Observables of String Field Theory

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Abstract

We study gauge invariant operators of open string field theory and find a precise correspondence with on-shell closed strings. We provide a detailed proof of the gauge invariance of the operators and a heuristic interpretation of their correlation functions in terms of on-shell scattering amplitudes of closed strings. We also comment on the implications of these operators to vacuum string field theory.
1 Introduction

There are two kinds of physically meaningful quantities one can compute in a gauge theory. One is on-shell scattering amplitude (the S-matrix), and the other is off-shell correlation functions of gauge invariant quantities. In perturbative string theory, however, we only know how to compute on-shell quantities. String field theory provides an interesting opportunity to explore off-shell issues along the line of standard gauge field theories. In this framework, we have an action which is invariant under a certain gauge transformation. It is then natural to raise the questions: what are the operators which are invariant under this gauge transformation, and what is the meaning of the correlation functions of these operators?

Different but equally interesting is the question: where are the closed strings in open string field theory? After all, open strings are not consistent by themselves since they can self interact to form a closed string. Indeed, shortly after Witten’s paper [?] it was shown that at one loop there are poles which can be related to closed strings [?]. Thus closed strings exist as a virtual state in the internal propagator of a generic string diagram. Unitarity then implies that they should also appear as asymptotic states, but how does one incorporate closed strings in a Feynman diagram computation of open string field theory?

One way to introduce closed strings is to include them from the beginning by considering an open-closed string field theory [?, ?]. This is a theory where both open and closed string fields are defined off-shell. Such a theory, however, is difficult to interpret especially in light of new insights provided by the AdS/CFT correspondence [?]. There, the on-shell observables of closed string theory in the bulk correspond to off-shell observables of the field theory on the boundary [?, ?]. It is however very difficult to see how off-shell closed string observables can have a dual boundary description. This implies that string field theory of closed strings should not exist, at least in an anti de Sitter background. What the AdS/CFT correspondence really teaches us is that the two questions in string field theory, one regarding the off-shell observables and the other regarding the status of closed strings, are closely related. The goal of this article is to explore this correspondence.

In fact, similar ideas can be exploited to derive the gauge invariant open Wilson lines of non-commutative gauge theories from string theory [?]. There, the way one obtains the straight open Wilson line of [?,?] is via a summation of all diagrams with one closed string and arbitrary number of open strings. In the appropriate decoupling limit, this infinite sum of higher order disk amplitudes resummed precisely to the path ordered exponential of gauge fields forming an open Wilson line.

One can imagine repeating the analysis of [?] and resum infinitely many disk diagrams to derive something resembling a Wilson line of open string field theory. It is natural to
expect that this will be far more complicated a task for a theory as complicated as string
field theory. Contrary to the expectation, the gauge invariant combination of open string
field which couples naturally to the closed string is far simpler. We find that a term linear in
open string field derived from a disk amplitude with one open and one closed string vertices
is invariant by itself. Formally, these operators define a set of Chern-Simons one-forms of
open string field theory.

The set of gauge invariant observables of string field theory therefore consist of the S-
matrix of external open string fields, correlation function of gauge invariant operators, and
combinations thereof. Since the gauge invariant operators of string field theory are in one to
one correspondence with the on-shell closed string vertex operator, these observables have
a natural interpretation as the S-matrix of both open and closed strings. Perhaps a useful
point of view to adopt is that the open strings are the fundamental degrees of freedom of
the theory, and that the closed strings are merely an artifact of insertions of off-shell gauge
invariant operators.

The paper is organized as follows. In section 2, we provide a schematic construction of
the gauge invariant operators and provide a formal proof that they are gauge invariant. We
also present a formal relation between the off-shell correlations function of the operators and
the on-shell scattering amplitudes of the closed strings. In section 3, we provide a rigorous
construction and a proof of gauge invariance by formulating these operators explicitly in
terms of world sheet oscillators. In section 4, we consider the issue of gauge invariant
observables for the vacuum string field theory.

2 Formal construction

In this section we construct the gauge invariant operators, prove that they are gauge invari-
ant, and interpret their correlation functions using formal arguments.

2.1 Gauge invariant operators

The cubic string field theory is defined by the action

$$ S = \int A \ast QA + \frac{2}{3} A \ast A \ast A , $$

(2.1)

which is invariant under the gauge transformation

$$ \delta A = QA + [A, A] . $$

(2.2)
Figure 1: The closed string is inserted at $\tau = 0$ at some $\sigma$. The open string is inserted as usual at $\tau = -\infty$. The dashed line represents the midpoint.

The fact that the action (2.1) resembles Chern-Simons theory is a useful hint in constructing gauge invariant operators, since one can consider other Chern-Simons invariants as long as we ignore the issue of ghost numbers. For example, the Chern-Simons one-form

$$\int A,$$ (2.3)

is also invariant under (2.2) and can be interpreted formally as a Wilson line around some one-cycle.

In string field theory, however, one must take the ghost number into account. We use the convention that the gauge parameter $\Lambda$ carries ghost number zero, and the physical open string field $A$ carries ghost number one. With that convention, the integral is non-vanishing only when the integrand has ghost number three. As a result, the Chern-Simons one-form (2.3) is missing two units of ghost number and trivially vanishes.

