Orientifolds and Slumps in $G_2$ and Spin(7) Metrics

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ABSTRACT

We discuss some new metrics of special holonomy, and their roles in string theory and M-theory. First we consider Spin(7) metrics denoted by $C_8$, which are complete on a complex line bundle over $\mathbb{CP}^3$. The principal orbits are $S^7$, described as a triaxially squashed $S^3$ bundle over $S^4$. The behaviour in the $S^3$ directions is similar to that in the Atiyah-Hitchin metric, and we show how this leads to an M-theory interpretation with orientifold D6-branes wrapped over $S^4$. We then consider new $G_2$ metrics which we denote by $C_7$, which are complete on an $\mathbb{R}^2$ bundle over $T^{1,1}$, with principal orbits that are $S^3 \times S^3$. We study the $C_7$ metrics using numerical methods, and we find that they have the remarkable property of admitting a $U(1)$ Killing vector whose length is nowhere zero or infinite. This allows one to make an everywhere non-singular reduction of an M-theory solution to give type IIA D6-branes wrapped over $S^2 \times S^2$, providing a new completely regular supergravity dual of $\mathcal{N} = 1$, $D = 4$ super-Yang-Mills theory. We also discuss some four-dimensional hyper-Kähler metrics described recently by Cherkis and Kapustin, following earlier work by Kronheimer. We show that in certain cases these metrics, whose explicit form is known only asymptotically, can be related to metrics characterised by solutions of the $su(\infty)$ Toda equation, which can provide a way of studying their interior structure.
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1 Introduction

Metrics of special holonomy have played an important role in many areas of string theory and M-theory. Compact metrics have been used extensively for Kaluza-Klein dimensional reduction to four, and other dimensions. Non-compact metrics of special holonomy have been used as generalisations of the usual flat metric on the space transverse to a p-brane, providing configurations with lesser supersymmetry, and by turning on additional fluxes, configurations that break other symmetries, such as the conformal symmetry in the usual AdS/CFT correspondence. In this paper, we shall investigate several examples involving new metrics with special holonomy.

Our first example is a family of eight-dimensional metrics of cohomogeneity one and Spin(7) holonomy, defined on a certain complex line bundle over $\mathbb{CP}^3$. The principal orbits in these metrics are $S^7$, viewed as a triaxially-squashed $S^3$ bundle over $S^4$. The system of four first-order equations following from requiring Spin(7) holonomy were obtained recently in [1, 2], and a numerical analysis in [2] indicated that complete asymptotically local conical (ALC) metrics, denoted by $C_8$ there, arise as solutions of these equations. There is a non-trivial parameter, characterising the degree of squashing of the $\mathbb{CP}^3$ bolt. These solutions were found to have a behaviour similar to that seen in the four-dimensional Atiyah-Hitchin metric [3, 4], with one direction on the $S^3$ fibres collapsing on the singular orbit at short distance, whilst a different direction in the $S^3$ stabilises to a constant radius at large distance [2].

If one takes the product of the Atiyah-Hitchin metric and seven-dimensional Minkowski spacetime, one obtains a solution of $D = 11$ supergravity which, after reduction on the circle of asymptotically-stabilised radius, admits an interpretation as a D6-brane orientifold plane [5]. We show in this paper that an analogous interpretation can be given to an M-theory solution of the product of a $C_8$ metric with 3-dimensional Minkowski spacetime, which from the type IIA viewpoint becomes D6-brane orientifold plane wrapped on $S^4$. We do this by making a detailed analysis of the coordinate identifications required by the regularity of the $C_8$ metrics, and by making a perturbative analysis of the large-distance asymptotics, to establish the mass of the solutions.

Our second example is concerned with seven-dimensional metrics of cohomogeneity one and $G_2$ holonomy. A wide class of metrics with $S^3 \times S^3$ principal orbits was studied in [2, 6], with six metric functions characterising homogeneous deformations of the $S^3 \times S^3$, and the first-order equations implying $G_2$ holonomy were obtained. In this paper, we study a class of solutions of these equations in which the principal $S^3 \times S^3$ orbits degenerate to a
bolt with the topology of the 5-dimensional space \( T^{1,1} \), as a consequence of the collapse of a circle at short distance. We obtain short-distance Taylor expansions for such solutions, and use them in order to integrate numerically to search for regular metrics. We find indications that there exist non-singular ALC metrics with a non-trivial parameter that characterises the degree of squashing of the \( T^{1,1} \) bolt. At the upper limit of the range of this parameter, the metric becomes asymptotically conical (AC). The ALC metrics, which we shall denote by \( C_7 \), are asymptotic to the product of a circle and an AC six-metric on the cone over \( T^{1,1}/Z_2 \). The \( Z_2 \) identification is a consequence of the requirement of regularity at the degenerate orbit at short distance. By contrast, the \( T^{1,1} \) of the bolt at short distance can be modified globally, if desired, to \( T^{1,1}/Z_N \) for any integer \( N \). (This does not affect the \( T^{1,1}/Z_2 \) structure at infinity.)

An intriguing feature of these new metrics, which are defined on an \( \mathbb{R}^2 \) bundle over \( T^{1,1} \), is that they admit a circle action whose length is everywhere finite and non-zero. This means that one can perform a completely non-singular Kaluza-Klein reduction on this circle, from a starting point of an M-theory solution comprising the \( G_2 \) metric times four-dimensional Minkowski spacetime. In ten dimensions, the solution acquires an interpretation as a D6-brane wrapped on \( S^2 \times S^2 \). The charge of the D6-brane is equal to the integer \( N \) characterising the \( T^{1,1}/Z_N \) geometry of the bolt. Owing to the fact that the dilaton stabilize everywhere, the large \( N \) limit of the solution is a valid supergravity approximation of the D6-branes wrapped around the \( S^2 \times S^2 \), providing a new regular supergravity dual of the \( D = 4, \mathcal{N} = 1 \) super Yang-Mills theory.

In the remainder of the paper, we study some new four-dimensional hyper-Kähler metrics, which have recently been discussed in \cite{7}.

1 Of course these are not of cohomogeneity one (all such metrics were well studied and fully classified long ago), and indeed in general the complete metrics discussed in \cite{7} would have no continuous symmetries at all. The asymptotic forms of the complete metrics are obtained explicitly in \cite{7}; these themselves would, if extended to short distance, lead to singularities, but it is argued that there exist smooth “resolutions” of these singularities. All the asymptotic metrics in \cite{7} admit a \( U(1) \times U(1) \) isometry that acts tri-holomorphically, and which would be lost in the core of the general complete solution.

In special cases the resolved metrics discussed in \cite{7} would still admit an exact \( U(1) \) isometry. This circle action would not act tri-holomorphically within the bulk of the so-

\[^1\] These metrics were also discussed a while ago by Kronheimer, but they did not appear explicitly in print.
lution. When such a \( U(1) \) isometry occurs, the hyper-Kähler metric must necessarily be describable within a class of metrics introduced in [8, 9], in which there is a function of three variables that satisfies an \( su(\infty) \) Toda equation. In principle, therefore, one can probe the properties as one moves in from the asymptotic region in this subclass of the complete metrics described by Cherkis and Kapustin, by perturbing around the asymptotic form after re-expressing it in the Toda-metric variables. We study this for certain special cases of the metrics in [7].

2 \( \mathbb{C}_8 \) metrics of Spin(7) holonomy, and orientifold planes

2.1 The construction of the \( \mathbb{C}_8 \) metrics

In [2], a numerical analysis was used to demonstrate the existence of new cohomogeneity one metrics of Spin(7) holonomy on the chiral spin bundle of \( S^4 \), which were denoted by \( \mathbb{C}_8 \). The principal orbits are \( S^7 \), viewed as an \( S^3 \) bundle over \( S^4 \) in which the 3-spheres are triaxially squashed. This leads to a metric ansatz involving four functions of the radial coordinate, given by [2]

\[
\mathbf{ds}^2 = \mathbf{dt}^2 + a_1^2 \mathbf{R}_i^2 + b^2 P_a^2 ,
\]

where the 1-forms \( R_i \) and \( P_a \) live in the coset \( S^7 = SO(5)/SO(3) \), and satisfy

\[
\begin{align*}
\mathbf{d}P_0 &= (R_1 + L_1) \wedge P_1 + (R_2 + L_2) \wedge P_2 + (R_3 + L_3) \wedge P_3 , \\
\mathbf{d}P_1 &= -(R_1 + L_1) \wedge P_0 - (R_2 - L_2) \wedge P_3 + (R_3 - L_3) \wedge P_2 , \\
\mathbf{d}P_2 &= (R_1 - L_1) \wedge P_3 - (R_2 + L_2) \wedge P_0 - (R_3 - L_3) \wedge P_1 , \\
\mathbf{d}P_3 &= -(R_1 - L_1) \wedge P_2 + (R_2 - L_2) \wedge P_1 - (R_3 + L_3) \wedge P_0 , \\
\mathbf{d}R_1 &= -2R_2 \wedge R_3 - \frac{1}{2}(P_0 \wedge P_1 + P_2 \wedge P_3) , \\
\mathbf{d}R_2 &= -2R_3 \wedge R_1 - \frac{1}{2}(P_0 \wedge P_2 + P_3 \wedge P_1) , \\
\mathbf{d}R_3 &= -2R_1 \wedge R_2 - \frac{1}{2}(P_0 \wedge P_3 + P_1 \wedge P_2) .
\end{align*}
\]

(The generators \( L_{AB} = -L_{BA} \) of \( SO(5) \), with \( A = 0, 1, 2, 3, 4 \), satisfy \( dL_{AB} = L_{AC} \wedge L_{CB} \), and are decomposed here as \( P_a = L_{a4} \), \( R_i = \frac{1}{2}(L_{0i} + \frac{1}{2} \epsilon_{ijk} L_{jk}) \) and \( L_i = \frac{1}{2}(L_{0i} - \frac{1}{2} \epsilon_{ijk} L_{jk}) \), where \( i = 1, 2, 3 \).) The first-order equations implying that (1) has Spin(7) holonomy are given by [1, 2]

\[
\begin{align*}
\dot{a}_1 &= \frac{a_1^2 - (a_2 - a_3)^2}{a_2 a_3} - \frac{a_1^2}{2b^2} ,
\end{align*}
\]
\[
\begin{align*}
\dot{a}_2 &= \frac{a_2^2 - (a_3 - a_1)^2}{a_3 a_1} - \frac{a_2^2}{2b^2}, \\
\dot{a}_3 &= \frac{a_3^2 - (a_1 - a_2)^2}{a_1 a_2} - \frac{a_3^2}{2b^2}, \\
\dot{b} &= \frac{a_1 + a_2 + a_3}{4b},
\end{align*}
\]  

(3)

It is not clear how to obtain an analytical solution for this system. For the restriction \(a_1 = a_2\), the equations were solved completely in [10], leading to new complete and non-singular asymptotically locally conical (ALC) metrics \(B_8\) and \(B^\pm_8\) on the chiral spin bundle of \(S^4\), and a new such metric, denoted by \(A_8\), on \(\mathbb{R}^8\). Thus one is now interested in looking for “triaxial” solutions such that \(a_1 \neq a_2 \neq a_3 \neq a_1\). A numerical analysis was carried out in [2], indicating the existence of new triaxial solutions, giving rise to complete ALC metrics on a line bundle over \(\mathbb{C}P^3\).

The procedure used in [2] for obtaining the new solutions involved first constructing a short-distance solution, in the form of a Taylor expansion in the neighbourhood of the bolt, or singular orbit. At this stage, one makes a choice about what the topology of the singular orbits will be. In the present case, regular possibilities can involve a collapse of \(S^7\), \(S^3\) or \(S^1\), giving \(\mathbb{R}^8\), an \(\mathbb{R}^4\) bundle over \(S^4\), or an \(\mathbb{R}^2\) bundle over \(\mathbb{C}P^3\) respectively. In fact, as shown in [2], the first case leads only to the \(A_8\) metric, and the second leads only to the \(B_8\) and \(B^\pm_8\) metrics, and the original AC Spin(7) metric of [11, 12]. The third possibility, with collapsing \(S^1\) orbits, leads to the new \(C_8\) metrics, which we shall discuss more fully here.

First, therefore, one constructs short-distance Taylor expansions for the metric functions, under the assumption that just one of the \(a_i\) goes to zero on the bolt at \(t = 0\). Without loss of generality, one may choose this to be \(a_1\). The expansions are then given by [2]

\[
\begin{align*}
a_1 &= 4t + \frac{(\lambda^4 - 40\lambda^2 - 48)}{12\lambda^2} t^3 + \cdots, \\
a_2 &= \lambda + (1 - \frac{1}{7}\lambda^2) t + \frac{(3\lambda^4 - 8\lambda^2 + 48)}{32\lambda} t^2 + \cdots, \\
a_3 &= -\lambda + (1 - \frac{1}{7}\lambda^2) t - \frac{(3\lambda^4 - 8\lambda^2 + 48)}{32\lambda} t^2 + \cdots, \\
b &= 1 + \frac{1}{16}(12 - \lambda^2) t^2 + \cdots.
\end{align*}
\]  

(4)

The bolt at \(t = 0\) has the topology of \(\mathbb{C}P^3\), viewed as an \(S^2\) bundle over \(S^4\). The constant \(\lambda\) characterises the scale of the \(S^2\) fibres relative to the \(S^4\) base, and thus parameterises different homogeneous squashed \(\mathbb{C}P^3\) metrics. In [2] it was shown numerically that the solution with these initial data will be regular everywhere, if \(\lambda^2 \leq 4\), with \(\lambda^2 = 4\) corresponding to the previously-known special case of the complex line bundle over the “round” \(\mathbb{C}P^3\) with its Fubini-Study metric (specifically, the fourth power of the Hopf bundle). Likewise, the
metrics with $\lambda^2 < 4$ also have the same topology of the complex line bundle over $\mathbb{CP}^3$. The metric for $\lambda^2 = 4$ is AC, and has $SU(4)$ holonomy, but for $\lambda^2 < 4$ the metrics are ALC, and have Spin(7) holonomy. They locally approach the product of a fixed-radius circle and the metric of $G_2$ holonomy on the cone over $\mathbb{CP}^3$ at large distance [2].

