The Master Ward Identity

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Abstract

In the framework of perturbative quantum field theory (QFT) we propose a new, universal (re)normalization condition (called 'master Ward identity') which expresses the symmetries of the underlying classical theory. It implies for example the field equations, energy-momentum, charge- and ghost-number conservation, renormalized equal-time commutation relations and BRST-symmetry.

It seems that the master Ward identity can nearly always be satisfied, the only exceptions we know are the usual anomalies. We prove the compatibility of the master Ward identity with the other (re)normalization conditions of causal perturbation theory, and for pure massive theories we show that the 'central solution' of Epstein and Glaser fulfills the master Ward identity, if the UV-scaling behavior of its individual terms is not relatively lowered.

Application of the master Ward identity to the BRST-current of non-Abelian gauge theories generates an identity (called 'master BRST-identity') which contains the information which is needed for a local construction of the algebra of observables, i.e. the elimination of the unphysical fields and the construction of physical states in the presence of an adiabatically switched off interaction.

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Contents

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1 Introduction

A perturbative interacting quantum field theory is usually constructed in terms of time ordered products ('T-products') $T(W_1,...,W_n)(x_1,...,x_n)$ of Wick polynomials $W_1(x_1),...,W_n(x_n)$. The T-products are non-unique for coinciding points. In the framework of the inductive construction of Bogoliubov [4] and Epstein/Glaser [18] ('causal perturbation theory') this can be formulated as follows: the T-products of n-factors are known by induction as operator-valued distributions up to the total diagonal $D_n \stackrel{\mathrm{def}}{=} \{(x_1,...,x_n) \mid x_1 = ... = x_n\}$. The problem of renormalization is located in the extension of the T-products to D_n,

for every n. This extension is always possible, but it is non-unique. The freedom is restricted by normalization conditions. They require that symmetries which are present outside D_n are maintained in the extension and that an interaction with mass dimension ≤ 4 yields a renormalizable theory (by power counting). Epstein/Glaser [18] (see also [5]) give a general formula (74) for the extension to D_n which satisfies the renormalizability requirement, but the other normalization conditions are not taken into account. So, *the main problem of perturbative renormalization is to prove that there is an extension which fulfills all normalization conditions*. In the framework of algebraic renormalization the corresponding problem is treated by means of the 'quantum action principle' [25], [24], [29], which states that the variation of Green's functions (under a change of coordinates, a variation of the fields or a variation of a parameter) is equal to the insertion of a (local or space-time integrated) composite field operator. Recently a local algebraic operator formulation of certain cases of the quantum action principle has been given by using causal perturbation theory, and the connection to our normalization conditions has been clarified [8].

The **master Ward identity** (we will use the abbreviation 'MWI') is a universal normalization condition supplementing the obvious ones. It is an explicit expression for

$$\partial_x^\nu T(W, W_1, ..., W_n)(x, x_1, ..., x_n) - T(\partial^\nu W, W_1, ..., W_n)(x, x_1, ..., x_n). \qquad (1)$$

Generally this difference cannot vanish for the following reason[1]: the Wick polynomials are built up from free fields, whereas the T-products are the building stones of the perturbative interacting fields [4]. However, the field equations of free and interacting fields are different.

Computing the difference (1) by means of the Feynman rules, it can be expressed solely by terms which contain the difference $\partial_x^\nu \langle \Omega, T(\phi, \chi)(x, x_l)\Omega\rangle - \langle \Omega, T(\partial^\nu \phi, \chi)(x, x_l)\Omega\rangle$ of Feynman propagators, where Ω is the Fock vacuum and ϕ, χ are free fields. The MWI requires that this structure is preserved in the process of renormalization (sect. 2). For tree diagrams this is automatically satisfied, but for loop diagrams it is a hard task to show that there exists a normalization which fulfills the MWI and the other normalization conditions (sect. 3). Unfortunately there are a few examples where this is impossible. However, the only obstructions we know are the usual, well-known anomalies of perturbative QFT (sect. 5).

The *master Ward identity expresses the inner symmetries of the underlying classical theory*[2]. In particular we will demonstrate that it implies

[1] In particular this argument, the name 'master Ward identity' and the application of the MWI to the computation of a rigorous substitute for the equal-time commutator of interacting fields (3) are due to Klaus Fredenhagen.

[2] In [9] we extensively work out the MWI in classical field theory. There the MWI can be formulated non-perturbatively: it is a consequence of the field equations and the fact that classical fields may be multiplied point-wise. Hence, the classical MWI holds always true. (This, together with the fact that in the perturbative expansion of classical fields solely tree diagrams appear, agrees with the triviality of the quantum MWI for tree diagrams.) The classical formulation of the MWI shows that there is a close connection to the Schwinger-Dyson equations.

- the field equations for the interacting fields (sect. 4.1),
- conservation of the energy-momentum tensor (sect. 5.2),
- charge conservation in the presence of spinor fields (sect.4.2),
- ghost number conservation in the presence of fermionic ghost fields (sect. 4.2), and
- the **master BRST-identity** (sect. 4.4-5), which contains the full information of BRST-symmetry [2] for massless and massive gauge fields.