To make proper sense out of (2.3), we need to introduce a new ingredient which supplies two units of ghost number. A natural candidate is the closed string vertex operator

$$V(\sigma_+, \sigma_-) = c_+(\sigma_+)c_-(\sigma_-)O,$$ (2.4)

where $O$ is a conformal primary operator of dimension (1,1) with ghost number zero.\(^1\) One can then consider a quantity

$$O_V = \int V(\sigma_+, \sigma_-)A.$$ (2.5)

From the open string field theory point of view, $V$ is simply an operator which acts on a string field, much like the BRST or any other operator. For each closed string vertex there is such an operator. Thus eq.(2.5) is a perfectly well defined quantity, in the sense that it has the correct ghost number and has a concrete oscillator realization. One can think of (2.5) as a Chern-Simons one-form where the role of the “one cycle” is played by the closed string vertex operators. In order to respect the time-ordering in the operator realization of the conformal field theory, we set $\tau = 0$ so that $\sigma_\pm = \pm i \sigma$. We therefore write the vertex operator simply as $V(\sigma)$. In terms of world sheet, this amounts to inserting a vertex operator

\(^1\)The dilaton is a special case where $O$ has ghost number zero but is not a primary conformal field [?].
Figure 2: World sheet description of the gauge invariant operator (2.6) in the coordinate where the metric is flat everywhere but at the insertion of the closed string vertex.

at coordinate $\sigma$ at time $\tau = 0$ on a state with quantum number of $A$ which has propagated from $\tau = -\infty$, as illustrated in figure 1.

Let us show that (2.5) with the on-shell closed string vertex operator inserted at the midpoint $\sigma = \frac{\pi}{2}$

$$O_V = \int V(\frac{\pi}{2})A,$$  \hspace{1cm} (2.6)

defines a gauge invariant operator of string field theory. The integral identifies the left and the right halves of a string, making the world sheet take on a geometry of the form illustrated in figure 2. These operators where first studied by Shapiro and Thorn [2, 3].

In order to demonstrate the gauge invariance of (2.6), let us consider the linear and the non-linear contribution to gauge transformation (2.2) separately. At the linear level, \(^2\) (2.6) transforms according to

$$\delta O_V = \int V(\sigma)Q\Lambda = \int QV(\sigma)\Lambda = 0,$$  \hspace{1cm} (2.7)

where we used the fact that the closed string vertex operator $V(\sigma)$ commutes with $Q$ provided that it is on-shell. Note that this part of the argument does not depend on $\sigma$. At the non-linear level, (2.6) transforms as

$$\delta O_V = \int V(\frac{\pi}{2})(A*\Lambda) - V(\frac{\pi}{2})(\Lambda*A).$$  \hspace{1cm} (2.8)

To make sense out of this expression, we need to understand what it means to act with an operator on a string field which is a $*$-product of a pair of string fields. In the coordinate where the metric is flat, the world sheet of the $*$-product of $A$ and $\Lambda$ looks like figure 3. As long as $\sigma$ is not exactly $\frac{\pi}{2}$, there is a neighborhood whose metric is identical to the metric of an ordinary strip. Therefore, there is a natural coordinate $\sigma_+$ and $\sigma_-$, as well as a meaningful notion of $\partial_+$ and $\partial_-$. The midpoint $\sigma = \frac{\pi}{2}$ appears to be a potentially singular point in this picture. Let us therefore consider inserting $V$ at $\sigma = \frac{\pi}{2} - \epsilon$. For concreteness, we take $\epsilon$ to be positive. Then, since the left side of $A*B$ is the left side of $A$ we get

$$V(\sigma)(A*\Lambda) = (V(\sigma)A)*\Lambda, \hspace{1cm} \sigma < \frac{\pi}{2}.$$  \hspace{1cm} (2.9)

Thus, eq.(2.8) reads

\(^2\)Gauge invariance of (2.6) at the linear level was demonstrated in [2, 3].
Figure 3: World sheet description of closed string vertex operator $V$ acting on a state which is a product of the form $A \ast \Lambda$. The geometry is such that the metric is flat away from the midpoint.

$$\delta O_V = \int (V(\sigma)A) \ast \Lambda - \int (V(\sigma)A) \ast A = \int (V(\sigma)A) \ast \Lambda - \int A \ast (V(\sigma)\Lambda).$$

(2.10)

Recalling that the string field theory integral identifies the left side of the string with the right side of the string, one has

$$\int A \ast V(\sigma)\Lambda = \int V(\pi - \sigma)A \ast \Lambda,$$

(2.11)

so that

$$\delta O_V = \int ((V(\sigma) - V(\pi - \sigma))A) \ast \Lambda,$$

(2.12)

which vanishes for $\sigma = \frac{\pi}{2}$. This shows that, much like in non-Abelian gauge theories, gauge invariance at the non-linear level imposes non-trivial constraints on objects which are invariant at the linear level.

We have therefore succeeded in demonstrating that (2.6) is gauge invariant both at the linear and the non-linear level provided that $V$ is an on-shell closed string vertex operator. However, this proof involved several formal manipulations which are quite subtle. For example, gauge invariance requires that $V$ be inserted at the midpoint even though the proof of gauge invariance works only if we approach the midpoint as a limit. Since the geometry of the world sheet illustrated in figure 3 is singular precisely at the midpoint, care must be taken to make sure that the limit exists. Closely related issue is the fact that operations such as integration and $\ast$-multiplication frequently involve insertion of ghost number at the midpoint. These insertions are often sources of anomalies whose cancellation is rather delicate. In order to address these issues, we will formulate and study the gauge transformations of (2.6) explicitly in terms of oscillators in section 3.

### 2.2 Correlation functions

In the previous subsection, we constructed a set of expressions of the form

$$O_V = \langle I|V(\frac{\pi}{2})|A \rangle,$$

(2.13)
Figure 4: Open string field theory description of one closed two open string disk amplitude. The closed string is described by a gauge invariant operator which propagate to the interaction point.

where $V(\sigma)$ is the closed string vertex operator

$$V(\sigma) = c_+ (\sigma) c_- (\sigma) O(\sigma), \quad (2.14)$$

and provided a formal proof of their gauge invariance. They are therefore the gauge invariant operators of string field theory. In this subsection, we will describe the interpretation of the correlation functions of these operators.