Specifically, at large proper distance $t$ the solution approaches $a_1 = a_2 \sim t$, $a_3 \to \text{const.}$ and $b \sim t$. The fact that $a_1$ and $a_2$ become equal at large $t$ implies that the solutions at large distance can be described as perturbations around the solutions with $a_1 = a_2$ that were obtained previously in [10]. The techniques for doing this are very similar to ones that can be employed for studying the asymptotic behaviour of the Atiyah-Hitchin metric. In that case, it can be viewed as a perturbation around the Taub-NUT metric. In order to set the scene for our analysis of the asymptotic behaviour of the $C_8$ metrics, it is therefore useful to carry out such an analysis for the Atiyah-Hitchin system. This is done in appendix A.3. Of course the results for this case are already known [3, 4], and in that case can be seen directly by studying the asymptotic properties of the elliptic functions that appear in the explicit solution. Our approach, which does not require knowledge of the explicit form for the solution, is rather simple and illuminating, and is appropriate for our discussion for the $C_8$ metrics, where no explicit solutions are known.

2.2 Asymptotic behaviour of the $C_8$ metrics

Here, we study the asymptotic behaviour of the $C_8$ metrics of Spin(7) holonomy, given by the ansatz (1) with the metric functions $a_i$ and $b$ satisfying (3), which were found numerically in [2] and are described in section 2.1.

Proceeding in the same vein as our discussion in appendix A.3, we note that the metric functions $a_1$ and $a_2$ become equal asymptotically in the $C_8$ solutions, implying that we can use the already-known exact solutions with $a_1 = a_2$ as starting points for perturbative analyses as large distance. These exact solutions, found in [10] and called $A_8$, $B_8$ and $B_{8}^\pm$ there, will play the same role, as the starting point for a perturbative large-distance analysis, as the Taub-NUT metric did for the perturbative analysis of the Atiyah-Hitchin metric in appendix A.3.

We are therefore led to make the following perturbative expansion at large distance, with

$$a_1 = 2A_0 + A_1, \quad a_2 = 2A_0 + A_2, \quad a_3 = 2B_0 + A_3, \quad b = C_0 + B,$$

where functions with the subscript 0 denote the unperturbed solutions in [10], while the
functions with the subscript 1 denote linearised perturbations. (The notation for the upper-case functions is chosen to fit with the symbols used in [10].) We find that $A_0$, $B_0$ and $C_0$ satisfy the first-order equations

$$
\dot{A}_0 = 1 - \frac{B_0}{2A_0} - \frac{A_0^2}{C_0^2}, \quad \dot{B}_0 = \frac{B_0}{2A_0} - \frac{B_0^2}{C_0^2}, \quad \dot{C}_0 = \frac{A_0}{C_0} + \frac{B_0}{2C_0},
$$

(6)
as in [10].

To solve the equations for the perturbations $A_i$ and $B$, it is useful to introduce a coordinate gauge function $h$ such that $dt = h \, dr$, where we also expand $h = h_0 + h_1$. We can substitute (5) into (3), thereby obtaining linearised equations for $A_i$ and $B$, given by

$$
A'_1 = A_1 \left( \frac{h_0}{B_0} - \frac{2A_0 h_0}{C_0^2} \right) - A_2 \left( \frac{h_0}{B_0} - \frac{B_0 h_0}{2A_0^2} \right) - A_3 \frac{h_0}{2A_0} + \frac{4A_0^2 h_0 B}{C_0^3} + 2A'_0 h_1,
$$

$$
A'_2 = -A_1 \left( \frac{h_0}{B_0} - \frac{B_0 h_0}{2A_0^2} \right) + A_2 \left( \frac{h_0}{B_0} - \frac{2A_0 h_0}{C_0^2} \right) - A_3 \frac{h_0}{2A_0} + \frac{4A_0^2 h_0 B}{C_0^3} + 2A'_0 h_1,
$$

$$
A'_3 = -A_1 \frac{h_0 B_0^2}{2A_0^3} - A_2 \frac{h_0 B_0^2}{2A_0^3} + A_3 h_0 \left( \frac{B_0^2}{A_0^2} - \frac{2B_0}{C_0^2} \right) + \frac{4B h_0 B_0^2}{C_0^3} + 2B'_0 h_1,
$$

$$
B' = \frac{h_0}{C_0} (A_1 + A_2 + A_3) - \frac{B h_0}{C_0^2} (A_0 + B_0) + C'_0 h_1,
$$

(7)

where a prime denotes $d/dr$.

It is useful to choose a gauge for $h$ such that $A_2 = -A_1$. This requires that

$$
h_1 = \frac{(8A_0 B - A_3 C_0^3) h_0}{2C_0 (2A_0^3 - 2A_0 C_0^2 + B_0 C_0^3)}.
$$

(8)

Then, the system admits one simple solution, with

$$
A_3 = 0, \quad B = 0, \quad h_1 = 0,
$$

(9)

and hence $A_1$ satisfies the rather simple equation

$$
2A'_1 = A_1 h_0 \left( \frac{4}{B_0} - \frac{B_0}{A_0^2} - \frac{4A_0}{C_0^2} \right).
$$

(10)

Thus we have

$$
A_1 = \exp \left[ \frac{1}{2} \int h_0 \left( \frac{4}{B_0} - \frac{B_0}{A_0^2} - \frac{4A_0}{C_0^2} \right) \right].
$$

(11)

The general solution to the zeroth-order equations (6) was found in [10]. This gave rise to an isolated regular metric, denoted by $\mathcal{A}_8$, on a manifold of topology $\mathbb{R}^8$, and to a family of metrics, denoted by $\mathcal{B}_8$, $\mathcal{B}_8^+$ and $\mathcal{B}_8^-$, characterised by a non-trivial parameter (not merely a scale), on the chiral spin bundle of $S^4$. The metric $\mathcal{A}_8$ and the metric $\mathcal{B}_8$, which is a particular case within the one-parameter family $\mathcal{B}_8^\pm$, are very simple in form, with metric coefficients that can be expressed as rational functions of a suitably-chosen radial variable.
The remainder of the $B_8^\pm$ family are more complicated in form, and are expressed in terms of hypergeometric functions. All of these solutions found in [10] have the feature that $a_3$ tends to a constant at large distance, while the remaining metric functions $a_1$, $a_2$ and $b$ have linear growth. Thus we have, at large distance, $h_0 \to 1$, $A_0 \sim r$, $C_0 \sim r$ and $B_0^2 \sim m^2$.

For the $A_8$ metric, with $r \geq \ell > 0$, we have [10]

$$
\begin{align*}
h_0 &= \frac{(r + \ell)}{\sqrt{(r + 3\ell)(r - \ell)}}, \quad A_0 = \frac{1}{2} \sqrt{(r + 3\ell)(r - \ell)}, \\
B_0 &= -\frac{\ell \sqrt{(r + 3\ell)(r - \ell)}}{(r + \ell)}, \quad C_0 = \frac{1}{\sqrt{2}} \sqrt{(r^2 - \ell^2)}.
\end{align*}
\tag{12}
$$

From (11) we therefore have the leading-order behaviour

$$
A_1 \sim \exp \left[2 \int \frac{1}{B_0} \right] = e^{-4r/\ell}.
\tag{13}
$$

It follows from this that our assumption that $A_1$ is a small perturbation also requires that we have $\ell > 0$.

The other simple solution, $B_8$, corresponds to changing the sign of $\ell$ in (12), and now taking $r \geq -3\ell > 0$. Defining the positive scale parameter $\tilde{\ell} \equiv -\ell$, we therefore have

$$
\begin{align*}
h_0 &= \frac{(r - \tilde{\ell})}{\sqrt{(r - 3\tilde{\ell})(r + \tilde{\ell})}}, \quad A_0 = \frac{1}{2} \sqrt{(r - 3\tilde{\ell})(r + \tilde{\ell})}, \\
B_0 &= \frac{\tilde{\ell} \sqrt{(r - 3\tilde{\ell})(r + \tilde{\ell})}}{(r - \tilde{\ell})}, \quad C_0 = \frac{1}{\sqrt{2}} \sqrt{(r^2 - \tilde{\ell}^2)},
\end{align*}
\tag{14}
$$

with $\tilde{\ell} > 0$. From (11) we now find the leading-order behaviour

$$
A_1 \sim e^{2\int \frac{1}{B_0}} = e^{4r/\tilde{\ell}}.
\tag{15}
$$

Thus in this case we find that we must have $\tilde{\ell} < 0$ in order to have a decaying perturbation at infinity. However, as we saw above, with this sign the unperturbed solution is really just the $A_8$ metric again.

An analogous calculation taking the whole non-trivial one-parameter family of metrics $B_8^\pm$ as the zeroth-order starting points would be more complicated. However, by continuity we can argue that the perturbations around these metrics would also require analogous asymptotic behaviour, with the same signs for the coefficients of the leading-order radial dependence as is implied by having $\tilde{\ell}$ negative in the particular case of the $B_8$ metric (14).

Footnote 2: The crucial distinction between the $A_8$ and $B_8$ cases is that in $A_8$, the first-order equations imply that $B_0$ has a minus-sign prefactor, as in (12), whereas in the $B_8$ case the first-order equations imply that $B_0$ has a plus-sign prefactor, as in (14).
We shall, for convenience, refer to this analogous (“trivial”) scale parameter in the general class of $B_8^\pm$ metrics as being the “$\tilde{\ell}$-parameter.” Since there is no family of “$A_8^\pm$ metrics” with a non-trivial parameter, we conclude that, viewed as perturbations around the large-distance limit of the general $B_8^\pm$ metrics, the $C_8$ metrics correspond to “negative-$\tilde{\ell}$” $B_8^\pm$ metrics that would be singular at short distance. The situation in these cases is somewhat analogous to the Atiyah-Hitchin metric, in that for the perturbation to be exponentially decaying at large distance, the sign of the scale parameter $\tilde{\ell}$ must be opposite to the sign that is needed for regular short-distance behaviour in the unperturbed metric.

The perturbative discussion above does not per se allow us to investigate directly how the perturbed solutions will extrapolate down to short distance. Rather, this information is contained in the details of the $C_8$ solutions themselves. The $C_8$ metrics behave asymptotically like $A_8$ with positive $\ell$, together with exponentially-small corrections. They also behave asymptotically like $B_8$ (or the $B_8^\pm$ generalisations) with negative $\tilde{\ell}$, again with exponentially-small corrections. Thus the $C_8$ metrics can in a sense be thought of as “resolutions” of (exponentially-corrected) “negative-\(\tilde{\ell}\)” $B_8^\pm$ metrics, in which the singularity that one would encounter at short distance in such a $B_8^\pm$ metric is smoothed out into a regular collapse to a $\mathbb{CP}^3$ bolt. Instead, the $C_8$ metrics can also be thought of as a class of alternative smooth inward extrapolations from the (exponentially-corrected) asymptotic behaviour of the $A_8$ metric with positive $\ell$. In this viewpoint the $C_8$ metrics are not “resolutions,” since the $A_8$ metric has positive $\ell$ and is already itself regular, but they do provide an alternative smooth short-distance behaviour, with a $\mathbb{CP}^3$ bolt instead of a NUT. An exception to the picture of $C_8$ as a resolution of a “negative-\(\tilde{\ell}\)” $B_8^\pm$ metric therefore occurs if we consider the special case of the simple $B_8$ metric (14), since in this particular case the negative-$\tilde{\ell}$ $B_8$ metric itself happens to be perfectly regular, being nothing but the $A_8$ metric.

It is not entirely clear how one should interpret the above results in terms of brane solutions. In appendix A.3, it was shown how the M-theory solution of a product of seven-dimensional Minkowski spacetime and the Atiyah-Hitchin metric acquires an interpretation as a negative-mass D6-brane orientifold in ten dimensions. However, when one is considering metrics that are asymptotically conical or locally conical, as opposed to asymptotically flat metrics such as Taub-NUT or Atiyah-Hitchin, the meaning of “mass” becomes less clear. Typically, in the AC or ALC cases, the metric coefficients approach constants at infinity with lesser inverse powers of the proper distance $\rho$ than would be the case for asymptotically flat metrics. Asymptotically flat $d$-dimensional metrics approach flatness like $\rho^{-d+3}$, and the “mass” is essentially a measure of the coefficient of such an asymptotic term relative to
the fiducial flat metric. It is not clear what the analogous zero-mass fiducial metric should be when one is considering an AC or ALC metric, and therefore it is unclear whether any coefficient in the asymptotic form of the metric characterises the “mass.”

Although the understanding of mass is problematical in the present case, we can, nevertheless, still discuss an orientifold interpretation. This forms the topic of section 2.3 below.

2.3 Orientifolds

2.3.1 Calabi metric on line bundle over $\mathbb{C}P^3$

Before discussing the more complicated $\mathbb{C}_8$ metrics with their Atiyah-Hitchin style “slumping,” it is helpful to make contact with a previously understood and simpler situation, namely the metric on the line bundle over $\mathbb{C}P^3$, first described in [13], which, as discussed in [2], is the $\lambda^2 = 4$ limit of the $\mathbb{C}_8$ metrics. The global structure was discussed in detail in [14]. The metric is Ricci flat and Kähler, with holonomy $SU(4) \equiv \text{Spin}(6) \subset \text{Spin}(7)$, and it is ALE, being asymptotic to $\mathbb{C}^4/\mathbb{Z}_4$. The isometry group is $U(4)$, which acts holomorphically. Indeed, in [14] a set of complex coordinates $Z^\alpha$ was introduced, in terms of which the $U(4)$ action is linear.