The field equations, conservation of the energy-momentum tensor, charge and ghost number conservation have already been proved by using other methods of renormalization (see e.g. [39] for the field field equation and [26] for the energy-momentum tensor, both are based on BPHZ-renormalization) or/and in the framework of causal perturbation theory [38],[12], [27], [7]. Using the normal products of Zimmermann [39], Lowenstein has proved that it is allowed to take a partial derivative out of a Green's function, if the degree of BPHZ-subtraction is lowered by one, see appendix B of [26]. However, we are not aware of a formulation of the MWI in its full generality in any method of renormalization.

Also the master BRST-identity is new to our knowledge. It is the answer to the obvious question: what results for

$$[Q_0, T(W_1, ..., W_n)(x_1, ..., x_n)]_\mp \tag{2}$$

if the MWI is satisfied? Thereby, $W_1, ..., W_n$ are arbitrary Wick monomials, Q_0 is the generator of the BRST-transformation of the free fields and $[\cdot, \cdot]_\mp$ means the $\mathbb{Z}_2$-graded (with respect to the ghost number) commutator. There have been other approaches to formulate BRST-symmetry in the framework of causal perturbation theory. In particular the 'perturbative gauge invariance' of [11], [14] and [33], which was further developed by [36], [20] and [15], suffices for a consistent construction of the S-matrix in the adiabatic limit, provided this limit exists. However, this assumption holds certainly not true in massless non-Abelian gauge theories, it seems that the confinement is out of the reach of perturbation theory. In massive non-Abelian gauge theories the instability of physical particles (W- and Z-bosons, muon and tau etc.) is an obstacle for an S-matrix description. Our way out is to *construct the observables locally* (i.e. with the interaction adiabatically switched off, sect. 4.5), as we have done it for QED [7]. For our operator formalism the BRST-charge operator of Kugo and Ojima [22] seems to be the adequate tool to define the BRST-transformation. But in contrast to this reference we do not perform the adiabatic limit and, hence, avoid the infrared divergences. The mentioned perturbative gauge invariance [11], [14] does not suffice for our local construction of observables in non-Abelian (massless or massive) gauge theories. But we show that ghost number conservation and the master BRST-identity contain all information which is needed for this construction. In particular we will see that the master BRST-identity implies the perturbative gauge invariance of [11], [14] (and even the generalization proposed in [17], which is called 'generalized (free perturbative operator) gauge invariance' in [15]).

In spite of all these important implications of the MWI, it is difficult to give a direct physical interpretation of this identity (in its full generality) or to

formulate the symmetry which is expressed by it. We give two partial answers:
- In classical field theory the MWI can be understood as a refined formulation of the field equations. But in QFT it contains much more information than only the field equations, because interacting quantum fields may not be multiplied point-wise, see [9].
- A particular case of the MWI is a formula for $\partial_{x_0} T(A_{\mathcal{L}}(x)B_{\mathcal{L}}(y)) - T(\partial_{x_0} A_{\mathcal{L}}(x)B_{\mathcal{L}}(y))$, where $A_{\mathcal{L}}, B_{\mathcal{L}}$ are interacting fields to the interaction $\mathcal{L}$ (non-local formal power series in free fields) and $T(...)$ means time ordering in x and y. This difference can be interpreted as a *rigorous substitute for the equal-time commutator of $A_{\mathcal{L}}$ and $B_{\mathcal{L}}$*:[3] defining heuristically $\tilde{T}(A_{\mathcal{L}}(x)B_{\mathcal{L}}(y)) \stackrel{\text{def}}{=} \Theta(x^0 - y^0)A_{\mathcal{L}}(x)B_{\mathcal{L}}(y) + \Theta(y^0 - x^0)B_{\mathcal{L}}(y)A_{\mathcal{L}}(x)$ we in fact obtain

$$\partial_{x_0} \tilde{T}(A_{\mathcal{L}}(x)B_{\mathcal{L}}(y)) - \tilde{T}(\partial_{x_0} A_{\mathcal{L}}(x)B_{\mathcal{L}}(y)) = \delta(x_0 - y_0)[A_{\mathcal{L}}(x), B_{\mathcal{L}}(y)]. \qquad (3)$$

However, $\tilde{T}(...)$ is problematic: $\Theta(x^0 - y^0)A(x)B(y)$ exists if A and B are free fields, but this does not hold for A and B being Wick polynomials, and for interacting fields the situation is even worse. In addition $\tilde{T}(...)$ is non-covariant: for a Lorentz covariant T-product (denoted by $T(...)$ in the following) and a free scalar field ϕ we must have the relation $\partial_\mu T(\phi(x)\partial_\nu\phi(y)) - T(\partial_\mu\phi(x)\partial_\nu\phi(y)) = Cg_{\mu\nu}\delta(x-y)$ (where C is an undetermined constant), which is obviously not satisfied by $\tilde{T}(...)$. But fortunately there is the possibility that the non-covariant terms (i.e. the terms coming from $\tilde{T}(...) - T(...)$) cancel out with other unwanted terms. This indeed happens for the (interacting) quark currents $j^\mu_{a\mathcal{L}}$ in QCD: the (heuristic) equal-time commutator $[j^0_{a\mathcal{L}}(t,\vec{x}), j^k_{a\mathcal{L}}(t,\vec{y})], k = 1,2,3$ has an 'anomalous' term, the Schwinger term:

$$[j^0_{a\mathcal{L}}(t,\vec{x}), j^k_{a\mathcal{L}}(t,\vec{y})] = if_{abc} j^k_{c\mathcal{L}}(t,\vec{x})\delta^{(3)}(\vec{x}-\vec{y}) + \sum_{l=1}^{3} S^{kl}_{ab}\partial^l\delta^{(3)}(\vec{x}-\vec{y}), \qquad (4)$$

where $S^{kl}_{ab} \in \mathbb{C}$ is constant. In formula 11-89 of [21] it is postulated that the non-covariant terms of $\tilde{T}(...)$ are compensated by the Schwinger terms:

$$\partial^x_\mu T(j^\mu_{a\mathcal{L}}(x) j^\nu_{b\mathcal{L}}(y)) - T(\partial_\mu j^\mu_{a\mathcal{L}}(x) j^\nu_{b\mathcal{L}}(y)) = if_{abc} j^\nu_{c\mathcal{L}}(x)\delta^{(4)}(x-y). \qquad (5)$$

We will show that this identiy is in fact a consequence of the MWI (sect. 4.3).

We return to the crucial question whether the MWI can be satisfied in agreement with all other normalization conditions. The compatibility with the other normalization conditions can be proved generally (sect. 3.2). If all fields are massive there is a distinguished normalization of the T-products, the so-called central solution [18]. We prove that the central solutions fulfil the MWI (and all other normalization conditions) if the UV-scaling behaviour of its individual terms is not relatively lowered (sect. 3.3). This assumption holds mostly true. However, e.g. for the axial and pseudo-scalar triangle-diagram (90) it is violated, and this makes possible the appearance of the axial anomaly.

[3] We recall the well-known fact that interacting fields to a sharp time do not exist, i.e. $\int d^3x\, f(\vec{x}) \int dx^0\, \delta(x_0 - t)A_{\mathcal{L}}(x)$, $t \in \mathbb{R}$, $f \in \mathcal{D}(\mathbb{R}^3)$, is mathematically ill-defined.

2 Formulation of the master Ward identity

2.1 The symbolical algebra with internal and external derivative

To each free quantum field ϕ_l which appears in the model ($(\phi_l)_l$ runs also through all partial derivatives of arbitrary order of these fields) we associate a symbol $\varphi_l = \mathrm{sym}(\phi_l)$ and *neglect* (for a moment) *the free field equations for the quantum fields*. Let $\mathcal{P}$ be the unital, Abelian $*$-algebra[4] generated by these symbols. Thereby the symbols corresponding to a free quantum field and to its derivatives are linearly independent. The $*$-operation in $\mathcal{P}$ corresponds to taking the adjoint of the free field operators: $\varphi_l^* \equiv \mathrm{sym}(\phi_l)^* \stackrel{\mathrm{def}}{=} \mathrm{sym}(\phi_l^+)$. We define an *internal derivative* $\partial^\mu : \mathcal{P} \to \mathcal{P}$ by $\partial^\mu \varphi_l \equiv \partial^\mu \mathrm{sym}(\phi_l) \stackrel{\mathrm{def}}{=} \mathrm{sym}(\partial^\mu \phi_l)$ and the requirements that ∂^μ is linear and a derivation. Now we divide $\mathcal{P}$ by the ideal J which is generated by the free field equations (with respect to the internal derivative) and denote the resulting unital, Abelian $*$-algebra by $\mathcal{P}_0$,

$$\mathcal{P}_0 \stackrel{\mathrm{def}}{=} \frac{\mathcal{P}}{J}. \tag{6}$$

Let π be the projection $\pi : \mathcal{P} \to \mathcal{P}_0 : A \to A + J$. Internal derivatives in $\mathcal{P}_0$ are defined by[5] $\partial^\mu \pi(A) \stackrel{\mathrm{def}}{=} \pi(\partial^\mu A)$, and in this sense the free field equations are valid in $\mathcal{P}_0$.

In addition we introduce an *external derivative*[6] $\tilde{\partial}^\mu$ on $\mathcal{P}_0$ which generates new symbols $\tilde{\partial}^a A$ ($A \in \mathcal{P}_0$, $a \in \mathbb{N}_0^4$, i.e. $\tilde{\partial}^a$ means a higher external derivative of order $|a| = a_0 + a_1 + a_2 + a_3$) and is required to be linear and a derivation. In particular we set $\tilde{\partial}^a \mathbf{1} \stackrel{\mathrm{def}}{=} 0$, $\forall a \neq 0$. The Abelian, unital $*$-algebra (anticommuting in the case of Fermi fields) which is generated by these new symbols is denoted by $\tilde{\mathcal{P}}_0$:

$$\tilde{\mathcal{P}}_0 \stackrel{\mathrm{def}}{=} \bigvee \{\tilde{\partial}^a A \,|\, A \in \mathcal{P}_0, a \in \mathbb{N}_0^4\}. \tag{7}$$