By construction, the operators (2.13) are in one to one correspondence with the closed string vertex operators. This suggests that the correlation functions of operators of the form (2.13) should be interpreted as the scattering amplitude of closed strings through world sheet with boundaries. Consider for example a two point function

$$\langle O_1 O_2 \rangle = \langle \int V_1 A \int V_2 A \rangle. \quad (2.15)$$

To leading order in perturbation theory, this amplitude is evaluated by contracting the open string field $A$'s using the propagator. In Feynman-Siegel gauge, this becomes

$$\langle O_1 O_2 \rangle = \langle \tilde{I} | O_1 | b_0 \int_{0}^{\infty} d\tau \exp(-\tau L_0) | O_2 | \tilde{I} \rangle, \quad (2.16)$$

which can be interpreted as the scattering amplitude of two closed strings on a disk similar to the ones considered in [?]. The $\tau$ integral is the integral over the moduli-space on the world sheet. We see that this space is one dimensional, appropriate for the amplitude of this type. It would be interesting to explicitly reproduce the result of [?] in full detail starting from (2.16).

Another interesting observable to consider is the amplitude of the type illustrated in figure 4, which is obtained by contracting (2.13) with the $A^3$ term in the string field theory action. This is analogous to the calculation of gluon scattering in ordinary gauge theories.
when the action is deformed by gauge invariant operators, say, $\text{Tr} F^4(p)$. The natural interpretation of this amplitude is the scattering of one closed and two open strings on a disk [?]. It would again be interesting to reproduce this result from string field theory. The dimension of moduli space is certainly in agreement with the expectation from string field theory.

The most striking feature of the operator (2.13) is the fact that it is linear in open string field. This is in marked contrast to the case of open Wilson lines in non-commutative gauge theories where terms higher order in non-commutative gauge fields were necessary to ensure gauge invariance at the non-linear level. The remarkably simple structure of (2.13) can be understood in light of the work by Zwiebach [?] who showed that a theory with an action of the form

$$S = \int A * QA + \frac{2}{3} A * A * A + \int A \Psi,$$

(2.17)

where $\Psi$ is a closed string field, can be understood as a special limit of open closed string field theory provided that $\Psi$ is on-shell. In particular, this theory covers the full moduli-space of the scattering amplitudes of open and closed strings with a boundary. The $A\Psi$ term is to be interpreted as (2.13). The fact that all such amplitudes can be generated by gluing interaction vertices of (2.17) through propagators suggests that (2.13) is the complete description of the coupling of closed strings.

These arguments appear to suggest that gauge invariant off-shell observables encode the closed string on-shell physics, very much along the lines of AdS/CFT correspondence. In other words, the standard perturbative on-shell scattering amplitudes involving both the open and the closed strings capture the complete set of the on-shell and the off-shell observables of string field theory. This shift in perspective may seem at first as nothing more than a complicated reformulation of well known perturbative string physics in terms of open string field theory. Nonetheless, this point of view may provide the critical insight in formulating string theory at the non-perturbative level. One important consequence of this new perspective is the fact that even though open strings are taken off-shell, the closed strings appear only on-shell. So far, these arguments have been presented only for amplitudes involving at least one boundary. It would be very interesting to see if viewing the closed strings as off-shell gauge invariant operators proves to be a useful point of view in thinking about purely closed string processes (such as the Virasoro-Shapiro amplitudes) in the framework of open string field theory.

### 3 Explicit oscillator representation

In the previous section, we provided an attractive picture of a correspondence between gauge invariant operators of open string field theory and the closed strings on shell. However, some
of the arguments used in the previous section to construct and prove the gauge invariance of these operators were somewhat formal and could suffer from number of technical problems when formulated more explicitly. There are in fact two potential sources of difficulties having to do with the fact that the closed string vertex operator is inserted at the midpoint of the open string world sheet. One is the fact that the closed string vertex operator inserts a ghost number at the midpoint. This is dangerous since cancellation of ghost related anomalies at the midpoint is generally subtle. The other is the fact that the insertion of closed string vertex operator at the midpoint of a product of two string field, as we did in establishing the gauge invariance at the non-linear level, is subtle. In the previous section, we defined this procedure by inserting the operator in such a way that it approaches the midpoint as a limit. This limit could potentially suffer from anomalies. In order to address these concerns, it is worthwhile to study these operators more explicitly.

Explicit computations in string field theory is performed using the oscillator formalism of \[?, ?\]. In this formalism, a string field is an element of the open string Hilbert space

\[ A \leftrightarrow |A\rangle, \tag{3.1} \]

obtained by acting on the vacuum with operators in the mode expansion of the string coordinates

\[
\begin{align*}
X_{\pm}^\mu(\sigma_{\pm}) & = \frac{1}{2}x_{0}^{\mu} + \frac{i}{2} \sqrt{2\alpha'} \sum_{m=1}^{\infty} \left( \frac{1}{\sqrt{m}}a_{m}^{\mu}e^{m(\sigma_{\pm})} - \frac{1}{\sqrt{m}}a_{-m}^{\mu}e^{-m(\sigma_{\pm})} \right), \\
x_{0} & = \frac{i}{2} \sqrt{2\alpha'}(a_{0} - a_{0}^{\dagger}), \\
b_{\pm}(\sigma_{\pm}) & = \sum_{m=-\infty}^{\infty} b_{m}e^{im(\sigma_{\pm})}, \\
c_{\pm}(\sigma_{\pm}) & = \sum_{m=-\infty}^{\infty} c_{m}e^{im(\sigma_{\pm})}. \tag{3.2}
\end{align*}
\]