This $SU(4)$ holonomy metric is a special solution of our equations with $a_2 = -a_3 = 2b$ (see equations (10), (11) and (12) in [2]). We have (setting the arbitrary scale $\ell = 1$)

$$ds^2 = \left(1 - \frac{1}{r^8}\right)^{-1} dr^2 + r^2 \left(1 - \frac{1}{r^8}\right) R_1^2 + r^2 (R_2^2 + R_3^2 + \frac{1}{4} P_a^2).$$ (16)

The metric function $a_1$ vanishes at $r = 1$, which is the zero section of the line bundle. Near $r = 1$ we have (with $dt = d(r - 1) (\sqrt{8} \sqrt{r - 1})^{-1}$),

$$ds^2 \sim dt^2 + 16l^2 R_1^2 + (R_2^2 + R_3^2 + \frac{1}{4} P_a^2).$$ (17)

By contrast, near infinity we have

$$ds^2 \sim dt^2 + t^2 (R_1^2 + R_2^2 + R_3^2 + \frac{1}{4} P_a^2).$$ (18)

The metric in round brackets in (18) is the round metric on $S^7$. The 1-forms $R_i$ span the $S^3 \equiv SU(2)$ fibres of the usual quaternionic Hopf fibration of $S^7$ with base $S^4$. The 1-form $R_1$ is tangent to the fibres of the $S^1 \equiv U(1) \subset SU(2) \equiv S^3$ complex Hopf fibration of $S^7$ with base $\mathbb{C}P^3$. The appearance of the factor 16 in the metric (17) near the zero section shows that the Hopf fibres must have one quarter their standard period. Thus if we
introduce Euler angles \((\tilde{\theta}, \tilde{\phi}, \tilde{\psi})\) for \(SU(2)\) near the bolt, and set \(R_1 = \tilde{\sigma}_3\), then the period of \(\tilde{\psi}\) will be \(\pi\).

In the asymptotic region, the metric is manifestly asymptotically locally flat when written in terms of the the complex coordinates \(Z^\alpha, \alpha = 1, 2, 3, 4\), and the points \(Z^\alpha\) and \(\sqrt{-1} Z^\alpha\) must be identified. The metric may therefore be viewed as a resolution or blow-up of the orbifold obtained by identifying flat Euclidean space \(\mathbb{E}^8 \equiv \mathbb{E}^2 \oplus \mathbb{E}^2 \oplus \mathbb{E}^2 \oplus \mathbb{E}^2\) under a simultaneous rotation through ninety degrees in four orthogonal two-planes. Note that the identification does not act on the \(S^4\) base of the fibration. This is also true of the identification in the Atiyah-Hitchin case. The identification map (a shift of \(\tilde{\psi}\) leaving the coordinates \(\tilde{\theta}\) and \(\tilde{\phi}\) on the \(S^2\) base invariant) does induce a rotation of the Cartan-Maurer 1-forms. (This is the identification induced by \(\tilde{I}_3\), as discussed in appendix A.) Thus if one seeks, as with the case of the Atiyah-Hitchin metric, to understand the identification at the Lie-algebra level, one must anticipate an induced action on the \(P_a\).

As discussed in appendix A, shifting \(\tilde{\psi}\) by \(\pi\) induces the action of the involution \(\tilde{I}_3\) on the Lie-algebra of \(SU(2)\), namely

\[
(R_1, R_2, R_3) \rightarrow (R_1, -R_2, -R_3). \tag{19}
\]

However this action of \(\tilde{I}_3\) alone is not an involution of the full algebra given in (2). Inspection reveals that one must supplement it with the following action on the \(P_a\) and \(L_i\):

\[
(P_0, P_1, P_2, P_3, L_1, L_2, L_3) \rightarrow (P_0, P_1, -P_2, -P_3, L_1, -L_2, -L_3). \tag{20}
\]

### 2.3.2 The \(\mathbb{C}_8\) metrics

The idea now is to use the results above to understand what happens in the ALC rather than ALE case, where “slumping” takes place. Equations (4) then apply near the bolt at \(t = 0\). The situation is somewhat more complicated than in the Atiyah-Hitchin case discussed in the appendix, because the left-invariant 1-forms \(R_i\) and \(P_a\) satisfy a more complicated algebra, given in (2). However, we can still think of the 1-forms \(R_i\) as being essentially like the standard left-invariant 1-forms \(\sigma_i\) of \(SU(2)\); taking account of normalisation factors, we shall have

\[
R_i \sim -\frac{1}{2}\sigma_i + \cdots, \tag{21}
\]

where the \(\sigma_i\) are defined as in appendix A.2, and the ellipses represent the BPST Yang-Mills \(SU(2)\) instanton connection terms that characterise the twisting of the \(S^3\) fibres over the \(S^4\) base spanned by \(P_a\).
As in appendix A.2, we can make two different “adapted” choices for Euler angles parameterising the $S^3$ fibres, with tilded angles near the $\mathbb{CP}^3$ bolt, and untilded angles at infinity. The vanishing of $a_1 \sim 4t$ forces an identification of the adapted Euler angle $\tilde{\psi}$, with period $\pi$. Passing to infinity, we find that $a_3$ tends to a constant. Thus if $(\theta, \phi, \psi)$ are adapted Euler angles at infinity, the identification that is forced is a reflection in $\psi$ and inversion in $(\theta, \phi)$ just as in equation (126) for the Atiyah-Hitchin case. We can think of $\psi$ as parameterising an M-theory circle, and so the reflection of $\psi$ corresponds to M-theory charge-conjugation, $C_{11}$.

The inversion and additional action on the $P_a$ given by (20) may be understood as follows. Reduction on the M-theory circle leads asymptotically to a 7-manifold of cohomogeneity one with principal orbits $\mathbb{CP}^3$, which we think of as an $S^2$ bundle over $S^4$. In fact $\mathbb{CP}^3$ is the ur-twistor space of $S^4$. Acting on the coordinates, our involution $\tilde{I}_3$ leaves the points on the $S^4$ base fixed and acts as the antipodal map on the $S^2$ fibres. This “real structure” plays an important rôle in Twistor constructions, although we shall not make use of it in that way here.

In the corresponding construction for the Atiyah-Hitchin metric itself, we have, upon reduction of the negative mass Taub-NUT limit near infinity, a 3-manifold which is a cone over $S^2$. This cone is the same as $\mathbb{R}^3$, and the antipodal map on $S^2$ induces the standard inversion (or parity) map $P$ on $\mathbb{R}^3$. The combination of the M-theory charge conjugation $C_{11}$ and the 3-parity $P$ amounts to inversion in four dimensions. Since Taub-NUT is thought of as a D6-brane [15], one therefore thinks of Atiyah-Hitchin as an orientifold plane.

In the case of the Spin(7) manifold, we need to take the product with $\mathbb{E}^{2,1}$ to get an eleven-dimensional Ricci-flat solution. We then interpret the M-theory quotient as a type IIA solution with a D6-brane wrapped over the Cayley 4-sphere. There are three transverse directions; the radius $r$, and the 2-sphere with coordinates $\theta$ and $\phi$. The six spatial world-volume directions consist of the 4-sphere and the two flat spatial coordinates in the $\mathbb{E}^{2,1}$ factor. The identification we are obliged to make is thus clearly an inversion in the transverse directions. The orientifold interpretation therefore goes through in a completely parallel way, and we now have an orientifold plane with a D6-brane wrapped around the $S^4$. 

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3 New $G_2$ metrics $\mathbb{C}_7$ with $S^3 \times S^3$ principal orbits

We now turn to an analogous discussion of a general class of solutions for 7-metrics with $G_2$ holonomy. Specifically, we shall consider the system of first-order equations for metrics of cohomogeneity one with $S^3 \times S^3$ principal orbits. A rather general ansatz involving six radial functions was considered in [16, 6], and the first-order equations for $G_2$ holonomy were derived. The six-function metric ansatz is given by

$$\begin{aligned}
ds^2 = dt^2 + a_i^2 (\sigma_i - \Sigma_i)^2 + b_i^2 (\sigma_i + \Sigma_i)^2,
\end{aligned}$$

where $\sigma_i$ and $\Sigma_i$ are left-invariant 1-forms for two $SU(2)$ group manifolds. It was found that for $G_2$ holonomy, $a_i$ and $b_i$ must satisfy the first-order equations

$$\begin{aligned}
\dot{a}_1 &= \frac{a_1^2}{4a_3 b_2} + \frac{a_2^2}{4a_2 b_3} - \frac{a_3}{4b_3} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2}, \\
\dot{a}_2 &= \frac{a_2^2}{4a_3 b_1} + \frac{a_3^2}{4a_1 b_3} - \frac{a_1}{4b_3} - \frac{b_1}{4a_3} - \frac{b_3}{4a_1}, \\
\dot{a}_3 &= \frac{a_3^2}{4a_2 b_1} + \frac{a_1^2}{4a_1 b_2} - \frac{a_2}{4b_2} - \frac{b_1}{4a_2} - \frac{b_2}{4a_1}, \\
\dot{b}_1 &= \frac{b_1^2}{4a_2 a_3} - \frac{b_2^2}{4b_3 b_2} - \frac{b_3}{4a_3} + \frac{b_2}{4a_2} + \frac{b_3}{4b_2}, \\
\dot{b}_2 &= \frac{b_1^2}{4a_3 a_1} - \frac{b_3^2}{4b_3 b_1} - \frac{b_1}{4a_3} + \frac{b_3}{4b_1}, \\
\dot{b}_3 &= \frac{b_2^2}{4a_1 a_2} - \frac{b_3^2}{4b_1 b_2} - \frac{b_1}{4a_2} + \frac{b_2}{4b_1}.
\end{aligned}$$

3.1 Analysis for $\mathbb{C}_7$ metrics with six-function solutions

We can look for regular solutions numerically, by first constructing solutions at short distance expanded in Taylor series, and then using these to set initial data just outside the bolt, for numerical integration. Once can consider various possible collapsing spheres, namely $S^1$, $S^2$ or $S^3$. The case $S^3$ has been studied previously, and the case $S^2$ gives no regular short-distance solutions. For a collapsing $S^1$, however, we find that there are regular short-distance Taylor expansions. The bolt at $t = 0$ will then be $(S^3 \times S^3)/S^1$ with the circle embedded diagonally; this is the space $T^{1,1}$. By analogy with the eight-dimensional $\mathbb{C}_8$ metrics of Spin(7) holonomy with a collapsing circle at short distance, we shall denote these new $G_2$ metrics by $\mathbb{C}_7$.

Taking the collapsing $S^1$, without loss of generality, to be in the $a_3$ direction, we find regular short-distance solutions of the form

$$a_1 = q_1 - \frac{(q_1^2 - q_2^2 + q_3^2)}{8q_1 q_3} t + O(t^2),$$

14
\[ a_2 = q_1 + \frac{(q_1^2 - q_2^2 - q_3^2)}{8q_2 q_3} t + O(t^2), \]
\[ a_3 = -t - \frac{[(q_1^2 - q_2^2)^2 + q_3^4 - 8(q_1^2 + q_2^2) q_3^2]}{96q_1^2 q_2^2 q_3^3} t^3 + O(t^5), \]
\[ b_1 = q_1 - \frac{(q_1^2 - q_2^2 - q_3^2)}{8q_2 q_3} t + O(t^2), \]
\[ b_2 = q_2 + \frac{(q_1^2 - q_2^2 + q_3^2)}{8q_1 q_3} t + O(t^2), \]
\[ b_3 = q_3 - \frac{[(q_1^2 - q_2^2)^2 - q_3^4]}{16q_1^2 q_2^2 q_3} t^2 + O(t^4). \] (24)

Here \((q_1, q_2, q_3)\) are free parameters in the solutions. They characterise the metric on the \(T^{1,1}\) space that forms the bolt. One of the three parameters corresponds just to setting the overall scale of the metric.

We have calculated the Taylor expansions (24) up to tenth order in \(t\), and used these in order to set initial data just outside the \(T^{1,1}\) bolt. Numerical integration can then be used in order to study the possibility of having solutions that remain regular at large \(t\). The numerical analysis of the system of six equations (23) seems to be somewhat delicate, and the results are rather less stable than one would wish, but they seem to suggest that if we fix, say, \(q_1\) and \(q_2\), then there exists a a range of values \(q_3 \leq K\), for some constant \(K\), that gives regular solutions. The indications are that the metrics will be ALC, for \(q_3 < K\), and AC for \(q_3 = K\). For the ALC metrics it is the function \(b_3\) that stabilises to a fixed radius at large distance. Thus at short distance the circle spanned by \((\sigma_3 - \Sigma_3)\) collapses, while at large distance the circle spanned by \((\sigma_3 + \Sigma_3)\) stabilises. This means that whilst the ALC metrics have a \(T^{1,1}\) bolt at short distance, and approach \(S^1\) times a cone over \(T^{1,1}\) at large distance, the \(T^{1,1}\) spaces in the two regions correspond to two different embeddings of \(S^1\) in \(S^3 \times S^3\). Note that the metric coefficient \(b_3^2\) remains finite and non-zero everywhere, both at short distance and large distance.

If we choose \(q_1 = q_2\), the equations (23) imply that we shall have \(a_1 = a_2\) and \(b_1 = b_2\) for all \(t\), and in fact the six-function system of equations can then be consistently truncated to a four-function system. As we describe in section 3.2 below, this truncated system of equations seems to give much more stable and reliable numerical results.

One can also repeat the perturbative analysis at large distance that we applied previously to the Atiyah-Hitchin metric (in appendix A) and to the Spin(7) metrics in section 2. We shall look here for solutions at large distance that are perturbations around the exact solution found in [6], for which we shall make the ansatz
\[ a_1 = \frac{\sqrt{\alpha}}{\sqrt{2\ell}} \sqrt{(r - \ell)(r + 3\ell)} + x, \quad a_2 = \frac{\sqrt{\alpha}}{\sqrt{2\ell}} \sqrt{(r - \ell)(r + 3\ell)} - x, \]
\[ b_1 = -\frac{\sqrt{3}}{4} \sqrt{(r + \ell)(r - 3\ell) + y}, \quad b_2 = -\frac{\sqrt{3}}{4} \sqrt{(r + \ell)(r - 3\ell) - y}, \quad (25) \]

\[ a_3 = -\frac{1}{2} r, \quad b_3 = \frac{\ell \sqrt{r^2 - 9\ell^2}}{\sqrt{r^2 - \ell^2}}, \quad (26) \]

where \( dr = -\frac{2b_3}{3\ell} dt \), and \( x \) and \( y \) are assumed to be small in comparison to the leading-order terms at infinity. At the linearised level at infinity, we find that the functions \( x \) and \( y \) are approximately given by

\[ x \sim \frac{1}{r^2} \left( c_1 e^{3r} + c_2 e^{-3r} \right), \quad y \sim \frac{6c_1}{\ell r} e^{2r} - \frac{c_2}{r^2} e^{-3r}. \quad (27) \]

Thus unlike the previous Spin(7) case, we cannot derive the sign of the mass at infinity, since with an appropriate choice of the constants \( c_1 \) and \( c_2 \), a small perturbation can give either sign.