If $(A_j)_{j \in \mathbb{N}}$ is a vector space basis of $\mathcal{P}_0$, then $(\tilde{\partial}^a A_j)_{j \in \mathbb{N}, a \in \mathbb{N}_0^4}$ is a vector space basis of $\tilde{\mathcal{P}}_0$. Next we extend the external and internal derivatives and the $*$-operation to maps $\tilde{\mathcal{P}}_0 \to \tilde{\mathcal{P}}_0$. For the former two we set

$$\tilde{\partial}^b \tilde{\partial}^a A \stackrel{\mathrm{def}}{=} \tilde{\partial}^{(b+a)} A, \qquad \partial^b \tilde{\partial}^a A \stackrel{\mathrm{def}}{=} \tilde{\partial}^a \partial^b A, \quad \forall A \in \mathcal{P}_0, \tag{8}$$

and require that $\tilde{\partial}^\mu$ and ∂^μ are linear and derivations. The $*$-operation is extended by $(\tilde{\partial}^a A)^* \stackrel{\mathrm{def}}{=} \tilde{\partial}^a (A^*) \quad (A \in \mathcal{P}_0)$ and by requiring the usual algebraic relations: anti-linearity, $(BC)^* = C^* B^*$ and $B^{**} = B$, $\forall B, C \in \tilde{\mathcal{P}}_0$.

[4] In the case of Fermi fields the symbols anticommute.

[5] Note that this definition is independent from the choice of the representative A.

[6] This external derivative has nothing to do with the external derivative of differential geometry.

Finally we introduce the space

$$\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0) \cong \tilde{\mathcal{P}}_0 \otimes \mathcal{D}(\mathbb{R}^4). \tag{9}$$

The internal and external derivatives are defined on this space as the operators $\partial^\mu \otimes \mathbf{1}$ and $\tilde{\partial}^\mu \otimes \mathbf{1}$.

Remark: There exists a surjective algebra $*$-homomorphism $\tilde{\sigma} : \tilde{\mathcal{P}}_0 \to \mathcal{P}$. This becomes clear from the formalism developed in [9]. Namely, we prove in appendix A of [9] that there exists a map $\sigma : \mathcal{P}_0 \to \mathcal{P}$ (i.e. 'from free fields to fields') with the properties:
(i) $\pi \circ \sigma = \mathbf{1}$.
(ii) σ is an algebra $*$-homomorphism, i.e. σ is linear, $\sigma(AB) = \sigma(A)\sigma(B)$ and $\sigma(A^*) = \sigma(A)^*$.
(iii) The Lorentz transformation commutes with $\sigma\pi$.
(iv) $\sigma\pi(\mathcal{P}_1) \subset \mathcal{P}_1$, where $\mathcal{P}_1$ is the sub vector space of $\mathcal{P}$ with basis $(\varphi_l)_l$, i.e. the 'one-factor symbols'.
(v) σ does not increase the mass dimension of the fields, i.e. $\sigma\pi(B)$ is a sum of terms with mass dimension $\leq \dim(B)$. In particular we find $\sigma\pi(\varphi) = \varphi$, if $\varphi \in \mathcal{P}_1$ corresponds to a free field without any derivative.
(vi) $\bigvee\{\partial^a \sigma(A) | A \in \mathcal{P}_0, a \in \mathbb{N}_0^d\} = \mathcal{P}$.
We now extend σ to a map $\tilde{\sigma} : \tilde{\mathcal{P}}_0 \to \mathcal{P}$ by setting

$$\tilde{\sigma}(\tilde{\partial}^a A) \stackrel{\text{def}}{=} \partial^a \sigma(A), \quad A \in \mathcal{P}_0, \tag{10}$$

and requiring that $\tilde{\sigma}$ is an algebra $*$-homomorphism. The property (vi) means that $\tilde{\sigma}$ is surjective. Let $\varphi \in \mathcal{P}_1$ correspond to a free Klein Gordon field (without any derivative). Usually it holds $\sigma\pi(\partial^\nu \varphi) = \partial^\nu \varphi$. Then, $\tilde{\sigma}(\tilde{\partial}^\nu \pi\varphi) = \partial^\nu \varphi = \tilde{\sigma}(\partial^\nu \pi\varphi)$, i.e. $\tilde{\sigma}$ is not injective. In [9] an additional symbol $\varphi^\mu \in \mathcal{P}_1 \subset \mathcal{P}$ is introduced and the free field equations read $\partial^\mu \varphi = \varphi^\mu$ and $\partial_\mu \varphi^\mu = -m^2 \varphi$. Then $\sigma\pi(\partial^\mu \varphi) = \gamma\varphi^\mu + (1-\gamma)\partial^\mu \varphi$, $\gamma \in \mathbb{R} \setminus \{0\}$. Since $\gamma \neq 0$ the mentioned example does not appear and the map $\sigma : \mathcal{P}_0 \to \mathcal{P}$ constructed in appendix A of [9] fulfills additionally:

$$\partial^a \sigma(A) \notin \operatorname{ran} \sigma, \ \forall\, 0 \neq a \in \mathbb{N}_0^d, A \in \mathcal{P}_0. \tag{11}$$

This relation implies that the corresponding $\tilde{\sigma}$ (10) is injective and hence $\tilde{\mathcal{P}}_0$ and $\mathcal{P}$ may be identified.