Formal operations such as integration, conjugation, and *-multiplication are defined in terms of special states \(|I\rangle, |V_{2}\rangle, |V_{3}\rangle\) so that

\[
\int A = \langle I |A\rangle, \quad 1\langle A\rangle = 12\langle V_{2}|A\rangle_{2}, \quad |A \ast B\rangle_{3} = 1\langle A\rangle_{2}2\langle B|V_{3}\rangle_{123}. \tag{3.3}
\]

The subscripts on the brackets label the open string Hilbert spaces when more than one are involved. Explicit expression for these states can be found in \[?, ?\]. Consider for example the identity element \(|I\rangle\) defined as

\[ |I\rangle = b_{+}(\frac{\pi}{2})b_{-}(\frac{\pi}{2})|\bar{I}\rangle, \tag{3.4} \]

where

\[ |\bar{I}\rangle = \exp \left( -\frac{1}{2} \sum_{n \geq 0} (-1)^{n}a_{n}^{\dagger}a_{n}^{\dagger} + \sum_{n \geq 1} c_{-n}b_{-n} \right) c_{0}c_{1}|0\rangle, \tag{3.5} \]
and $|0\rangle$ is the vacuum invariant with respect to $SL(2, R)$ subgroup of the Virasoro algebra. $|\tilde{I}\rangle$ satisfies the relations

\[
\begin{align*}
(a_m + (-1)^m a_{-m})|\tilde{I}\rangle &= 0, \\
(c_m + (-1)^m c_{-m})|\tilde{I}\rangle &= 0, \\
(b_m - (-1)^m b_{-m})|\tilde{I}\rangle &= 0,
\end{align*}
\]  

(3.6)

whereas $|I\rangle$ violates (3.6) for the $c$ fields due to the presence of extra factors $b_+ (\frac{\pi}{2}) b_- (\frac{\pi}{2})$. With these extra factor of $b$ fields, $|I\rangle$ is BRST invariant

\[
Q|I\rangle = 0.
\]  

(3.7)

Using $\langle I|$ we can evaluate

\[
\int VA = \langle I|V(\frac{\pi}{2})|A\rangle,
\]  

(3.8)

explicitly in terms of the oscillators. Formal argument for gauge invariance at the linearized level follows immediately from the fact that $[Q, V] = 0$ and $\langle I|Q = 0$. There are some subtleties, however, that needs to be addressed before this expression can be made completely well defined. To be concrete, let us take $V$ to be the vertex operator of the closed string tachyon

\[
V(\sigma_+, \sigma_-) = c_+(\sigma_+)c_-(\sigma_-) : e^{ikx_+(\frac{\sigma_+}{2})} : : e^{ikx_-(\frac{\sigma_-}{2})} :.
\]  

(3.9)

One of the subtleties arises from the fact that the insertion of $c_+(\frac{\pi}{2})c_-(\frac{\pi}{2})$ of $V$ and $b_+(\frac{\pi}{2})b_-(\frac{\pi}{2})$ of $\langle I|$ collides on the world sheet. One way to avoid this subtlety is to simply interpret

\[
\langle \tilde{I}| b_+(\frac{\pi}{2})b_-(\frac{\pi}{2})c_+(\frac{\pi}{2})c_-(\frac{\pi}{2})O(\frac{\pi}{2}) = \mathcal{N}\langle \tilde{I}|O(\frac{\pi}{2}),
\]  

(3.10)

where $\mathcal{N} = \delta^2(0)$, is an infinite factor obtained from evaluating the commutator of conjugate fields at the same point. Since this is just an overall multiplicative factor, we might as well not include it in our definition of the gauge invariant operator. In other words, we define a quantity of the form

\[
O = \langle \tilde{I}|O(\frac{\pi}{2})|A\rangle.
\]  

(3.11)

This quantity is well defined and can be evaluated for any given closed string vertex operator. For example, using the closed string tachyon, this evaluates to

\[
O_T = \langle \tilde{I}|e^{ikx(\frac{\pi}{2})/2}|A\rangle
\]

\[
= \langle 0|c_0c_{-1} \exp \left(-\frac{1}{2} \sum_{n \geq 0} (-1)^n a_n a_n + \sum_{n \geq 1} c_n b_n - \sqrt{2\alpha'}k \left(\frac{a_0}{2} + \sum_{n \geq 1} \sqrt{\frac{1}{n}}(-1)^n a_{2n}\right)\right)|A\rangle
\]  

(3.12)

where $X = X_+ + X_-$. This expression is in a complete agreement with the open-closed interaction vertex of Shapiro and Thorn [?, ?] for the closed string tachyon.
We need to verify that this quantity is invariant with respect to gauge transformation at
the linear and the non-linear level. At the linear level, we need to verify that

\[ Q e^{ikX(\pi/2)/2} |\tilde{I}\rangle = Q \exp \left( \sqrt{2\alpha'k} \left( \frac{a_0}{2} + \sum_{n \geq 1} \sqrt{\frac{1}{n}} (-1)^n a_{2n} \right) \right) |\tilde{I}\rangle = 0. \] (3.13)

This follows from the fact that

\[ Q|\tilde{I}\rangle = -16 \sum_{n \geq 1} n(-1)^n c_{2n}|\tilde{I}\rangle, \] (3.14)

and

\[ [Q, \exp \left( \sqrt{2\alpha'k} \left( \frac{a_0}{2} + \sum_{n \geq 1} \sqrt{\frac{1}{n}} (-1)^n a_{2n} \right) \right)] |\tilde{I}\rangle = \exp \left( \sqrt{2\alpha'k} \left( \frac{a_0}{2} + \sum_{n \geq 1} \sqrt{\frac{1}{n}} (-1)^n a_{2n} \right) \right) 2(2\alpha'k^2) \sum_{n \geq 1} n(-1)^n c_{2n}|\tilde{I}\rangle, \] (3.15)

so that they vanish for

\[ 2\alpha'k^2 = 8 \] (3.16)

which is precisely the on-shell condition for the closed string tachyon. This establishes the
gauge invariance of (3.11) at the linear level.