### 3.2 Analysis for \( \mathbb{C}_7 \) metrics for the four-function truncation

#### 3.2.1 Numerical results

The six-function set of equations (23) can be consistently truncated to a four-function system, by setting corresponding pairs of the \( a_i \) and \( b_i \) equal. Choosing, for example, \( a_2 = a_1 \) and \( b_2 = b_1 \), we obtain

\[ \begin{align*}
\dot{a}_1 &= \frac{a_1^2}{4a_3 b_1} - \frac{a_3}{4b_1} - \frac{b_1}{4a_3} - \frac{b_3}{4a_1}, \\
\dot{b}_1 &= \frac{b_1^2}{4a_1 a_3} - \frac{a_1}{4a_3} - \frac{a_3}{4a_1} + \frac{b_3}{4b_1}, \\
\dot{a}_3 &= \frac{a_3^2}{2a_1 b_1} - \frac{a_1}{2b_1} - \frac{b_1}{2a_1}, \\
\dot{b}_3 &= \frac{b_3^2}{4a_1^2} - \frac{b_1^2}{4b_1}. \end{align*} \quad (28) \]

This system is easier to analyse, and for the purposes of a numerical analysis, we find that we get much more stable and reliable results.

We shall again look for solutions where there is a collapsing \( S^1 \) at short distance. The Taylor expansions are in fact those following from (24) by setting \( q_1 = q_2 \). Without loss of generality we shall make a scale choice, and set \( q_1 = q_2 = 1 \), and take \( q_3 = q \), giving

\[ \begin{align*}
a_1 &= 1 - \frac{1}{8} q t + \frac{1}{128} (16 - 3q^2) t^2 + \cdots, \\
a_3 &= -t - \frac{1}{128} (q^2 - 1) t^3 + \cdots, \\
b_1 &= 1 + \frac{1}{8} q t + \frac{1}{128} (16 - 3q^2) t^2 + \cdots, \\
b_3 &= q + \frac{1}{16} q^3 t^2 + \cdots. \end{align*} \quad (29) \]

Numerical analysis indicates that the solution is regular if \( |q| \leq q_0 = 0.917181 \cdots \), with \( |q| = q_0 \) giving an AC solution, and \( |q| < q_0 \) giving ALC solutions in which \( b_3 \) becomes
constant at infinity.\(^3\) Again we see that at short distance the circle spanned by \((\sigma_3 - \Sigma_3)\) collapses, while at large distance the circle spanned by \((\sigma_3 + \Sigma_3)\) stabilises. Thus here too we see that the ALC metrics have a \(T^{1,1}\) bolt at short distance, and approach \(S^1\) times a cone over \(T^{1,1}\) at large distance, but the \(T^{1,1}\) spaces in the two regions correspond to two different embeddings of \(S^1\) in \(S^3 \times S^3\). Again, we have the feature that the metric coefficient \(b_3^2\) is finite and non-zero everywhere, including short distances and large distances.

### 3.2.2 Global structure of \(\mathbb{C}_7\) metrics

To understand the effect of this new type of “slump” in the \(G_2\) manifolds, it is useful to express the \(SU(2)\) left-invariant 1-forms in terms of Euler angles:

\[
\begin{align*}
\sigma_1 &= \cos \psi_1 \, d\theta_1 + \sin \psi_1 \sin \theta_1 \, d\phi_1, \quad \Sigma_1 = \cos \psi_2 \, d\theta_2 + \sin \psi_2 \sin \theta_2 \, d\phi_2, \\
\sigma_2 &= -\sin \psi_1 \, d\theta_1 + \cos \psi_1 \sin \theta_1 \, d\phi_1, \quad \Sigma_2 = -\sin \psi_2 \, d\theta_2 + \cos \psi_2 \sin \theta_2 \, d\phi_2, \\
\sigma_3 &= d\psi_1 + \cos \theta_1 \, d\phi_1, \quad 
\Sigma_3 = d\psi_2 + \cos \theta_2 \, d\phi_2.
\end{align*}
\]

(Eq. 30)

Making the following redefinitions,

\[
\begin{align*}
\psi &= \psi_1 + \psi_2, \\
\tilde{\psi} &= \psi_1 - \psi_2,
\end{align*}
\]

(Eq. 31)

we have

\[
\begin{align*}
ds_7^2 &= dt^2 + (a_1^2 + b_1^2) \left( d\theta_1^2 + d\theta_2^2 + \sin^2 \theta_1 \, d\phi_1^2 + \sin^2 \theta_2 \, d\phi_2^2 \right) \\
&\quad - 2(a_1^2 - b_1^2) \left( \sin \tilde{\psi} \left( \sin \theta_1 \, d\theta_2 \, d\phi_1 - \sin \theta_2 \, d\theta_1 \, d\phi_2 \right) \\
&\quad \quad + \cos \tilde{\psi} \left( \sin \theta_1 \, d\theta_2 + \sin \theta_2 \, d\theta_1 \, d\phi_2 \right) \\
&\quad \quad + a_3^2 \left( d\tilde{\psi} + \cos \theta_1 \, d\phi_1 - \cos \theta_2 \, d\phi_2 \right)^2 \\
&\quad \quad + b_3^2 \left( d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2 \right)^2. \quad \text{(Eq. 32)}
\end{align*}
\]

At small distance, the geometry is an \(\mathbb{R}^2\) bundle over a squashed \(T^{1,1}/\mathbb{Z}_N\), with \(\psi\), the fibre coordinate, having any period of the form \(4\pi/N\). Regularity of the collapsing circles requires that \(\tilde{\psi}\) must have period of \(2\pi\). At large distance, since the maximum allowed period for \(\tilde{\psi}\), which forms the fibre coordinate in the \(T^{1,1}\) there, would have been \(4\pi\), this means that we have a \(T^{1,1}/\mathbb{Z}_2\) at large distance.

It is worth remarking that from (32) we can see that \(K \equiv \partial/\partial \psi\) is an exact Killing vector everywhere. Furthermore, as we already noted, the metric coefficient \(b_3^2\) is finite

\(^3\)The metric on the \(T^{1,1}\) bolt at \(t = 0\) would itself be Einstein if \(q = 2/\sqrt{3} \sim 1.154701 \cdots\), and so all of the non-singular seven-dimensional metrics correspond to situations where the \(T^{1,1}\) bolt is squashed along its \(U(1)\) fibres, relative to the length needed for the Einstein metric.
and non-vanishing everywhere, which means that the length of $K$ is everywhere finite and non-zero. As far as we are aware, there is no other known example of a Ricci-flat metric with such a $U(1)$ isometry, and in which the circle is not a metric product. In fact there are necessary conditions that must be satisfied by any space that admits such a property, which we can verify are indeed satisfied by $\mathbb{C}_7$. Firstly, the Euler number must be zero. Since $\mathbb{C}_7$ is an $\mathbb{R}^2$ bundle over $T^{1,1}$, and $T^{1,1}$ is itself topologically $S^2 \times S^2$, it follows, since $S^3$ has no relevant non-trivial topology, that $\mathbb{C}_7$ is topologically the product of $S^3$ with an $\mathbb{R}^2$ bundle over $S^2$. Since $S^3$ has zero Euler number, it follows that $\mathbb{C}_7$ has also. A second necessary condition is that the $U(1)$ Killing vector must not be hypersurface orthogonal. In other words, viewed as a 1-form, we must have $K \wedge dK \neq 0$. It is easily verified that indeed $K \wedge dK$ is non-zero in our case.

### 3.2.3 Wrapped D6-branes

Let us consider an eleven-dimensional “vacuum” comprising the direct product of four-dimensional Minkowski spacetime and a $\mathbb{C}_7$ manifold of $G_2$ holonomy. We observed in section 3.2.2 that the length of the Killing vector $K = \partial / \partial \psi$ is everywhere finite and non-zero, and this means that we can perform a Kaluza-Klein reduction on the circle parameterised by $\psi$ that will be completely non-singular.

By reducing on the coordinate $\psi$ we then obtain wrapped D6-branes of type IIA theory in the conifold, with metric

$$ds_{10}^2 = e^{-\frac{4}{3}\phi} dx^\mu dx_\mu + e^{\frac{4}{3}\phi} ds_6^2. \tag{33}$$

With the period of $\psi$ taken to be $4\pi / N$, the resulting D6-branes carry $N$ charges, with a string coupling constant given by

$$g_{str} = e^{\phi} = \left( \frac{b_3}{N} \right)^{3/2}. \tag{34}$$

Thus the string coupling constant is finite and non-vanishing everywhere. The geometry of the wrapped D6-brane is $\mathbb{R}^2 \times S^2 \times S^2$ at small distance, and the cone of $T^{1,1}$ at large distance. Note that this is a very different resolution of a D6-brane wrapped around the $T^{1,1}$ cycles from those that have been obtained previously. In previous examples, the non-singular resolution is manifest only from an eleven-dimensional viewpoint; in $D = 10$, the previous solutions were either singular, or suffered from an instability of the dilaton. Here, by contrast, the ten-dimensional solution is completely regular, with a stable dilaton. It follows that this supergravity solution at large $N$ limit is a good approximation everywhere
for $N$ D6-branes wrapped around the $S^2 \times S^2$, providing another good supergravity dual for the $D = 4, \mathcal{N} = 1$ super Yang-Mills theory.

The circle parameterised by $\psi$ in this $G_2$ manifold also provides a stable $S^1$ that can be used for performing a T-duality transformation between the type IIA and type IIB theories, without encountering any singularity.

To close this discussion of the dimensionally-reduced metric, we remark that, as discussed in [16], the $\sigma_i$ and $\Sigma_i$ left-invariant $SU(2)$ 1-forms can be embedded within the left-invariant 1-forms $L_{AB}$ of $SO(4)$ as

$$
\begin{align*}
\sigma_1 &= L_{42} + L_{31}, & \sigma_2 &= L_{23} + L_{41}, & \sigma_3 &= L_{34} + L_{21}, \\
\Sigma_1 &= L_{42} - L_{31}, & \Sigma_2 &= L_{23} - L_{41}, & \Sigma_3 &= L_{34} - L_{21},
\end{align*}
$$

(35)

where $dL_{AB} = L_{AC} \wedge L_{CB}$. In terms of this basis, the six-dimensional metric in (33) takes the form

$$
\begin{align*}
ds_6^2 &= dt^2 + \tilde{a}^2 L_{11}^2 + \tilde{b}^2 L_{22}^2 + \tilde{c}^2 L_{12}^2,
\end{align*}
$$

(36)

where $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are functions of a radial variable $t$, and $i = 3, 4$. The metric (36) is written precisely in the form used in [17] for writing the the Stenzel metric on the cotangent bundle of $S^3$ (i.e. the deformation of the conifold, first constructed in [18]). Thus the wrapped D6-brane can be recognised as a generalisation of the metric on the product of $(\text{Minkowski})_4 \times (\text{Stenzel})_6$, in which the dilaton and the R-R 2-form field strength of the type IIA theory are also excited.

## 4 ALG gravitational instantons and periodic monopoles

### 4.1 ALG solutions

In a recent paper, Cherkis and Kapustin [7] have discussed ”ALG” gravitational instantons. These are asymptotically locally invariant under a tri-holomorphic $T^2$ action and in some respects resemble the Atiyah-Hitchin metric. The reason for their choice of the name is presumably that G follows F in the alphabet. Exact expressions for these metrics are not known, but asymptotically, an ALG metric may be cast in the form

$$
\begin{align*}
ds^2 &= \frac{1}{\tau_2} |dy_1 + \tau dy_2|^2 + \tau_2 dz d\bar{z},
\end{align*}
$$

(37)

where $y_1$ and $y_2$ are real, and $\tau = \tau(z) = t_1 + i \tau_2$ is a holomorphic or anti-holomorphic function of $z$. Thus $\tau_2$ is a harmonic function, and the metric (37) is of the standard adapted form with respect to the tri-holomorphic Killing vector field $\partial/\partial y_1$. The Killing
vector field $\partial/\partial y_2$ is also tri-holomorphic, and passing to coordinates adapted to it involves a Legendre transformation, which may be regarded as an example of mirror symmetry. The two coordinates $y_1$ and $y_2$ are periodically identified, and so locally the metric (37) admits an elliptic fibration, i.e. a fibration by tori labelled by the complex coordinate $z$, and carrying a unimodular metric parameterised by $\tau(z)$.

It should be emphasised that neither the asymptotic form (37) nor the tri-holomorphic $T^2$ isometry group remains valid in the interior of the manifold. In fact in general, the exact interior metric will have no Killing vectors at all. In special cases, or at special points in the modulus space, it may admit a non-tri-holomorphic circle action. In what follows we shall use $\theta$ for the angle conjugate to this Killing field, which in adapted coordinates takes the form $\partial/\partial \theta$.

One of the metrics considered in [7] is the relative modulus space for two BPS Yang-Mills monopoles on $\mathbb{E}^2 \times S^1$ with the product metric, the circle having radius $L$. In the limit that $L$ goes to infinity it approaches the Atiyah-Hitchin metric, which is the relative modulus space of two BPS Yang-Mill monopoles on $\mathbb{E}^3$. Because the system is invariant under the Euclidean group acting on $\mathbb{E}^2$, the relative modulus space is invariant under the action of $SO(2) \subset E(2)$, which has the effect of rotating the monopoles about their centre of mass. In the limit that $L$ tends to infinity, the symmetry is enhanced to $SO(3) \supset SO(2)$. The $SO(2)$ action is not tri-holomorphic and, as we shall see in more detail later, this has the consequence that the metric can be written in terms of a solution of the $su(\infty)$ Toda equation.