2.2 Inductive construction of time ordered products, basic normalization conditions (N0)-(N3)

The time-ordered product T_n (also called 'T-product') is a linear, symmetrical[7] map from $\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)^{\otimes n}$ into the (unbounded) operators on the Fock space of the

[7]To distinguish the symmetry of T_n from other symmetries we sometimes call it 'permutation symmetry'.

free quantum fields[8]. All T-products $T_n(f_1 \otimes ... \otimes f_n),\ f_j \in \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0),\ n \in \mathbb{N}$, have the same domain $\mathcal{D}$ which is a dense subspace of the Fock space and which is invariant under all T-products [18]. Sometimes we use 'unsmeared T-products', which are defined by

$$\int dx_1...dx_n\, T_n(V_1, ..., V_n)(x_1, ..., x_n) g_1(x_1)...g_n(x_n) \stackrel{\text{def}}{=} T_n(V_1 g_1 \otimes ... \otimes V_n g_n), \tag{12}$$

where $g_1, ..., g_n \in \mathcal{D}(\mathbb{R}^4),\ V_1, ..., V_n \in \tilde{\mathcal{P}}_0$.

Let $\tilde{\mathcal{P}}_0 \ni V = \prod_k \tilde{\partial}^{a^{(k)}} \text{sym}(\phi_{j_k})$. Then we define T_1 by

$$T_1(Vg) \stackrel{\text{def}}{=} \int dx\ :\prod_k \partial^{a^{(k)}} \phi_{j_k} : (x) g(x), \quad T_1(\mathbf{1}g) \stackrel{\text{def}}{=} \int dx\, g(x), \quad g \in \mathcal{D}(\mathbb{R}^4), \tag{13}$$

and by linearity, where the double dots mean normal ordering of the free field operators. We point out that T_1 is not injective, because $T_1((\tilde{\partial}^a V)Wg) = T_1((\partial^a V)Wg), V, W \in \tilde{\mathcal{P}}_0$. However, T_1 is injective if it is restricted to $\mathcal{D}(\mathbb{R}^4, \mathcal{P}_0)$.

The T-products are required to satisfy causal factorization[9]

$$\begin{aligned} &(\textbf{Causality}) \quad T_n(f_1 \otimes ... \otimes f_n) = T_k(f_1 \otimes ... \otimes f_k) T_{n-k}(f_{k+1} \otimes ... \otimes f_n) \\ &\text{if} \quad \big(\text{supp}\, f_1 \cup ... \cup \text{supp}\, f_k\big) \cap \big((\text{supp}\, f_{k+1} \cup ... \cup \text{supp}\, f_n) + \bar{V}_-\big) = \emptyset, \end{aligned} \tag{14}$$

where $\bar{V}_-$ is the closed backward light cone in Minkowski space. Causality enables us to construct inductively the T-products of higher orders $n \geq 2$: if the time ordered products of less than n factors are everywhere defined, the time ordered product of n factors is uniquely determined up to the total diagonal $D_n \stackrel{\text{def}}{=} \{(x_1, ..., x_n) \mid x_1 = ... = x_n\}$. Thus renormalization amounts to an extension, for every n, of time ordered products to D_n. This extension is always possible, but it is non-unique. It can be done such that the following **normalization conditions** hold. (Note that these conditions are automatically fulfilled on $\mathcal{D}(\mathbb{R}^{4n} \setminus D_n)$ due do the inductive procedure and causal factorization.)

• **Poincare covariance**: Let U be a unitary positive energy representation of the Poincare group $\mathcal{P}_+^\uparrow$ in Fock space. U induces an automorphic action α of $\mathcal{P}_+^\uparrow$ on $\mathcal{D}(\mathbb{R}^4, \mathcal{P}_0)$ by the definition

$$T_1(\alpha_L(f)) \stackrel{\text{def}}{=} \text{Ad}\, U(L)(T_1(f)), \qquad \forall f \in \mathcal{D}(\mathbb{R}^4, \mathcal{P}_0),\ L \in \mathcal{P}_+^\uparrow, \tag{15}$$

because T_1 is injective on this subspace. We extend α_L to $\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)$ by the prescription that $\tilde{\partial}^a f$ transforms in the same way as $\partial^a f$. More precisely let

[8] In [9] and [10] the arguments of T_n are elements of $\mathcal{D}(\mathbb{R}^4, \mathcal{P})^{\otimes n}$. According to the above Remark, the two formalisms essentially agree.

[9] This is the reason for the name 'time ordered product'.

$f=\sum_i V_i\otimes g_i$, $V_i\in\mathcal{P}_0$, $g_i\in\mathcal{D}(\mathbb{R}^4)$. From (15) we know the transformation of $\partial^a f$, which can be written in the form

$$\alpha_{(\Lambda,a)}\partial^a f=\sum_{i,j}\partial^a V_i\otimes D(\Lambda)_{ji}g_{(\Lambda,a)\,j}\,,\quad L\equiv(\Lambda,a)\in\mathcal{P}_+^\uparrow, \tag{16}$$

where $g_{(\Lambda,a)}(x)\stackrel{\mathrm{def}}{=}g(\Lambda^{-1}(x-a))$. Then we define

$$\alpha_{(\Lambda,a)}\tilde{\partial}^a f\stackrel{\mathrm{def}}{=}\sum_{i,j}\tilde{\partial}^a V_i\otimes D(\Lambda)_{ji}g_{(\Lambda,a)\,j}\,,\quad(\Lambda,a)\in\mathcal{P}_+^\uparrow. \tag{17}$$