There is a rather subtle point here. One could very easily reach a different conclusion by
performing the calculation in a different order. Namely

\[ Q e^{ikX(\pi/2)/2} |\tilde{I}\rangle = [Q, e^{ikX(\pi/2)/2}] |\tilde{I}\rangle + e^{ikX(\pi/2)/2} Q |\tilde{I}\rangle \] (3.17)

\[ = (2\alpha'k^2 - 16) \sum_{n \geq 1} n(-1)^n c_{2n}|\tilde{I}\rangle. \]

This leads to the condition \( 2\alpha'k^2 = 16 \) for BRST invariance which is off by a factor of
two compared to the on-shell condition of the closed string tachyons. The origin of this
discrepancy can be traced to an anomaly in associativity\(^3\)

\[ Q(e^{ikX(\pi/2)/2} |\tilde{I}\rangle) \neq (Qe^{ikX(\pi/2)/2}) |\tilde{I}\rangle. \] (3.18)

Since we are interested in understanding the BRST invariance of a state, \( |T, p\rangle = e^{ikX(\pi/2)/2} |\tilde{I}\rangle \),
we should show that the left hand side, and not the right hand side, of (3.18) is zero. Computing
\( e^{ikX(\pi/2)/2} |\tilde{I}\rangle \) \textit{before} acting with \( Q \) essentially amounts to normal ordering this expression so
that \( Q e^{ikX(\pi/2)/2} |\tilde{I}\rangle \) becomes (3.15). Put differently, understanding that (3.8) or (3.11) should
be normal ordered defines \( |T, p\rangle \) unambiguously, and the condition for BRST invariance of
\( |T, p\rangle \) becomes the on-shell condition of the closed string tachyon.

\(^3\)Associativity anomaly in string field theory was originally formulated in [\textsuperscript{10}].
The other main subtleties are in the proof of gauge invariance at the non-linear level. This is equivalent to showing
\[ \langle \tilde{I} | \mathcal{O} | A \ast \Lambda \rangle - \langle \tilde{I} | \mathcal{O} | A \ast \Lambda \rangle = 0 , \] (3.19)
or more explicitly,
\[ 123 \langle V_3 | \mathcal{O} | \tilde{I} \rangle_3 = 123 \langle V_3 | \mathcal{O} | \tilde{I} \rangle_3 = 0 . \] (3.20)
Where \(|V_3\rangle\) is the three string vertex whose details will be given shortly. The subtleties arose from the ambiguity in inserting an operator at the midpoint which is precisely the point where the local coordinate of the glued world sheet of the form illustrated in figure 3 is singular. One can again regulate this potential singularity by “point splitting”
\[ O = \lim_{\epsilon \to 0} \langle \tilde{I} | \mathcal{O} (\frac{\pi}{2} - \epsilon) | A \rangle . \] (3.21)
For this quantity, establishing the gauge invariance at the non-linear level amounts to showing that
\[ \lim_{\epsilon \to 0} 123 \langle V_3 | \mathcal{O} (\frac{\pi}{2} - \epsilon) | \tilde{I} \rangle_3 = 0 . \] (3.22)
This quantity, establishing the gauge invariance at the non-linear level amounts to showing that
\[ \lim_{\epsilon \to 0} 3 \langle \tilde{I} | \mathcal{O} (\frac{\pi}{2} - \epsilon) | V_3 \rangle = 0 . \] (3.23)

4 The way we prove this is by showing that the matter part is symmetric while the ghost part is anti-symmetric.

Let us start with the matter part. For simplicity we consider the closed string tachyon. Using the mode expansion and eq.(3.6), one finds
\[ \langle \tilde{I} | e^{i k X(\sigma)/2} = \langle \tilde{I} | \exp \left[ \frac{i}{2} \sqrt{2 \alpha' k} \beta_n(\sigma)(1 + C)_{nm} a_m \right] , \] (3.24)