The other metrics considered in [7] have $n$ additional Dirac-type monopoles present, with $n = 1, 2, 3, 4$. They show, using the techniques of [4], that the appropriate choice for $\tau$ is

$$\tau = i \left( \frac{\tau_{\text{ren}} - ib}{2} + \frac{1}{\pi} \log(\overline{z}) - \frac{1}{4\pi} \sum_{j=1}^{n} \log(z - m_j) \right).$$

(38)

The asymptotic metric is complete as $|z|$ goes to infinity. The volume growth is quadratic: the volume inside a region of proper radius $r$ grows as $r^2$. If $n \neq 4$, the circle in the $y_1$ direction collapses in size while the length of the circle in the $y_2$ direction blows up. If $n = 4$, the metric is (up to identifications) asymptotic to the flat metric on $\mathbb{C} \times T^2$. If $m_i = 0$ for $i = 1, 2, 3, 4$, the asymptotic metric is exactly flat. In general the metric has a non-trivial “monodromy,” meaning that traversing a large circle in the base induces an $SL(2, \mathbb{Z})$ transformation of the torus. From a Kaluza-Klein perspective, the solution behaves like a cosmic string carrying Kaluza-Klein (or in the string context, Ramond-Ramond) flux, associated to the two asymptotic $U(1)$ isometries. This monodromy phenomenon is therefore
related to the work of [19] on “Kaluza-Klein vortices.” In the case \( n = 4 \), the fluxes vanish and hence there is no monodromy.

4.2 Monodromy and the Heisenberg group

The metric (37) with \( n = 0 \) admits a mono-holomorphic axial symmetry. Together with the two tri-holomorphic symmetries we get a three-dimensional isometry group acting on three-dimensional orbits. In other words the metric is of cohomogeneity one. In fact the metric is of Bianchi type II, i.e. it is invariant under the action of the Heisenberg group. Related metrics occur as supersymmetric domain walls [20]. However, by contrast with the domain-wall metrics, for which the Heisenberg action is tri-holomorphic, in the present case the action is mono-holomorphic. Roughly speaking, the distinction between the two cases is whether the group action commutes with supersymmetry. Another way of expressing the difference is that if the curvature is self-dual, and \( K \) is a Killing vector with \( k = K_a \, dx^a \) the associated 1-form, then it is tri-holomorphic if \( dk \) is self-dual, or mono-holomorphic if \( dk \) is anti-self-dual. If \( n \neq 0 \) the metric (37) is not exactly axisymmetric, unless \( m_i = 0 \), but it becomes so near infinity. The limiting metric therefore also admits a Heisenberg action. To see this explicitly, note that at large distances (or exactly, if \( n = 0 \)) we may take (after convenient rescalings and choices of inessential constants)

\[
\tau_1 = \left(1 - \frac{n}{4}\right) \theta, \quad \tau_2 = \left(1 - \frac{n}{4}\right) \log r. \tag{39}
\]

Let

\[
\sigma_1 = dy_2, \quad \sigma_2 = d\theta, \quad \sigma_3 = dy_1 + \left(1 - \frac{n}{4}\right) \theta \, dy_2. \tag{40}
\]

One has

\[
d\sigma_1 = d\sigma_2 = 0, \quad d\sigma_3 = -\left(1 - \frac{n}{4}\right) \sigma_1 \wedge \sigma_2. \tag{41}
\]

Thus \((\sigma_1, \sigma_2, \sigma_3)\) are left-invariant 1-forms on the Heisenberg group, and the metric may be cast in the standard triaxial form

\[
ds^2 = a^2 b^2 c^2 \, d\eta^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2, \tag{42}
\]

with

\[
r = e^\eta, \quad a^2 = c^{-2} = \tau_2, \quad b^2 = r^2 \tau_2. \tag{43}
\]

The Killing vectors are

\[
e_1 = \partial_{y_1}, \quad e_2 = \partial_{y_2}, \quad e_3 = \partial_\theta - \left(1 - \frac{n}{4}\right) y_2 \partial_{y_1}. \tag{44}
\]
The only non-vanishing bracket is
\[ [e_2, e_3] = -(1 - \frac{n}{4}) e_1. \] (45)

One may check that the metric (42) with (43) coincides with the triaxial self-dual Bianchi II metric (see equation (24b) of the first reference in [21], after setting \( \lambda_2 = 1 \)). The triholomorphic case (which has \( a^2 = c^2 \)) is obtained by instead setting \( \lambda_2 = 0 \). This admits an extra \( U(1) \) isometry. These two cases are analogous to the Atiyah-Hitchin metric, which is triaxial (being invariant merely under \( SO(3) \)), and the Eguchi-Hanson metric, with its isometry group \( U(2) \). The latter is biaxial and admits an additional \( U(1) \) isometry. The first-order equations implying \( SU(2) \) holonomy can be derived from a superpotential. For the sake of completeness, we shall present the superpotentials for the both the Bianchi IX system, when \( d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k \), and the Heisenberg or Bianchi II case, which can be taken to have \( d\sigma_1 = d\sigma_2 = 0 \), \( d\sigma_3 = -\sigma_1 \wedge \sigma_2 \) (a simple rescaling of (41)).

The first-order equations implying \( SU(2) \) holonomy can be derived from a superpotential. For the sake of completeness, we shall present the superpotentials for the Bianchi IX system, when \( d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k \), and the Heisenberg or Bianchi II case, which can be taken to have \( d\sigma_1 = d\sigma_2 = 0 \), \( d\sigma_3 = -\sigma_1 \wedge \sigma_2 \) (a simple rescaling of (41)).

The conditions for Ricci-flatness for the metrics (42) can be derived from the Lagrangian \( L = T - V \) where both for Bianchi IX and Bianchi II, the kinetic terms are given by
\[
T = \frac{1}{2} g_{ij} \frac{d\alpha^i}{d\eta} \frac{d\alpha^j}{d\eta}, \quad g_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},
\] (46)
where \((a, b, c) = (e^\alpha_1, e^\alpha_2, a^3)\). The potentials are

Bianchi IX: \( V = \frac{1}{4} (a^4 + b^4 + c^4 - 2b^2 c^2 - 2a^2 b^2) \),
Bianchi II: \( V = \frac{1}{4} c^4 \).

These potentials can be derived from superpotentials, \( V = -\frac{1}{2} g^{ij} \partial W / \partial \alpha^i \partial W / \partial \alpha^j \), with

Bianchi IX (1): \( W = \frac{1}{2} (a^2 + b^2 + c^2) \),
Bianchi IX (2): \( W = \frac{1}{2} (a^2 + b^2 + c^2 - 2b c a - 2a b) \),
Bianchi II (1): \( W = \frac{1}{2} c^2 + k a c \),
Bianchi II (2): \( W = \frac{1}{2} c^2 + k b c \),

where, for the Bianchi II cases, \( k \) is an arbitrary constant. Note that only for \( k = 0 \) can the Bianchi II superpotential be obtained as a scaling limit of the Bianchi IX superpotentials (which both yield the same \( k = 0 \) limit).

The corresponding first-order equations, re-expressed in terms of the proper-distance coordinate \( t \), for which \( dt = a b c d\eta \), are

Bianchi IX (1): \( \dot{a} = \frac{b^2 + c^2 - a^2}{2b c} \), and cyclic,
Bianchi IX (2): \[ \dot{a} = \frac{(b - c)^2 - a^2}{2bc}, \quad \text{and cyclic}, \]
Bianchi II (1): \[ \dot{a} = \frac{c}{2b}, \quad \dot{b} = \frac{c}{2a} + k, \quad \dot{c} = \frac{-c^2}{2ab}, \]
Bianchi II (2): \[ \dot{a} = \frac{c}{2b} + k, \quad \dot{b} = \frac{c}{2a}, \quad \dot{c} = \frac{-c^2}{2ab}, \quad (49) \]

Case (1) for Bianchi IX is the system of equations whose only complete non-singular solution is the Eguchi-Hanson metric, for which two of the three metric functions are equal; however, a general triaxial solution, albeit singular, also exists [22]. Case (2) for Bianchi IX has the Atiyah-Hitchin metrics as its general regular solution [3], with Taub-NUT as a special regular solution if any two of the metric functions are set equal.

For Bianchi II (or Heisenberg), the two cases are equivalent, modulo a relabelling of \(a\) and \(b\). Up to an overall scale, all non-vanishing choices for the constant \(k\) are equivalent. If we take \(k = 0\), the solution has the domain-wall form

\[ ds_4^2 = y dy^2 + y (\sigma_1^2 + \sigma_2^2) + \frac{1}{y} \sigma_3^2 \]

that was discussed in [23, 24, 20]. If, on the other hand, we take \(k = 1\), the solution has the form

\[ ds_4^2 = a_0^2 b_0^2 ye^{2\alpha_0 y} dy^2 + 2a_0^2 y \sigma_1^2 + 2b_0^2 ye^{2\alpha_0 y} \sigma_2^2 + \frac{2}{y} \sigma_3^2. \quad (51) \]

If we set \(y = 2 \log r\), and take the constants to be \(a_0 = b_0 = \frac{1}{2}\), we get the \(n = 0\) metric of Cherkis and Kapustin,

\[ ds^2 = \log r dr^2 + \log r \sigma_1^2 + r^2 \log r \sigma_2^2 + \frac{1}{\log r} \sigma_3^2. \quad (52) \]

Note that the rotational invariance is not manifest because the coordinate \(\theta\) appears explicitly in the metric. This may be avoided by changing to a new variable

\[ y_1 \longrightarrow y_1 + \left(1 - \frac{n}{4}\right) \theta y_2. \quad (53) \]

Using this new coordinate, one may put the metric in canonical form with respect to the non-tri-holomorphic Killing field \(\partial/\partial \theta\), and extract the relevant solution of the Toda equation.

The monodromy phenomenon rests on the fact that, with suitable identifications, one may think of the Heisenberg group as a \(T^2\) bundle over \(S^1\). Traversing a closed loop in the base space leads to an \(SL(2, \mathbb{Z})\) action on the \(T^2\) fibres, whose coordinates are \((y_1, y_2)\). In the case \(n = 0\), going half way around the circle corresponds to interchanging two identical monopoles, and so one must identify \((z, y)\) with \((-z, -y)\).
4.3 Masses

It is important to note that the strength of the logarithmic potential associated with the monopole is $-4$ times the strength of the logarithmic potential associated with each of the Dirac singularities. A similar factor of $-4$ arises in the case of the asymptotic metric of the $D_n$ ALF gravitational instantons obtained by Sen [5]. These have an asymptotic tri-holomorphic circle action, with associated harmonic function on $E^3$ given by

$$1 - \frac{16M}{|x|} + \frac{4M}{|x - x_j|} + \frac{4M}{|x + x_j|}.$$

The first non-constant term is needed to get the asymptotic Atiyah-Hitchin metric (i.e. Taub-NUT with negative mass). In stringy language this is the orientifold 6-plane with negative tension. The remaining terms represent D6-branes with positive tension, with the ratio $(-4)$ of charges being derived from string theory. The form of the contribution of the D6-branes is that needed to make the $CP$ identification. Note that in the case of the $D_2$ metric we have zero mass. This case appears to coincide with the Page-Hitchin metric [25, 26, 27]. The total Kaluza-Klein monopole moment vanishes, and the boundary is an identified product $(S^1 \times S^2)/\Gamma$.

A similar interpretation follows for the asymptotic metrics obtained in [7]. It is natural to regard them as corresponding to orientifolds wrapped over a 2-torus, together with $n$ wrapped D6-branes.

As we remarked earlier, there is in general no reason to suppose that an ALG metric admits any Killing vectors at all, except in the asymptotic large-distance limit. Certainly there can be no tri-holomorphic isometries. As noted above, there is an $SO(2)$ action in the case $n = 0$, but this may not be manifest from the asymptotic form of the metric because of the phenomenon of monodromy. The absence of tri-holomorphic isometries means that the Kaluza-Klein or Ramond-Ramond electric charges associated with the two asymptotic $U(1)$ isometries will not be exactly conserved. Processes in the core will lead to their violation (see [28]).

4.3.1 Olber-Seeliger paradox and negative masses

In this section we point out that approximate constructions using multi-centre metrics [29, 30] will typically involve negative mass-points. Sufficiently close to the negative mass points, the metric signature will become negative definite. This, of course, signals a breakdown of the multi-centre approximation, while the exact solution that the multi-centre metric
approximates will be perfectly regular. However, if the multi-centre approximation is good at large distances, the long-range fields of the negative mass-points may show up there.

To see why negative mass-points are typically required, recall that constructions using blow ups and Eguchi-Hanson metrics may be summarised as follows: One considers $\mathbb{C}^2 \equiv (z, y)/\Lambda$ and blows up the singularities, where $\Lambda$ is a lattice (a discrete abelian group of translations) and $\Gamma$ is an involution which acts as before. At one extreme, if $\Lambda$ has rank zero we get the Eguchi-Hanson metric itself. At the other extreme, if $\Lambda$ has maximal rank, i.e. rank 4, we get the Kummer construction [31, 32]. As intermediate cases, if $\Lambda$ has rank 2 we get the $\mathcal{D}_4$ metrics, whilst if $\Lambda$ has rank 1 we get Page’s periodic but non-stationary instanton [25, 26, 33]. If $\Lambda$ has rank 3 we perhaps get something like the quasi-periodic gravitational instantons of [34].

If one uses the harmonic function ansatz, one is likely to run into the problem that if all the terms are taken to be positive, then the associated expression for the potential is that of a periodic array of charges all of the same sign and the same magnitude, and this sum will not converge. This is essentially the gravitational version of Olber’s (or more strictly Halley’s) paradox in cosmology, and in the gravitational context is usually ascribed to Seeliger. One way of circumventing it is to use a hierarchical distribution along the lines of [35]. Another way is to introduce negative, as well as positive, masses. It is easily seen from the expressions in [34] and [36] that this is indeed how they arrange to get convergent periodic potentials.

### 4.4 Cosmic string solutions

In this sub-section we shall relate the ALG metrics to Stringy Cosmic Strings [37], Dirichlet Instanton corrections [36, 38], the seven-brane of type IIB theory [39], periodic gravitational instantons [34], and work on Kaluza-Klein vortices [19].