One easily verifies $\alpha_{L_1L_2}=\alpha_{L_1}\alpha_{L_2}$ and that equation (15) holds true for the extended α_L, i.e. for all $f\in\mathcal{D}(\mathbb{R}^4,\tilde{\mathcal{P}}_0)$. The normalization condition expressing the Poincare covariance of the time ordered products reads

(N1) $\mathrm{Ad}\,U(L)(T(f_1\otimes...\otimes f_n))=T(\alpha_L(f_1)\otimes...\otimes\alpha_L(f_n)),\quad L\in\mathcal{P}_+^\uparrow.$

For pure massive theories the so-called 'central solution/extension' (see [18] and sect. 3.3) is Poincare covariant. For theories with massless fields the existence of a Poincare covariant extension has been proved (in [37] and in the second paper of [11]) by tracing it back to a cohomological problem; an explicit solution has been given in [6].

• **Unitarity**: To explain what we mean by 'unitarity' we introduce the S-matrix (as a formal power series) which is the generating functional of the T-products

$$S(f)=\mathbf{1}+\sum_{n=1}^\infty\frac{i^n}{n!}T_n\big(f\otimes...\otimes f\big),\quad f\in\mathcal{D}(\mathbb{R}^4,\tilde{\mathcal{P}}_0). \tag{18}$$

Since the zeroth order term does not vanish, it has a unique inverse in the sense of formal power series

$$S(f)^{-1}=\mathbf{1}+\sum_{n=1}^\infty\frac{(-i)^n}{n!}\bar{T}_n\big(f\otimes...\otimes f\big), \tag{19}$$

where the 'anti-chronological products' $\bar{T}(...)$ can be expressed in terms of the time ordered products

$$\bar{T}_n\big(f_1\otimes...\otimes f_n\big)\stackrel{\mathrm{def}}{=}\sum_{P\in\mathcal{P}(\{1,...,n\})}(-1)^{|P|+n}\prod_{p\in P}T_{|p|}\big(\otimes_{j\in p}f_j\big)\ . \tag{20}$$

(Here $\mathcal{P}(\{1,...,n\})$ is the set of all ordered partitions of $\{1,...,n\}$, $|P|$ is the number of subsets in P and $|p|$ is the number of elements of p). The reason for the word 'anti-chronological' is that the $\bar{T}$-products satisfy anti-causal factorization, which means (14) with reversed order of the factors on the r.h.s.. Unitarity of the S-matrix is expressed by

$$S(f)^+=S(f^*)^{-1} \tag{21}$$

($^+$ means the adjoint on $\mathcal{D}$, $(\phi, B\psi) = (B^+\phi, \psi)$, $\quad \phi, \psi \in \mathcal{D}$.) Hence, for the T-products we require the normalization condition

$$(\mathbf{N2}) \qquad T_n\big(f_1 \otimes ... \otimes f_n\big)^+ = \bar{T}_n\big(f_1^* \otimes ... \otimes f_n^*\big),$$

which can easily be satisfied by symmetrizing an arbitrary normalized T-product (see [18]).

• **Relation to T-products of sub-polynomials**: Let $\mathcal{G} \subset \mathcal{P}_0$ be a linearly independent set of generators of $\mathcal{P}_0$, i.e. $\mathcal{G}$ is a (vector space) basis of $\pi\mathcal{P}_1$ (see the Remark in sect. 2.1 for the definition of $\mathcal{P}_1$). Then $\tilde{\mathcal{G}} \stackrel{\mathrm{def}}{=} \{\tilde{\partial}^a\varphi \,|\, \varphi \in \mathcal{G}, a \in \mathbb{N}_0^4\}$ is a set of linearly independent generators of $\tilde{\mathcal{P}}_0$. We define the commutator 'function' $\Delta_{\varphi,\chi}$ by

$$i\int dx\, dy\, h(x)g(y)\Delta_{\varphi,\chi}(x-y) \stackrel{\mathrm{def}}{=} [T_1(\varphi h), T_1(\chi g)], \quad \varphi, \chi \in \tilde{\mathcal{G}}. \tag{22}$$

Every $V \in \mathcal{P}_0$ can uniquely be written as a polynomial in the generators $\mathcal{G}$. By partial differentiation in this sense we obtain a sub-polynomial $\frac{\partial V}{\partial \varphi}$, $\forall \varphi \in \mathcal{G}$. For $f(x) = \sum_i f_i(x)V_i,\ f_i \in \mathcal{D}(\mathbb{R}^4),\ V_i \in \mathcal{P}_0$, we set

$$\frac{\partial f}{\partial \varphi} \stackrel{\mathrm{def}}{=} \sum_i f_i(x)\frac{\partial V_i}{\partial \varphi}. \tag{23}$$