4To see this note that
\[ 123 \langle V_3 | \mathcal{O} | \tilde{I} \rangle_3 = 14 \langle V_2 |_{25} \langle V_2 |_{3} \langle \tilde{I} | \mathcal{O} | V_3 \rangle_{345} . \] Thus under the exchange 1 ↔ 2,
\[ 132 \langle V_3 | \mathcal{O} | \tilde{I} \rangle_3 = 24 \langle V_2 |_{15} \langle V_2 |_{3} \langle \tilde{I} | \mathcal{O} | V_3 \rangle_{345} . \]
Combining this with [7]
\[ \langle V_2 | = 12 (|c_+ \rangle_1 \langle 0 | c_-) \exp \left[ -a_+^2 C_{nm} b_m^2 - (c_+^2 C_{nm} b_m^2 + c_+^2 C_{nm} b_m^2) \right] (c_0^2 + c_0^2) \]
implies that
\[ 24 \langle V_2 |_{15} \langle V_2 |_{3} \langle \tilde{I} | \mathcal{O} | V_3 \rangle_{345} = -15 \langle V_2 |_{24} \langle V_2 |_{3} \langle \tilde{I} | \mathcal{O} | V_3 \rangle_{345} = -14 \langle V_2 |_{25} \langle V_2 |_{3} \langle \tilde{I} | \mathcal{O} | V_3 \rangle_{354} . \] Therefore, if 123 \langle V_3 | \mathcal{O} | \tilde{I} \rangle_3 is even under 1 ↔ 2, then 3 \langle \tilde{I} | \mathcal{O} | V_3 \rangle_{123} must be odd.
where
\[ \beta_0(\sigma) = \frac{i}{2}, \quad \beta_n(\sigma) = i\sqrt{\frac{1}{m}}\cos(m\sigma), \quad (3.25) \]
and \( C \) is defined like in [?]
\[ C_{nm} = \delta_{nm}(-)^n. \quad (3.26) \]
Without loss of generality we can assume that \( \sigma \leq \frac{\pi}{2} \). Then, since \( C_{nm}\beta_m(\sigma) = \beta_n(\pi - \sigma) \), we have
\[ \langle \tilde{I} | e^{i k X(\sigma)/2} = \langle \tilde{I} | \exp \left[ \frac{i}{2} \sqrt{2\alpha'k} (\beta_n(\sigma) + \beta_n(\pi - \sigma))a_m \right]. \quad (3.27) \]
To compute (3.22) one uses the definition of the three-string vertex
\[ |V_3\rangle = \exp \left[ - \sum_{n,m \geq 0} \frac{1}{2} a^r_n V_{nm} a^s_m \right] |0\rangle. \quad (3.28) \]
Here \( n \) and \( m \) run over the stringy modes while \( r \) and \( s \) specify the string field state. That is, \( r, s = 1, 2, 3 \). We would like to study the behavior under \( 1 \leftrightarrow 2 \). Thus it is convenient to separate the 3 sector
\[ |V_3\rangle = \exp \left[ \frac{1}{2}(a^{t1} a^{t2}) \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} a^{t1} \\ a^{t2} \end{pmatrix} - (a^{t1}, a^{t2}) \begin{pmatrix} V_{13} \\ V_{23} \end{pmatrix} a^{t3} - \frac{1}{2} a^{t3} V_{33} a^{t3} \right] |0\rangle_{123}. \quad (3.29) \]
With the help of standard relation
\[ \langle 0 | \exp(\lambda_i a_i - \frac{1}{2} P_{ij} a_i a_j) \exp(\mu_i a_i^\dagger - \frac{1}{2} Q_{ij} a_i^\dagger a_j^\dagger)|0\rangle = \det(K)^{-\frac{1}{2}} \exp(\mu K^{-1} - \frac{1}{2} \lambda Q K^{-1} - \frac{1}{2} \mu K^{-1} P \mu), \quad K = 1 - PQ, \quad (3.30) \]
and using some identities [?]
\[ M_{rs} = CV_{rs}, \]
\[ M_{11} + M_{12} + M_{21} = 1, \quad M_{12} M_{21} = M_{11}^2 - M_{11}, \quad (3.31) \]
\[ M_{12}^2 + M_{21}^2 = 1 - M_{11}^2, \quad M_{12}^3 + M_{21}^3 = (1 - M_{11})^2 (1 + 2 M_{11}), \]
we find\(^5\)
\[ 3\langle \tilde{I} | e^{i k X(\sigma)/2} |V_3\rangle_{123} = \exp \left[ - (a^{t1} a^{t2}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \frac{M_{12} - M_{21}}{1 - M_{11}} \right] \frac{1}{2} \sqrt{2\alpha'k} (\beta(\sigma) + \beta(\pi - \sigma)) \]
\[ - \frac{1}{2}(a^{t1}, a^{t2}) \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} a^{t1} \\ a^{t2} \end{pmatrix} |0\rangle_{12}. \quad (3.32) \]
\(^5\)We have dropped the overall determinant factor of \( \det(1 - M_{11})^{-1} \) since it does not affect the proof of gauge invariance.
After some manipulations which are explained in the appendix, one finds that
\[
\frac{M_{12} - M_{21}}{1 - M_{11}} (\beta(\sigma) + \beta(\pi - \sigma)) = \beta(\sigma) - \beta(\pi - \sigma), \quad \sigma < \frac{\pi}{2}.
\] (3.33)

Therefore,
\[
3\langle \hat{I} | e^{ikX(\sigma)/2} | V_3 \rangle_{123} = \exp \left[ -\frac{1}{2} \sqrt{2a'} k(a'^1_1 \beta(\sigma) + a'^2_2 \beta(\pi - \sigma)) - a'^1_1 C a'^2_2 \right] |0\rangle_{12} ,
\] (3.34)
which is indeed symmetric in 1 and 2 in the limit $\sigma \to \frac{\pi}{2}$.

The calculation for the ghost part goes along similar lines. We would like to compute
\[
3\langle \hat{I} | V_3 \rangle_{123},
\] (3.35)
and show that it is antisymmetric with respect to the exchange $1 \leftrightarrow 2$. We begin with the explicit expressions for $\langle \hat{I} \rangle$ and $|V_3\rangle$ in terms of oscillators [? , ?]
\[
\langle \hat{I} \rangle = \langle V_2 | \hat{I} \rangle = \langle 0 | c_{-1} c_0 \exp [- \sum_{n,m \geq 1} c_n c_{nm} b_m],
\]
\[
|V_3\rangle_{123} = \exp \left[ -\sum_{n \geq 1} \sum_{m \geq 0} c^r_{-n} \tilde{V}^r_{nm} b^r_{-m} \right] (c_0 c_1 |0\rangle)_{123}.
\] (3.36)

In order to contract along $b^3$ and $c^3$, we separate the 3 sector as we did in the matter part. We also analyze the zero mode and the non-zero mode parts of $|V_3\rangle$ separately.

\[
|V_3\rangle = \exp \left[ -\sum_{n,m \geq 1} \left\{ c^3_{-n} \tilde{V}^3_{nm} b_{-m} + c^3_{-n} (\tilde{V}^{31}_{nm} \tilde{V}^{32}_{nm}) \begin{pmatrix} b^1_{-m} \\ b^2_{-m} \end{pmatrix} \\
+ (c^1_{-n} c^2_{-n}) \begin{pmatrix} \tilde{V}^{13}_{nm} \\ \tilde{V}^{23}_{nm} \end{pmatrix} b^3_{-m} + (c^1_{-n} c^2_{-n}) \begin{pmatrix} \tilde{V}^{11}_{nm} \\ \tilde{V}^{21}_{nm} \end{pmatrix} \begin{pmatrix} \tilde{V}^{12}_{nm} \\ \tilde{V}^{22}_{nm} \end{pmatrix} \begin{pmatrix} b^1_{-m} \\ b^2_{-m} \end{pmatrix} \right\} \right] (c_0 c_1 |0\rangle)_{123}.
\] (3.37)