From a type IIB perspective, we write the ten-dimensional metric in Einstein gauge as

$$ds^2 = -dt^2 + (dx_9)^2 + (dx_8)^2 + (dx_7)^2 + (dx_6)^2 + (dx_5)^2 + (dx_4)^2 + (dx_3)^2 + e^\phi dz d\bar{z}. \quad (55)$$

The static equations arise from the two-dimensional Euclidean Lagrangian

$$L = R - \frac{(\partial \tau_1)^2 + (\partial \tau_2)^2}{2\tau_2^2}, \quad (56)$$

where $\tau = \tau_1 + i \tau_2 = a + i e^{-\Phi}$ gives a map into the fundamental domain of the modular group $SL(2, \mathbb{Z}) \backslash SO(2, \mathbb{R})/SO(2)$. We may regard the two-dimensional spatial sections as a Kähler manifold, and the harmonic map equations are thus satisfied by the holomorphic
ansatz $\tau = \tau(z)$. We must also satisfy the Einstein condition. Using the formula for the Ricci scalar of the two-dimensional metric, and the holomorphic condition, this reduces to the linear Poisson equation

$$\partial \bar{\partial} (\phi - \log \tau_2) = 0.$$  \hfill (57)

To get the fundamental string one interprets the axion and dilaton as coming from the NS-NS sector. One therefore chooses

$$\phi = \Phi, \quad \tau \propto \log z.$$  \hfill (58)

In four spacetime dimensions the fundamental string is “super-heavy,” and it is not asymptotically conical at infinity.

To get the seven brane, which does correspond to a more conventional cosmic string, one picks

$$j(\tau(z)) = f(z) = \frac{p(z)}{q(z)},$$  \hfill (59)

where $j(\tau)$ is the elliptic modular function and $f(z)$ is a rational function of degree $k$.

The appropriate solution for the metric is

$$e^\phi = \tau_2 \eta^2 \bar{\eta}^2 \left| \prod_{i=1}^{k} (z - z_i)^{-1/12} \right|^2,$$  \hfill (60)

where $\eta(\tau)$ is the Dedekind function. Asymptotically

$$e^\phi \sim (z \bar{z})^{-k/12},$$  \hfill (61)

and therefore the spatial metric is that of a cone with deficit angle

$$\delta = \frac{4k\pi}{24}.$$  \hfill (62)

This may also be verified using the equations of motion and the Gauss-Bonnet theorem. As a result, one can have up to 12 seven-branes in an open universe. To close the universe one needs 24 seven-branes.

The solution has however the following purely gravitational interpretation. One considers the metric

$$ds^2 = g_{ij} dy^i dy^j + e^\phi dz d\bar{z},$$  \hfill (63)

where $g_{ij}$ is the previous unimodular metric on the torus $T^2$ with coordinates $y^i$:

$$g_{ij} = \begin{pmatrix} \tau_2^{-1} & \tau_1 \tau_2^{-1} \\ \tau_1 \tau_2^{-1} & \tau_1^2 \tau_2^{-1} + \tau_2 \end{pmatrix}. $$  \hfill (64)
This differs from our previous expression (37) essentially by a conformal transformation, i.e. a holomorphic coordinate transformation on the base metric of this elliptic fibration.

The metric (63) is self-dual, or hyper-Kähler. If one takes 24 seven-branes one gets an approximation to a K3 surface elliptically fibred over \(\mathbb{CP}^1\). Essentially this suggestion is used in [36] to give an interpretation in terms of D-Instantons. This proposed construction of certain limits of K3 metrics has been vindicated mathematically in [38].

4.5 The \(\mathcal{D}_4\) metric

In this section we shall discuss the ALG metrics in the special case when \(n = 4\). The metric is then asymptotic to the flat metric on \(\mathbb{C} \times T^2/\mathbb{Z}_2\). We shall begin by describing an orbifold model obtained by setting \(m_j = 0, j = 1, 2, 3, 4\), in the asymptotic metric. We shall go on to discuss deformations of the orbifold, and how the orbifold might be blown up. Then we shall discuss an approach to the metric based on the \(\mathfrak{su}(\infty)\) Toda equation.

4.5.1 The orbifold model

The following description owes much to a lecture by Kronheimer several years ago. As far as we are aware, Kronheimer’s work has not appeared explicitly in print. There is considerable overlap with the papers of Cherkis and Kapustin.

We begin by taking \(\mathbb{C} \times T^2\) with complex co-coordinates \((z, y)\), where \(y = y_1 + \tau y_2\). The points \((z, y)\) and \((z, y + Rn_1 + iRn_2)\) with \(n_1, n_2 \in \mathbb{Z}\) and \(R \in \mathbb{R}\) are to be identified. The metric is obtained by setting \(\tau = \text{constant}\) in (37). Note that to specify the metric we need three real parameters, namely \(R\) and \(\tau\), which characterise the size and shape respectively of the 2-torus.

We now quotient by the holomorphic involution

\[
\Gamma : (z, y) \rightarrow (-z, -y),
\]

which in polar coordinates defined by \(z = re^{i\theta}\) with \(0 < r < \infty, 0 \leq \theta < 2\pi\), becomes

\[
\Gamma : (r, \theta, y_1, y_2) \rightarrow (r, \theta + \pi, -y_1, -y_2).
\]

The involution \(\Gamma\) does not act freely, but rather has 4 fixed points:

\[
(0, 0, 0, 0), \quad (0, 0, \frac{R}{2}, 0), \quad (0, 0, 0, \frac{R}{2}), \quad (0, 0, \frac{R}{2}, \frac{R}{2}).
\]

Thus \(\mathcal{M}^{\text{sing}} = \mathbb{C} \times T^2/\Gamma\) has 4 singular points, each locally being isomorphic to \(\mathbb{C}^2/(\pm 1)\). At the fixed points \(\Gamma\), being holomorphic, acts as a self-dual rotation, and so it leaves invariant
anti-self-dual 2-forms and negative chirality spinors. Consequently \( \mathcal{M}^{\text{sing}} \) is locally flat, but it has holonomy given by \( \Gamma \) and hence may be thought of as having distributional self-dual curvature localised at the 4 singular points.

The covering space \( \mathbb{C} \times T^2 \) is a trivial torus bundle over \( \mathbb{C} \). The quotient manifold \( \mathbb{C} \times T^2/\Gamma = \mathcal{M}^{\text{sing}} \) may also be thought of as a torus bundle. In the language of algebraic geometry \( \mathcal{M}^{\text{sing}} \) admits a (singular) elliptic fibration by elliptic curves

\[
T^2 \rightarrow M \xrightarrow{f} \mathcal{M}^{\text{sing}} \\
\downarrow \pi \quad \quad \downarrow \pi_f \\
\mathbb{C} \quad \quad B
\]

where the projection map is \( \pi : (z, y) \rightarrow (z, 0) \). The base space of the induced fibration of \( \mathcal{M}^{\text{sing}} \) is \( \mathbb{C}/(\pm 1) \), since \((\Gamma, \theta)\) and \((r, \theta + \pi)\) must be identified; i.e. it is a cone with deficit angle \( \pi \). The singular fibre lying above the vertex of the cone \( z = 0 \) is not a torus but a tetrahedron with vertices at \( y_1 = 0, y_2 = 0; y_1 = \frac{R}{2}, y_2 = 0; y_1 = R, y_2 = 0; y_1 = R, y_2 = \frac{R}{2} \). The metric on the tetrahedron is flat except at these vertices where there is a deficit angle of \( \pi \). At infinity the geometry of a large boundary surface of fixed radius is \( (S^1 \times T^2)/(\pm 1) \), i.e. we have a twisted torus bundle over \( S^1 \).

The continuous isometries of \( \mathcal{M}^{\text{sing}} \) consist of the isometries of \( \mathbb{C} \times T^2 \) that commute with \( \Gamma \). This leaves only rotations around the vertex of the cone,

\[
\theta \rightarrow \theta + \text{constant},
\]

with Killing vector \( \mathbf{m} = \partial/\partial \theta \). By contrast the Killing vectors \( \partial/\partial y_1 \) and \( \partial/\partial y_2 \), which generate translations in the torus, do not commute with \( \Gamma \).

### 4.5.2 The resolved solution

We now consider “physical picture” of the blown up metric in the sense of Page [32], who elaborated a construction first described in [31] by which the K3 metric is built up from the orbifold \( T^4/(\pm 1) \) with its 16 singular points. Each singular point is locally like \( \mathbb{C}^2/(\pm 1) \).

One may “blow up” these singular points, replacing them by copies of \( \mathbb{C} \mathbb{P}^1 \). Now the blow up of \( \mathbb{C}^2/(\pm 1) \) is the cotangent bundle of \( \mathbb{C} \mathbb{P}^1 \), i.e. \( T^* (\mathbb{C} \mathbb{P}^1) \), and this carries the self-dual Eguchi-Hanson metric. To specify the Eguchi-Hanson metric one needs to give three real parameters, comprising one length scale and two orientations. Equivalently, one must give a self-dual 2-form in \( \mathbb{C}^2 \). This contributes \( 16 \times 3 = 48 \) parameters in total. In addition, to specify the torus one needs to give a further 10 real parameters, making \( 48 + 10 = 58 \) in all.

On the other hand, the specification of a self-dual metric on K3 requires 58 real parameters,
and thus the counting makes it plausible that the metric on K3 may be approximated by replacing a small spherical neighbourhood of each singular point by an Eguchi-Hanson manifold.

This physical picture has been vindicated by subsequent work by mathematicians (see [27] for a review). The passage from the smooth K3 surface to the orbifold limit is referred to as a “type I degeneration,” and convergence is shown in the Gromov-Hausdorff topology.

Let us now apply the idea to the $\mathbb{D}_4$ orbifold $\mathbb{C}^2 \times T^2/(\pm 1)$, which may be regarded as a limit of the $T^4/(\pm 1)$ orbifold. There are 3 real parameters for $T^2$ and $3 \times 4 = 12$ for the four Eguchi-Hanson metrics, making 15 in all. Each $\mathbb{C}P^1$ has self-intersection number +2, and the torus gives a fifth homology 2-cycle, which intersects each of the four $\mathbb{C}P^1$’s at one point. The intersection form is therefore given by the extended Dynkin diagram $\mathbb{D}_4$: the rank and signature are both equal to five, and therefore $b_2^+ = 5$. This agrees with our parameter count, since the number of zero modes of the Lichnerowicz operator is $3 b_2^+ = 15$ [40]. Since the metric is self-dual there are no negative modes, which suggests that these configuration are at worst neutrally stable.

5 The Toda equation

The metrics (37) are flat at infinity, and since the general ideas of Kaluza-Klein theory suggest that deviations from flatness are governed by the Laplace operator with a mass term, the approach to flatness should be exponentially fast. As mentioned above, the metric cannot be expected to have more than one Killing vector. In the orbifold case, $m = m^\alpha \partial / \partial x^\alpha = \partial / \partial \theta$ is a Killing vector, and this can be expected to survive. In terms of the coordinates $z$ and $y$, $\partial / \partial \theta$ is holomorphic, but this complex structure, let us call it $\eta^3$, is privileged. The Killing vector $\partial / \partial \theta$ is not tri-holomorphic; the $U(1)$ it generates will rotate the 2 orthogonal complex structures $\eta^1$ and $\eta^2$ into each other. Thus the Killing vector $m_\alpha$ is anti-self-dual: $m_{[\alpha;\beta]} = - \ast m_{[\alpha;\beta]}$. It follows that the metric may be cast in the form [23]

$$ds^2 = \hat{\nu}^{-1} (2d\theta + \nu_u dv - \nu_v du)^2 + \hat{\nu} [dt^2 + e^\nu (du^2 + dv^2)] ,$$

(70)

where $\nu$ satisfies the Toda equation

$$(e^\nu)^\cdot + \nu_{uu} + \nu_{vv} = 0 ,$$

(71)

with $\cdot$ denoting $\partial / \partial t$. The interpretation of the coordinate $t$ is that it is the moment map associated to $\partial / \partial \theta$ regarded as a Hamiltonian vector field with respect to the privileged
symplectic structure $\omega^3$. The complex coordinate $w = u + iv$ parameterises the symplectic quotient of the 4-manifold by the $U(1)$ action, thus:

$$\omega^3 = \nu e^\nu du \wedge dv - dt \wedge (2d\theta + \nu_u dv - \nu_v du),$$

(72)

and

$$\mathcal{L}_m \eta^3 = 0 \implies i \frac{\partial}{\partial \theta} \eta^3 = 2dt.$$  

(73)

Note that the closure of $\eta^3$ is equivalent to the Toda equation (71).

The geometric picture is as follows. Locally, the manifold is foliated by level sets $t = \text{constant}$. The orbits of the $SO(2)$ action lie in these level sets. The coordinates $(u, v)$ or $(y_1, y_2)$ parameterise the two-dimensional space $\Sigma_2 \equiv T^2$ of orbits. The symplectic form $\eta^3$ descends to give a symplectic form, or area form, on the symplectic quotient $\Sigma_2$. The freedom to choose canonical or Darboux coordinates on the quotient gives rise to the gauge group $SDiff(T^2)$, whose Lie algebra $sdiff(T^2)$ is also known as $su(\infty)$, $A_\infty$ or $w_\infty$.

To summarise the above discussion, we saying that the exact non-singular ALG metrics, whose asymptotic forms are given by (37), will, in certain special cases, admit a non-triholomorphic circle action, and that such ALG metrics must necessarily be contained within the class of metrics (70), where $\nu$ satisfies the Toda equation (71). Although explicitly solving the Toda equation is difficult we may, nevertheless, be able to use it in order to make a large-distance perturbative analysis. Thus we may take an explicit large-distance solution of the form (37) as a zeroth-order starting point, re-express it in the Toda metric form (70), and then look for perturbations around it that satisfy the Toda equation (71). The asymptotic solution of the form (37) that provides the zeroth-order starting point must be one that is expected to extend to an exact solution with a non-triholomorphic circle action. Two natural candidates present themselves, namely the solutions with the holomorphic function $\tau(z)$ given by (38) for $n = 0$, or for $n = 4$ with $\overline{m}_j = 0$. This latter example is in fact nothing but the flat metric, and we shall study it first.