For $\psi \in \tilde{\mathcal{G}}$ we analogously define $\frac{\partial}{\partial \psi}$ to be a linear derivation $\tilde{\mathcal{P}}_0 \to \tilde{\mathcal{P}}_0$ ($\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0) \to \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)$ resp.) with

$$\frac{\partial(\tilde{\partial}^a\chi)}{\partial(\tilde{\partial}^b\varphi)} \stackrel{\mathrm{def}}{=} \delta_{a,b}\frac{\partial \chi}{\partial \varphi} = \delta_{a,b}\delta_{\chi,\varphi}, \quad \chi, \varphi \in \mathcal{G}. \tag{24}$$

The derivation property of the commutator $[\cdot, T_1(\chi g)]$ implies

$$[T_1(f), T_1(\chi g)] = i\sum_{\psi \in \tilde{\mathcal{G}}} T_1\Big(\frac{\partial f}{\partial \psi}\Delta_{\psi,\chi} \star g\Big), \quad \forall f \in \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0),\ \chi \in \tilde{\mathcal{G}},\ g \in \mathcal{D}(\mathbb{R}^4), \tag{25}$$

where $\star$ means convolution.

We now generalize the normalization condition **(N3)** of [7] to the present framework: we require

$$(\mathbf{N3}) \qquad [T_n(f_1 \otimes ... \otimes f_n), T_1(\chi g)] = i\sum_{l=1}^{n}\sum_{\psi \in \tilde{\mathcal{G}}} T_n(f_1 \otimes ... \otimes \frac{\partial f_l}{\partial \psi}\Delta_{\psi,\chi} \star g \otimes ... \otimes f_n) \tag{26}$$

where $f_1, ..., f_n \in \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0), \chi \in \tilde{\mathcal{G}}$. The r.h.s. is well-defined because $\Delta_{\psi,\chi} \star g$ is a smooth function.

We point out that the defining properties of the T-products given so far (linearity, symmetry, causality, **(N1)**, **(N2)** and **(N3)**) are purely algebraic conditions, they are independent from the choice of a state. In the realization of the T-products as operators in Fock space, **(N3)** becomes equivalent to the 'causal Wick expansion' of Epstein and Glaser[10], see [18] sect. 4.

• **Scaling degree**: **(N3)** gives the relation to time ordered products of sub-polynomials. Once these are known (in an inductive procedure), only the C-number part of the T-product (which is equal to the Fock vacuum expectation value of the T-product) has to be fixed. Due to translation invariance this scalar distribution depends on the relative coordinates only. Hence, the extension of the (operator valued) T-product to D_n is reduced to the extension of a C-number distribution $t_0 \in \mathcal{D}'(\mathbb{R}^{4(n-1)} \setminus \{0\})$ to $t \in \mathcal{D}'(\mathbb{R}^{4(n-1)})$. (We call t an extension of t_0 if $t(f) = t_0(f)$, $\forall f \in \mathcal{D}(\mathbb{R}^{4(n-1)} \setminus \{0\})$). The singularity of $t_0(y)$ and $t(y)$ at $y = 0$ is classified in terms of Steinmann's scaling degree [35, 5]

$$\mathrm{sd}(t) \stackrel{\mathrm{def}}{=} \inf\{\delta \in \mathbb{R}\,,\ \lim_{\lambda \downarrow 0} \lambda^\delta t(\lambda x) = 0\}. \tag{27}$$

Note

$$\mathrm{sd}(\partial^a t) = \mathrm{sd}(t) + |a| \qquad \text{and} \qquad \mathrm{sd}(\delta^{(m)}) = m, \tag{28}$$

where $\delta^{(m)}$ denotes the m-dimensional δ-distribution. By definition $\mathrm{sd}(t_0) \leq \mathrm{sd}(t)$, and the possible extensions are restricted by requiring

$$(\mathbf{N0}) \qquad\qquad \mathrm{sd}(t_0) = \mathrm{sd}(t). \tag{29}$$

Then the extension is unique for $\mathrm{sd}(t_0) < 4(n-1)$, and in the general case there remains the freedom to add derivatives of the δ-distribution up to order $(\mathrm{sd}(t_0) - 4(n-1))$. In formula:

$$t(y) + \sum_{|a| \leq \mathrm{sd}(t_0) - 4(n-1)} C_a \partial^a \delta(y) \tag{30}$$

is the general solution, where t is a special extension [5, 28, 18], and the constants C_a are restricted by **(N1)**, **(N2)**, permutation symmetries and the normalization conditions ($\tilde{\mathbf{N}}$) (normalization of time ordered products of symbols with external derivative) and **(N)** (MWI) below. For an interaction $\mathcal{L}$ with UV-dimension $\dim(\mathcal{L}) \leq 4$ the requirement (29) implies renormalizability by power counting, i.e. the number of indeterminate constants C_a in $(T_n((g\mathcal{L})^{\otimes n}))_n$ ($g \in \mathcal{D}(\mathbb{R}^4)$) does not increase by going over to higher perturbative orders n.

In the seminal paper [18] Epstein and Glaser prove that there exists an extension to D_n which fulfills the normalization conditions **(N0)-(N3)**, but they say only few about further symmetries which should be maintained in the extension, e.g. the field equations or gauge invariance. The MWI is a universal normalization condition which summarizes the request for most of this 'further symmetries'.

[10]Epstein/Glaser do not use this name, but it appears e.g. in [5].