We can now compute the contraction over the non-zero modes of $b^3$ and $c^3$ using the relation given in [?]
\[
\langle 0 | c_{-1} c_0 \exp [-c_n P_{nm} b_m] \exp [-c_{-n} Q_{nm} b_{-m} - \lambda_n b_{-n} - c_{-n} \mu_n] c_0 c_1 |0\rangle
= \det (1 + PQ) \langle 0 | c_{-1} c_0 \exp [\lambda (1 + PQ)^{-1} P \mu] c_0 c_1 |0\rangle,
\] (3.38)
where
\[ M = C, \quad N = \tilde{V}^{33}, \quad \lambda = (c_{-n}^1 c_{-n}^2) \left( \frac{\tilde{V}_{nm}^{13}}{\tilde{V}_{nm}^{23}} \right), \]
\[ \mu = (\tilde{V}^{31}_{nm} \tilde{V}^{32}_{nm}) \left( \begin{pmatrix} b_{-m}^1 \\ b_{-m}^2 \end{pmatrix} \right) + (\tilde{V}^{31}_{n0} \tilde{V}^{32}_{n0} \tilde{V}^{33}_{n0}) \left( \begin{pmatrix} b_0^1 \\ b_0^2 \end{pmatrix} \right). \] (3.39)

Now, using the variables
\[ \tilde{M}^{rs} = -C\tilde{V}^{rs}, \] (3.40)
and the relations [?]
\[ \tilde{M}_{11} + \tilde{M}_{12} + \tilde{M}_{21} = 1, \quad \tilde{M}_{12} \tilde{M}_{21} = \tilde{M}_{11}^{2} - \tilde{M}_{11}, \]
\[ \tilde{M}_{12}^{2} + \tilde{M}_{21}^{2} = 1 - \tilde{M}_{11}^{2}, \quad \tilde{M}_{12}^{3} + \tilde{M}_{21}^{3} = (1 - \tilde{M}_{11})^{2}(1 + 2\tilde{M}_{11}), \] (3.41)
\[ \tilde{V}_{0}^{1} + \tilde{V}_{0}^{2s} + \tilde{V}_{0}^{2s} = 0, \quad \tilde{V}_{0}^{r1} + \tilde{V}_{0}^{r2} + \tilde{V}_{0}^{r3} = 0, \]
\[ \tilde{V}_{0}^{21} = -\frac{\tilde{M}_{12}}{1 - \tilde{M}_{11}} \tilde{V}_{0}^{11}, \quad \tilde{V}_{0}^{12} = -\frac{\tilde{M}_{21}}{1 - \tilde{M}_{11}} \tilde{V}_{0}^{11}, \]

one can show that
\[ 3 \langle \tilde{I} | V_{3} \rangle_{123} = 3 \langle 0 | c_{-1}^{1} c_{0}^{3} \exp \left[ c_{1}^{1} C b_{0}^{2} + c_{2}^{2} C b_{0}^{1} + (c_{1}^{1} - c_{2}^{2})\Omega(b_{0}^{1} - b_{0}^{2}) + (c_{1}^{1} - c_{2}^{2})\Gamma b_{0}^{3} \right] (c_{0} c_{1} | 0 \rangle)_{123} \] (3.42)

where
\[ \Omega = -\frac{1}{1 - \tilde{M}_{11}} \tilde{V}_{0}^{11}, \]
\[ \Gamma = (\tilde{V}_{0}^{12} - \tilde{V}_{0}^{21}). \] (3.43)

Following the notation of [?], we have suppressed the subscript \( n \) and \( m \) which take on integer values greater than or equal to one in expressions like in expressions like \( c_{n}^{r}, b_{n}^{r}, \tilde{V}_{nn}^{rs} \) and \( \tilde{V}_{nn}^{rs} \). Finally, recalling that \( \langle 0 | c_{-1} c_{0} c_{1} | 0 \rangle = 1 \), we arrive at
\[ 3 \langle \tilde{I} | V_{3} \rangle_{123} = (c_{1}^{1} - c_{2}^{2})\Gamma \exp \left[ c_{1}^{1} C b_{0}^{2} + b_{2}^{2} C b_{0}^{1} + (c_{1}^{1} - c_{2}^{2})\Omega(b_{0}^{1} - b_{0}^{2}) \right] (c_{0} c_{1} | 0 \rangle)_{123}, \] (3.44)
which is clearly odd under the exchange \( 1 \leftrightarrow 2 \).

This rigorously establishes the gauge invariance of (3.11) when \( \mathcal{O} \) is the closed string tachyon. It would be interesting to extend this calculation to other closed string modes.

4 Vacuum string field theory

In this paper we focused on Witten’s open string field theory. Recently, some compelling numerical evidence [? , ? , ?] that there exists a non-trivial saddle point in this theory was presented. This saddle point has a non-vanishing tachyon expectation value and is conjectured
to describe the vacuum where the D25-brane have decayed away. However, the description of the fluctuation around this new saddle point in terms of Witten’s original string fields is very complicated and to date have been explored only using numerical techniques [?]. To overcome this challenge, a new theory, known as vacuum string field theory [?], was conjectured to describe the fluctuations about the non-trivial vacuum of Witten’s original theory. We close this paper by commenting on the nature of gauge invariant observables for this theory.6

The action of vacuum string field theory is given by

\[ S = \int A^\ast Q A + \frac{2}{3} A^\ast A^\ast A , \]  

(4.1)

where \( Q \) consists only of ghosts

\[ Q = a_0 c_0 + \sum_{n=1} a_n (c_n + (-1)^n c_n) , \]  

(4.2)

and \( a_n \) are in general arbitrary numerical coefficients. Because the operator \( Q \) has trivial cohomology, there are no perturbative open string states in this theory, justifying the conjecture.