5.1 Perturbation around the $n = 4$ flat metric

This first example is obtained by setting $n = 4$ in (38), and taking the parameters $\overline{m}_j$ all to be zero. This implies that $\tau(z)$ will be a constant. By making a convenient choice of moduli for the resulting metric, in which we set $\tau = i$ to get a square torus, we see that the metric (37) becomes

$$ds^2 = dy_1^2 + dy_2^2 + dz \overline{dz}.$$  

(74)
Within the Toda class of metrics (70), the flat metric corresponds to taking

\[ e^\nu = At, \quad (75) \]

where \( A \) is a constant which may, by suitable rescaling of \( u \) and \( v \), be set to 4. We may then identify \( 2(u, v) \) in (70) with \( (y_1, y_2) \) in (74). The metric (70) becomes

\[ ds^2 = 4t \, d\theta^2 + \frac{1}{t} \, dt^2 + |dy|^2. \quad (76) \]

The radial proper distance is \( \rho = 2\sqrt{t} \), and if \( 0 \leq \theta < 2\pi \), we get a flat metric on a cone with deficit angle \( \pi \). Writing

\[ e^\nu = 4t + \lambda, \quad (77) \]

where \( \lambda \) is small compared with \( t \), we find that (71) becomes

\[ t \, \lambda_{tt} + \lambda_{uu} + \lambda_{vv} = 0. \quad (78) \]

If we make the ansatz

\[ \lambda = f(t) \exp(i(k_1 u + k_2 v)), \quad (79) \]

then if \( \lambda \) is small compared with \( t \) near infinity, the regular solution takes the form \( f \sim \rho \, K_1(k \rho) \) where \( K_1 \) is a modified Bessel function, and so asymptotically we have

\[ f \sim (\text{const}) \rho^{\frac{3}{2}} e^{-\rho r}, \quad (80) \]

where

\[ k \equiv \sqrt{k_1^2 + k_2^2}. \quad (81) \]

Thus indeed the deviations from flatness, which necessarily entail a \( U(1) \times U(1) \) symmetry violation, fall off exponentially as claimed above.

The terms \((\nu_u \, dv - \nu_v \, du)\) in the metric (70) correspond to the presence of magnetic fields along the \( x^3 \) direction. These fall off exponentially away from the core region.

Later, in section 5.2.1, we shall discuss the \( n = 0 \) metric in the class described by (37) with \( \tau \) given by (39).

### 5.2 Metrics with both self-dual and anti-self-dual Killing vectors

In this case the metric depends upon a free function of two variables. It is easy to specify this function if one uses the harmonic function description adapted to the tri-holomorphic Killing vector \( \partial/\partial \psi \) say. It appears to be difficult to translate this explicitly into the solution of the Boyer-Finley [8, 9] equation depending upon two variables that one obtains if one uses
the Toda description adapted to the mono-holomorphic Killing vector $\partial/\partial \theta$ and imposes translational symmetry.

Suppose the metric with the tri-holomorphic Killing vector is also axisymmetric, and thus it takes the form

$$ds^2 = V^{-1} (d\psi + \omega d\theta)^2 + V (dz^2 + d\rho^2 + \rho^2 d\theta^2),$$

where $V$ and $\omega$ depend only on $\rho$ and $z$. We have

$$\rho V_z = \omega, \quad \rho V_{\rho} = -\omega z.$$ (83)

The privileged symplectic form is

$$\eta^3 = (d\psi + \omega d\theta) \wedge dz + V \rho d\theta \wedge d\rho,$$ (84)

and is closed by virtue of the second equation in (83). The closure guarantees the local existence of the moment map $t$ which satisfies

$$i_{\partial/\partial \theta} \eta^3 = 2 dt = \omega dz + \rho V d\rho.$$ (85)

Thus

$$2t_z = \omega, \quad 2t_{\rho} = \rho V.$$ (86)

whence Note that adding a constant $c$ to $V$ shifts $t$ by $c \rho$. We also have another closed one-form given by

$$2du = V dz - \frac{\omega}{\rho} d\rho.$$ (87)

If we now set $e^\nu = \rho^2$ and use the chain rule we find that $\nu = \nu(t, u)$ will satisfy the Toda equation with no $v$ dependence. If we set $2v = \tau$ we can pass between the two forms of the metric. In particular

$$\frac{\partial \nu}{\partial t} = \frac{2V}{\omega^2 + \rho^2 V^2}$$ (88)

and

$$\frac{\partial \nu}{\partial u} = \frac{2\omega}{\omega^2 + \rho^2 V^2}.$$ (89)

A different approach [41] to relating solutions of the Toda equation to harmonic functions starts with an axisymmetric harmonic function $H(\rho, z)$. If

$$T = \frac{1}{2} \rho H_{\rho}, \quad x = -H_z,$$ (90)

and $f = \log(\rho^2/4)$ then we get a translation-invariant solution of the Toda equation

$$(e^f)_{TT} + f_{xx} = 0.$$ (91)
One may reverse the steps. Given \( f \), one may obtain \( H \) and then construct a metric with triholomorphic \( U(1) \) by setting \( V = 1 + H_z \) and \( \omega = - \rho H_\rho \). Note that starting with a given \( V \) and \( \omega \) one gets in general two different solutions of the Toda equation. Only the first approach leads to the conventional Toda form of the metric and the moment map \( t \).

### 5.2.1 The \( n = 0 \) Cherkis-Kapustin metric

After appropriate rescalings and elimination of inessential constants, in this case we can take the holomorphic function \( \tau(z) \) in (37) to be given by (39) with \( n = 0 \), implying

\[
\tau_1 = \theta, \quad \tau_2 = \log r.
\]

(92)

After the transformation (53), and converting to the notation of (82) the metric (37) then becomes

\[
ds^2 = \frac{1}{\log \rho} (d\psi - z d\theta)^2 + \log \rho (dz^2 + d\rho^2 + \rho^2 d\theta^2),
\]

(93)

with \( V = \log \rho \) and \( \omega = -z \).

The transformation to coordinates of the Toda metrics (70) can be effected by noting from (83) that we have the two exact 1-forms

\[
dt = w dz + V \rho d\rho = -z dz + \rho \log \rho d\rho,
\]

\[
du = V dz - \frac{\omega}{\rho} d\rho = \log \rho dz - \frac{z}{\rho} d\rho,
\]

(94)

which can be integrated to give

\[
t = -\frac{1}{2} z^2 + \frac{1}{4} \rho^2 (2 \log \rho - 1), \quad u = z \log \rho.
\]

(95)

We also make the identification \( 2v = \tau \). It is easily verified that the function \( \nu \) in the Toda form of the metric is given by

\[
e^\nu = \rho^2,
\]

(96)

and, by use of the chain rule, that this indeed satisfies the Toda equation (71).

### 5.2.2 The Calderbank-Tod-Nutku-Sheftel solution

This metric [42, 43] is a solution of the Toda or Boyer-Finley equation depending upon two holomorphic functions \( a(u + i v) \) and \( b(u + i v) \), one of which is essentially the gauge freedom to make holomorphic coordinate transformations of \( u + i v \). Thus in effect it depends on one analytic function, or one free function of two variables, which is just the freedom in giving

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4We thank Paul Tod and Richard Ward for helpful discussions on this and related points.
one of the metrics in the previous section. However the two solutions are distinct [42]. The Calderbank-Tod-Nutku-Sheftel solution is

\[ e^\nu = \frac{|(t + a(u + i v)) b'(u + i v)|^2}{(1 + |b(u + i v)|^2)^2}. \]  

(97)

5.2.3 Separable solutions

These are of the form

\[ e^\nu = (t^2 + 2\beta t + \gamma) e^{\phi(u, v)}, \]  

(98)

where \( \beta \) and \( \gamma \) are constants, and \( \phi \) is a solution of the Liouville equation

\[ (\partial_u \partial_u + \partial_v \partial_v) \phi = 2e^{\phi}. \]  

(99)

The general solution of Liouville’s equation leads to

\[ e^\nu = \frac{(t^2 + 2\beta t + \gamma) |b'(u + i v)|^2}{(1 + |b(u + i v)|^2)^2}, \]  

(100)

where \( b(u + i v) \) is again an arbitrary holomorphic function. As before, the freedom to change coordinates means that one might as well set \( b(u + i v) = u + iv \), and shifting \( t \) allows one to set \( \beta = 0 \). One gets the Eguchi-Hanson metric in this way.

5.2.4 Further solutions

Other known solutions have been recast in the Toda form, including the Atiyah-Hitchin metric. Because the \( SO(3) \) isometry group does not act tri-holomorphically, there is a solution of the Toda equation for every \( SO(2) \) subgroup. These have been worked out by Bakas and Sfetsos [44] and may be useful in suggesting an ansatz for the more complicated ALG case.

As explained in [7] and noted earlier by Kronheimer and Nakajima, one may obtain ALG metrics from the modulus spaces of solutions of the Hitchin equations on certain Higgs bundles over surfaces. The latter may be considered as the dimensional reduction of the self-dual Yang-Mills equations on \( \mathbb{E}^2 \times S \), where \( S \) is a Riemann surface.

5.3 Some properties of the Toda system: Lax pair and spinors

We recall a few useful properties of the Toda equation. Firstly, we may introduce a quantity \( \phi \) by \( \nu = -\ddot{\phi} \). The equation (71) then becomes

\[ \partial \bar{\partial} \phi = \exp(-\ddot{\phi}). \]  

(101)
This is manifestly of Toda form if one thinks of the operator \(-\frac{\partial^2}{\partial t^2}\) as the limit to zero step-size of the leap-frog difference operator \(-\phi_{n-1} + 2\phi_n - \phi_{n+1}\). Considered as an infinite matrix, this is the Cartan matrix \(A_\infty\) which is associated to the Lie algebra \(su(\infty)\).

Secondly, a Lax pair may be constructed by considering two time-dependent operators \(L\) and \(N\) acting on \(\mathbb{C}\)-valued functions \(f = f(y_1, y_2)\):

\[
\begin{align*}
N f &= (\partial \nu) f, \\
L f &= \overline{\partial} f + (\dot{\nu} e^\nu) f. 
\end{align*}
\] (102)

One has

\[
[N, L] f = - (\partial \overline{\partial} \nu) f, \tag{103}
\]

and hence

\[
\dot{L} = [N, L]. \tag{104}
\]

The operator \(L\) is a modified Dirac operator acting on spinors defined over the 2-dimensional quotient manifold \(\Sigma_2 \equiv T^2\).

### 6 Conclusion

In this paper, we have studied several examples of new metrics with special holonomy, and their significance in string theory and M-theory. We first considered the \(C_8\) metrics whose existence was demonstrated by numerical analysis in [2]. They have cohomogeneity one, with \(S^7\) principal orbits, described as triaxially-squashed \(S^3\) bundles over \(S^4\), which degenerate to \(\mathbb{C}P^3\) on a bolt at short distance. The \(\mathbb{C}P^3\) itself is viewed as an \(S^2\) bundle over \(S^4\), and a non-trivial parameter \(\lambda\) in the \(C_8\) metrics characterises the squashing of the \(\mathbb{C}P^3\). Regular metrics arise for \(\lambda^2 \leq 4\), with the limiting case \(\lambda^2 = 4\) being the standard asymptotically conical (AC) Ricci-flat Kähler Calabi metric on the complex line bundle over the Fubini-Study metric on \(\mathbb{C}P^3\). For \(\lambda^2 < 4\) the metrics have \(\text{Spin}(7)\) holonomy, and they are asymptotically locally conical (ALC). At large distance they are locally of the form of a product of a circle of stabilised radius and an AC 7-metric of \(G_2\) holonomy. The metric functions in the three directions in the \(S^3\) fibres in \(\mathbb{C}8\) behave similarly to those in the Atiyah-Hitchin metric, and we find that this leads to a natural orientifold picture for an M-theory solution (Minkowski)\(_4 \times \mathbb{C}_8\) reduced on the asymptotic circle, with D6-branes wrapped over \(S^4\).

We next considered a new class of complete \(G_2\) metrics, which are of cohomogeneity one with \(S^3 \times S^3\) principal orbits. By starting from a short-distance Taylor expansion,
and then studying the evolution to large distances numerically, we established that there exist regular metrics, which we denote by $\mathbb{C}_7$, in which the principal orbits degenerate to a $T^{1,1}$ bolt at short distance. The metrics are ALC, locally approaching the product of a stabilised circle and the deformed six-dimensional conifold over $T^{1,1}/\mathbb{Z}_2$ at large distance. An intriguing feature of these metrics is that the circle whose radius stabilises at large distance also remains non-singular at short distance. Thus if we perform a Kaluza-Klein dimensional reduction of an M-theory solution (Minkowski)$^4 \times \mathbb{C}_7$ on this circle, we get a type IIA solution with an everywhere non-singular dilaton. It admits an interpretation as a D6-brane wrapped over $S^2 \times S^2$.

We then considered a new class of four-dimensional hyper-Kähler metrics described by Cherkis and Kapustin [7], following earlier related work by Kronheimer. Only the asymptotic form of these metrics is known explicitly; at large distance they are characterised by a holomorphic or anti-holomorphic function, and they admit a tri-holomorphic $T^2$ action. In the interior, the metrics will not in general admit any Killing symmetries at all, although in special cases there can be a mono-holomorphic $U(1)$ symmetry. In such cases the metric falls into a class described in terms of a function satisfying the $su(\infty)$ Toda equation, and this can provide a way of studying the behaviour of the metric in the interior region.

