O(a) \) QCDSR-CM fits for the vector channel at \( \kappa = 0.1530 \).](image-url)
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Fit Z M $\chi^2$/d.o.f

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<th>Type</th>
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Table 1
Values of Z, M and $\chi^2$ for the fitting ansatz.

4. Wilson Correlators

In this section, the work of [1–3] and [4] is combined by applying a QCDSR-CM approach to the continuum limit expansion of the Wilson lattice propagator, i.e. we use $S_q^{OPE} = S_q^{WF}$ in Eq.(3).

To obtain a closed expression for $G_2$ we first expand the modified Bessel functions for short distances and then perform the spatial integral. This procedure was performed for all mesonic channels, with general (non-degenerate) constituent quarks, and for the momenta, $p = 0$ and $p = (1,1,1)$. We obtain, e.g., for the zero-momentum pseudoscalar channel the following expression:

$$G_2^{PS}(t)^{OPE} = -\frac{3}{4\pi^2 t^3}(1 + 2\text{ram}) + \frac{3m^2}{4\pi^2 t}(1 - \text{ram}).$$

Values for the $Z$ and $M$ fitting parameters together with the $\chi^2$ for the pseudoscalar and vector channels are given in Table 1. The QCDSR-CM fits for the Wilson correlators are labelled by $\mathcal{O}(a)$. The fitting window for the 1-exp fit was $t = [1/2, 20]$ and for the 2-exp and $\mathcal{O}(a)$ fits was $t = [2, 20]$. Note that the $\mathcal{O}(a)$ fits perform better than the 2-exp fit in every case, and that the improvement is more significant for the vector channel particularly as the mass increases.

5. Conclusions

In this work we have derived lattice mesonic correlation functions for the Wilson action, including $\mathcal{O}(a)$ effects. Our Ansatz fit lattice Monte Carlo data better than conventional “2-exponential” fits. Future work includes fitting of heavy mesons, and deriving analogous formulae for the Clover action.

6. Acknowledgements

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REFERENCES

4. B.M.S. Paladini and J. Sexton, hep-lat/9805021, B.M.S. Paladini, PhD Thesis, University of Dublin Trinity College