If the conjecture of [?] is correct and the vacuum string field theory is indeed equivalent to Witten’s theory around the shifted vacuum, we expect to find the same set of gauge invariant operators in vacuum string field theory as we did in Witten’s theory. After all, these operators correspond to closed strings which continue to exist in the absence of D-branes. Let us therefore examine how the argument for gauge invariance of (3.11) is modified when the action is (4.1).

In vacuum string field theory, gauge transformation acts on the string fields according to

\[ \delta A = Q \Lambda + [A, \Lambda] . \]  

(4.3)

Consider an expression of the form (3.11) considered in the previous section. Just as in the previous section, we are interested in whether (3.11) is invariant with respect to the linear and the non-linear contributions to the gauge transformations. Since the vacuum string field theory differs only in the choice of \( Q \), the only possible difference from the analysis of the previous section will arise from the effect of the linear term.

In Witten’s cubic theory, gauge invariance of (3.11) at the linear level required the BRST operator to commute with the closed string vertex. This constrained the momentum of (3.11) to lie on the mass shell of the corresponding closed string. In the case of the vacuum string

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6Gauge invariant observables of vacuum string field theory was also considered in [?]. The relation between these operators and the operators described in this paper is not clear.
field theory, however, this constraint is much simpler since \( Q \) acts trivially on the matter sector. Consider for example the tachyon vertex operator (3.9). Since (3.9) does not contain any dependence on the \( b \) field, it commutes with \( Q \) trivially. We therefore see that there are more gauge invariant operators in vacuum string field theory.

The fact that the on-shell condition is relaxed in the vacuum string field theory giving rise to a mismatch in the spectrum of gauge invariant observables appears to be a generic feature. The structure of gauge transformations involving \( Q \) made purely out of ghosts is simply not restrictive enough. This mismatch appears to point to the conclusion that the conjecture of [?] is at best singular. In order for \( Q \) to have the possibility of imposing mass shell condition on the (3.11), \( Q \) must depend at some degree on matter fields. The challenge is to find \( Q \) with trivial cohomology that imposes the correct on-shell condition on the gauge invariant operators. One way to approach this difficulty is [?] to regulate \( Q \) with an explicit dependence on the matter part. It would be very interesting to see if such a prescription does in fact give rise to the correct physics.

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**Appendix A: Proof of eq.(3.33)**

To prove eq.(3.33) we start with some relations found in [?]

\[
V_{11} = \frac{1}{3}(C + U + \bar{U}), \\
V_{12} = \frac{1}{6}(2C - U - \bar{U}) + \frac{1}{6}i\sqrt{3}(U - \bar{U}), \\
V_{21} = \frac{1}{6}(2C - U - \bar{U}) - \frac{1}{6}i\sqrt{3}(U - \bar{U}).
\]  
\text{(A.1)}

Where \( \bar{U} = CUC \) and \( U \) satisfy

\[
(1 - Y)E(1 + U) = 0, \quad (1 + Y)\frac{1}{E}(1 - U) = 0, \quad Y = -\frac{1}{2}C + \frac{1}{2}\sqrt{3}X.
\]  
\text{(A.2)}

With

\[
(E^{-1})_{nm} = \delta_{nm} \sqrt{\frac{1}{2}n + \delta_{n0}\delta_{m0}},
\]  
\text{(A.3)}
and $X_{nm}$ are the Fourier components of the operator

$$X(\sigma, \sigma') = i \left( \Theta \left( \frac{\pi}{2} - \sigma \right) - \Theta \left( \sigma - \frac{\pi}{2} \right) \right) \delta(\sigma + \sigma' - \pi), \quad (A.4)$$

which can be written explicitly as

$$X_{0m} = \frac{i \sqrt{2}}{\pi m} (1 - (-1)^m)(-1)^{(m-1)/2},$$
$$X_{nm} = \frac{i}{\pi} (-1)^{(n-m)/2} (1 - (-1)^{n+m}) \left( \frac{1}{n + m} + \frac{(-1)^m}{n - m} \right). \quad (A.5)$$

When decompose these matrices into two by two block matrices associated with odd and even indices they take the form $\begin{bmatrix}\bar{C} & X \\ X^* & U^* \end{bmatrix}$, $X = \begin{bmatrix} X_{00} & X_{0e} \\ X_{eo} & X_{ee} \end{bmatrix}$, $U = \begin{bmatrix} U_{oo} & U_{oe} \\ U_{eo} & U_{ee} \end{bmatrix}$, $\bar{U} = \begin{bmatrix} U_{oo} & -U_{oe} \\ -U_{eo} & U_{ee} \end{bmatrix}$ \quad (A.6)

From the oo and oe components of (A.2) we have $\begin{bmatrix}\beta_n(\sigma) + \beta_n(\pi - \sigma) \end{bmatrix}$ is non-vanishing only for even $n$, we can write

$$\frac{M_{12} - M_{21}}{1 - M_{11}} = \begin{bmatrix} 0 & EX_{oe}E^{-1} \\ E^{-1}X_{eo}E \end{bmatrix}. \quad (A.12)$$

Now using (A.3) and (A.5), we get eq.(3.33).

\footnote{Note that we use $F$ to denote what was defined as $M$ in \cite{?} to avoid confusion with $M_{11}$, $M_{12}$ and $M_{21}$.}