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**Appendices**

**A Atiyah-Hitchin metric**

In this appendix, we review some basic results about the Atiyah-Hitchin metric. We begin in section A.1 with a review of the solution itself, as discussed in [3]. Then, in section A.2, we give a discussion, based on one in [4], of the global structure of the Atiyah-Hitchin metric, focusing especially on the identifications of the angular coordinates on the principal orbits.
A.1 The solution

We write the metric in the form

$$ds_4^2 = dt^2 + \frac{1}{4}(a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + a_3^2 \sigma_3^2),$$  \hspace{1cm} (105)

where the metric functions satisfy the first-order equations

$$\dot{a}_1 = \frac{a_2^2 - (a_2 - a_3)^2}{a_2 a_3}, \quad \dot{a}_2 = \frac{a_3^2 - (a_3 - a_1)^2}{a_3 a_1}, \quad \dot{a}_3 = \frac{a_1^2 - (a_1 - a_2)^2}{a_1 a_2}. \hspace{1cm} (106)$$

These can be solved by defining a new coordinate $\tau$, related to $t$ by $dt = -\frac{1}{8}(a_1 a_2 a_3) u^{-2} d\tau$, with $u$ being a solution of

$$\frac{d^2 u}{d\tau^2} + \frac{1}{4} u \cosec^2 \tau = 0. \hspace{1cm} (107)$$

It can then be verified that the solution is given by

$$a_1 = 2 \sqrt{\frac{w_2 w_3}{w_1}}, \quad a_2 = 2 \sqrt{\frac{w_3 w_1}{w_2}}, \quad a_3 = -2 \sqrt{\frac{w_1 w_2}{w_3}}, \hspace{1cm} (108)$$

where

$$w_1 = -u u' - \frac{1}{2} u^2 \cosec \tau, \quad w_2 = -u u' + \frac{1}{2} u^2 \cot \tau, \quad w_3 = -u u' + \frac{1}{2} u^2 \cosec \tau, \hspace{1cm} (109)$$

and $u'$ means $du/d\tau$. The solution of (107) is taken to be

$$u = \frac{1}{\pi} \sqrt{\sin \tau} K(\sin^2 \frac{1}{2} \tau), \hspace{1cm} (110)$$

where

$$K(k) \equiv \int_0^{\pi/2} \frac{d\phi}{(1 - k \sin^2 \phi)^{1/2}} \hspace{1cm} (111)$$

is the complete elliptic integral of the first kind. The coordinate $\tau$ ranges from $\tau = 0$ at the bolt, to $\tau = \pi$ at infinity.

Near $\tau = 0$ we have

$$a_1 = \frac{1}{8} \tau^2 + \frac{1}{192} \tau^2 + \cdots, \hspace{1cm} a_2 = 1 + \frac{1}{32} \tau^2 + \cdots, \hspace{1cm} a_3 = -1 + \frac{1}{32} \tau^2 + \cdots. \hspace{1cm} (112)$$

In terms of the proper-distance coordinate $t = \frac{1}{32} \tau^2 + \cdots$, we see that near the bolt we shall have

$$ds_4^2 \sim dt^2 + 4t^2 \sigma_1^2 + \frac{1}{4} (\sigma_2^2 + \sigma_3^2). \hspace{1cm} (113)$$
Near $\tau = \pi$, which is the asymptotic region at infinity, one finds that $a_1$ and $a_2$ grow linearly with proper distance, while $a_3$ tends to a constant. To see this, we use the facts that

\[
K(1 - \epsilon) \sim -\frac{1}{2} \log \epsilon - \log 2 + O(\epsilon \log \epsilon),
\]

\[
E(1 - \epsilon) \sim 1 - \frac{1}{4}(\log 2 + 2 \log 2) \epsilon + O(\epsilon^2 \log \epsilon),
\]

as $\epsilon$ goes to zero, where $E(k) = \int_0^{\pi/2} \sqrt{1 - k \sin^2 \phi} d\phi$ is the complete elliptic integral of the second kind. In particular, we see that $a_3$ approaches the value

\[
a_3(\infty) = -\frac{2}{\pi}
\]

at infinity, while $a_1$ and $a_2$ have the asymptotic form $a_1 \sim a_2 \sim \pi^{-1} \log \epsilon$, where $\epsilon = \cos^2 \frac{1}{2}\tau$.

We also find $u \sim -e^{-(\sqrt{2}/2) \epsilon^{1/4} \log \epsilon}$, and hence $\epsilon \sim e^{2\pi t}$. At leading order, the metric at infinity therefore approaches

\[
ds^2_4 \sim dt^2 + t^2 (\sigma_1^2 + \sigma_2^2) + \frac{1}{\pi^2} \sigma_3^2.
\]

### A.2 Global considerations

We shall represent the left-invariant 1-forms $\sigma_i$ in terms of Euler angles

\[
\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi,
\]

\[
\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi,
\]

\[
\sigma_3 = d\psi + \cos \theta d\phi.
\]

These are the natural ones to use in the asymptotic region near infinity, where it is $\sigma_3^2$ that has a coefficient tending to a constant.\(^5\) However, near the bolt, where the coefficient of $\sigma_1^2$ goes to zero, it is more natural to consider a redefined set of 1-forms $\tilde{\sigma}_i$, where

\[
\sigma_1 = \tilde{\sigma}_3, \quad \sigma_2 = \tilde{\sigma}_1, \quad \sigma_3 = \tilde{\sigma}_2.
\]

These can be represented in terms of tilded Euler angles:

\[
\tilde{\sigma}_1 = -\sin \tilde{\psi} d\tilde{\theta} + \cos \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi},
\]

\[
\tilde{\sigma}_2 = \cos \tilde{\psi} d\tilde{\theta} + \sin \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi},
\]

\[
\tilde{\sigma}_3 = d\tilde{\psi} + \cos \tilde{\theta} d\tilde{\phi},
\]

\(^5\)The roles of $\sigma_1$ and $\sigma_2$ are reversed compared with our usual conventions. This is in order to maintain the same conventions as are customarily used when discussing the Atiyah-Hitchin metric. In particular, we now have $d\sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k$.\]
An explicit relationship between the tilded and untilded Euler angles can be given, but we do not need that here.

It is clear from (113) that regularity near the bolt requires that \( \tilde{\psi} \) should have period \( \pi \). We now need to express this in terms of the original untilded Euler angles, which we shall be using in the asymptotic region. To do this, let us begin by assuming that the \( \sigma_i \), and hence also the \( \tilde{\sigma}_i \), are left-invariant 1-forms on \( SO(3) \) rather than \( SU(2) \), meaning that the periods of the fibre coordinates will be \( \Delta \psi = 2\pi \), and also \( \Delta \tilde{\psi} = 2\pi \). We see that this means we should identify \( \tilde{\psi} \) further, under a discrete transformation \( \tilde{I}_3 \) defined by

\[
\tilde{I}_3: \quad \tilde{\theta} \longrightarrow \tilde{\theta}, \quad \tilde{\phi} \longrightarrow \tilde{\phi}, \quad \tilde{\psi} \longrightarrow \pi + \tilde{\psi}.
\] (120)

Observe that \( \tilde{I}_3 \) has the following action on the three \( \tilde{\sigma}_i \) 1-forms:

\[
\tilde{I}_3(\tilde{\sigma}_1) = -\tilde{\sigma}_1, \quad \tilde{I}_3(\tilde{\sigma}_2) = -\tilde{\sigma}_2, \quad \tilde{I}_3(\tilde{\sigma}_3) = +\tilde{\sigma}_3.
\] (121)

Note that this leaves the algebra, \( d\tilde{\sigma}_i = \frac{1}{2} \epsilon_{ijk} \tilde{\sigma}_j \wedge \tilde{\sigma}_k \), invariant. One can also define permuted analogues of this discrete operation, which instead leave either \( \tilde{\sigma}_1 \) or \( \tilde{\sigma}_2 \) fixed in sign, while reversing the signs of the other two. These are defined by

\[
\tilde{I}_1(\tilde{\sigma}_1) = +\tilde{\sigma}_1, \quad \tilde{I}_1(\tilde{\sigma}_2) = -\tilde{\sigma}_2, \quad \tilde{I}_1(\tilde{\sigma}_3) = -\tilde{\sigma}_3, \\
\tilde{I}_2(\tilde{\sigma}_1) = -\tilde{\sigma}_1, \quad \tilde{I}_2(\tilde{\sigma}_2) = +\tilde{\sigma}_2, \quad \tilde{I}_2(\tilde{\sigma}_3) = +\tilde{\sigma}_3.
\] (122)

It can be seen from (119) that the corresponding actions of \( \tilde{I}_1 \) and \( \tilde{I}_2 \) on the tilded Euler angles will be

\[
\tilde{I}_1: \quad \tilde{\theta} \longrightarrow \pi - \tilde{\theta}, \quad \tilde{\phi} \longrightarrow \pi + \tilde{\phi}, \quad \tilde{\psi} \longrightarrow -\tilde{\psi}, \\
\tilde{I}_2: \quad \tilde{\theta} \longrightarrow \pi - \tilde{\theta}, \quad \tilde{\phi} \longrightarrow \pi + \tilde{\phi}, \quad \tilde{\psi} \longrightarrow \pi - \tilde{\psi}.
\] (123)

We want to see how \( \tilde{I}_3 \) acts on the untilded Euler angles. Since the tilded and untilded quantities are related by permutation, as in (118), it follows that there will be an identical permutation relation for the discrete operators, namely

\[
I_1 = \tilde{I}_3, \quad I_2 = \tilde{I}_1, \quad I_3 = \tilde{I}_2,
\] (124)

where the \( I_i \) act on untilded quantities exactly as \( \tilde{I}_i \) act on tilded quantities. Thus we have

\[
I_1: \quad \theta \longrightarrow \pi - \theta, \quad \phi \longrightarrow \pi + \phi, \quad \psi \longrightarrow -\psi, \\
I_2: \quad \theta \longrightarrow \pi - \theta, \quad \phi \longrightarrow \pi + \phi, \quad \psi \longrightarrow \pi - \psi, \\
I_3: \quad \theta \longrightarrow \theta, \quad \phi \longrightarrow \phi, \quad \psi \longrightarrow \pi + \psi.
\] (125)
In particular, we see that the identification $I_3$ that we needed in order to get a metric regular near the bolt implies, in terms of the untilded coordinates that it is natural to use near infinity, that we must identify under $I_1$, namely

$$I_1: \quad \theta \rightarrow \pi - \theta, \quad \phi \rightarrow \pi + \phi, \quad \psi \rightarrow -\psi.$$  \hfill (126)

It can be noted that one is also free to impose in addition an identification under $I_3$, since this has no fixed points on the bolt. But it should be emphasised that the identification under $I_1$ is obligatory, whilst the further identification under $I_3$ is optional.

A.3 Asymptotic behaviour of the Atiyah-Hitchin metric

The first-order equations for the Atiyah-Hitchin system are given by (106). In terms of a new radial coordinate $r$, defined by $dt = h dr$, a simple known solution is the Taub-NUT metric, given by

$$a_1 = a_2 = 2(r^2 - m^2)^{1/2}, \quad a_3 = 4m \left( \frac{r - m}{r + m} \right)^{1/2}, \quad h = \left( \frac{r + m}{r - m} \right)^{1/2}. \hfill (127)$$

We now look for a solution perturbed around the Taub-NUT metric, in which $a_1$ and $a_2$ are unequal, but where they approach equality asymptotically at large $r$. We make a perturbative expansion

$$a_1 = A_0 + A_1, \quad a_2 = A_0 + A_2, \quad a_3 = B_0 + A_3, \quad h = h_0 + h_1,$$  \hfill (128)

where $A_0$, $B_0$ and $h_0$ will now be the zeroth-order solution above, i.e.

$$A_0 = 2(r^2 - m^2)^{1/2}, \quad B_0 = 4m \left( \frac{r - m}{r + m} \right)^{1/2}, \quad h_0 = \left( \frac{r + m}{r - m} \right)^{1/2}, \hfill (129)$$

and we shall work to linear order in the functions $A_1$, $A_2$, $A_3$ and $h_1$.

We can choose a gauge for $h_1$ such that $A_2 = -A_1$, by imposing

$$h_1 = \frac{h_0 A_3}{2A_0 - B_0}. \hfill (130)$$

The linearised equations for $A_1$ and $A_3$ then become

$$A'_1 = \frac{4h_0 A_1}{B_0} - \frac{h_0 B_0 A_1}{A_0^2},$$

$$A'_3 = \frac{h_0 B_0 (4A_0 - B_0) A_3}{A_0^2 (2A_0 - B_0)}. \hfill (131)$$

A simple solution can be obtained by taking $A_3 = 0$. We see that the solution for $A_1$ is given by

$$A_1 \sim e^{\int \left( \frac{4h_0}{B_0} - \frac{h_0 B_0}{A_0^2} \right)}.$$

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From the zeroth-order solution (129) we therefore find that the dominant large-$r$ behaviour for $A_1$ will be

$$A_1 \sim e^{\int \frac{4m}{B_0} } \sim e^{r/m} .$$

(133)

It follows that our assumption that $A_1$ is a small perturbation will be valid only if

$$m < 0 ,$$

(134)

and hence the Atiyah-Hitchin metric is asymptotic to Taub-NUT with a negative mass.

To see this explicitly, consider the M-theory solution

$$d\hat{s}_{11}^2 = dx^\mu dx_\mu + ds_4^2 ,$$

(135)

where $dx^\mu dx_\mu$ is the metric on 8-dimensional Minkowski spacetime. We reduce this to the Einstein-frame metric in $D = 10$, using the standard Kaluza-Klein reduction

$$d\hat{s}_{11}^2 = e^{-\frac{1}{6} \phi} ds_{10}^2 + e^{\frac{4}{3} \phi} (dz + A_{(1)})^2 .$$

(136)

The reduction is performed on the circle parameterised by $\psi$ in (117), we can be done at large distance because $a_1$ and $a_2$ become equal asymptotically, and so $\partial/\partial \psi$ asymptotically becomes a Killing vector. Thus we get

$$ds_{10}^2 = a_3^{1/4} dx^\mu dx_\mu + a_3^{1/4} (dt^2 + a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2) .$$

(137)

The metric coefficient $g_{00}$ for $dx^0 dx^0$ is therefore given by

$$g_{00} = -a_3^{-1/4} .$$

(138)

For the Taub-NUT metric we therefore have

$$g_{00} = (4m)^{1/4} \left(1 - \frac{m}{4r} + \cdots\right) ,$$

(139)

allowing us to read off the mass as $M = \frac{1}{4} m$. Thus our perturbative discussion above has shown that the mass $M$ is negative in the Atiyah-Hitchin solution.

References