2.3 Normalization of time-ordered products of symbols with external derivative

The aim of this subsection is to fix the normalization of time-ordered products of symbols with external derivative(s) in terms of time-ordered products without external derivative. This fixation is a necessary ingredient of the formulation of the MWI, because T-products of symbols with external derivatives unavoidably appear in the MWI. Heuristically the external derivative is a derivative which acts *after* having done the time-ordered contractions of the corresponding symbols (free fields resp.), e.g.

$$\begin{gathered} T_{n+1}((\tilde{\partial}^\nu V)g \otimes W_1 f_1 \otimes ... \otimes W_n f_n) = \\ \int dx\, dx_1 ... dx_n\, g(x) f_1(x_1) ... f_n(x_n) \partial^\nu_x T(V, W_1, ..., W_n)(x, x_1, ..., x_n) \equiv \\ -T_{n+1}(V \partial^\nu g \otimes W_1 f_1 \otimes ... \otimes W_n f_n), \end{gathered} \tag{31}$$

where $W_1, ..., W_n \in \tilde{\mathcal{P}}_0$. However, there are other time ordered products involving factors with external derivatives such as $(\tilde{\partial}^\nu V)W$ which cannot be defined in this way in terms of time ordered products of factors without any external derivative. Hence we proceed in an alternative, recursive way: we give an explicit expression for the difference

$$T_{n+1}((\tilde{\partial}^\nu V)Wg \otimes W_1 f_1 \otimes ... \otimes W_n f_n) - T_{n+1}((\partial^\nu V)Wg \otimes W_1 f_1 \otimes ... \otimes W_n f_n) \tag{32}$$

where $V, W, W_1, ..., W_n \in \tilde{\mathcal{P}}_0$. For this purpose we introduce some notations: by means of the Feynman propagator $\Delta^F_{\chi,\psi}$

$$\int dx\, dy\, f(x) g(y) i\Delta^F_{\chi,\psi}(x-y) \stackrel{\text{def}}{=} \langle \Omega, T_2(\chi f \otimes \psi g)\Omega\rangle, \quad \chi, \psi \in \tilde{\mathcal{G}},\ f, g \in \mathcal{D}(\mathbb{R}^4), \tag{33}$$

(where Ω denotes the Fock vacuum) we define

$$\delta^\mu_{\chi,\psi} \stackrel{\text{def}}{=} \partial^\mu \Delta^F_{\chi,\psi} - \Delta^F_{\partial^\mu \chi,\psi}, \quad \chi, \psi \in \tilde{\mathcal{G}}. \tag{34}$$

Inserting the causal factorization of $T_2(...)(x,y)$ for $x \neq y$ we find that $\delta^\mu_{...}$ is a local distribution, hence it has the form

$$\delta^\mu_{\chi,\psi}(z) = \sum_{a \in \mathbb{N}_0^4} C^\mu_{\chi,\psi;a} \partial^a \delta(z), \tag{35}$$

where the $C^\mu_{\chi,\psi;a} \in \mathbb{C}$ are constant numbers. Then we define

$$\begin{aligned} \Delta^\mu_{\chi,\psi} \quad &: \quad \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)^{\times 2} \longrightarrow \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0) \\ \Delta^\mu_{\chi,\psi}(Vg, Wf) \quad &= \quad \sum_a C^\mu_{\chi,\psi;a} (-1)^{|a|} \sum_{0 \leq b \leq a} \\ &\qquad \frac{a!}{b!(a-b)!} (\tilde{\partial}^b V) W (\partial^{(a-b)} g) f \end{aligned} \tag{36}$$

where $|a| = a_0 + a_1 + a_2 + a_3$ and $a! \equiv \prod_\mu a_\mu!$. This formula is motivated by the identity

$$\int dx\, \partial^a \delta(x-y) V(x) g(x) W(y) f(y) =$$

$$\sum_{0 \le b \le a} (-1)^{|a|} \frac{a!}{b!(a-b)!} (\partial^b V)(y) W(y) (\partial^{(a-b)} g)(y) f(y), \tag{37}$$

where $V(x)$ and $W(y)$ are here Wick polynomials (cf. (13)). The subtle point in the definition (36) is that the derivative on V on the r.h.s. is an external one. This results from the derivation of the MWI in classical field theory [9]. And, if the derivative on V would be an internal one, we would get wrong results, e.g. for the BRST-transformation of the interacting gauge field in non-Abelian models (179).[11] Note that $\Delta^\mu_{\chi,\psi}$ is not invariant with respect to the exchange of its arguments.

We are now going to compute the difference (32) on a *heuristic level* according to our prescription that the external derivative acts after contracting. Let[12] $V = \prod_{k=1}^m \varphi_k$, $W = \prod_{k=m+1}^p \varphi_k$, $\varphi_k \in \tilde{\mathcal{G}}$, $\varphi_k = \tilde{\partial}^{a_k} \mathrm{sym}(\phi_k)$. We consider the sum of diagrams in which $\phi_1, ..., \phi_l$ ($l \le m$) and $\phi_{m+1}, ..., \phi_q$ ($q \le p$) are contracted and $\phi_{l+1}, ..., \phi_m, \phi_{q+1}, ..., \phi_p$ are not. By means of the Feynman rules we compute the contribution of this sum of diagrams to the first T-product in (32):

$$T_{n+1}((\tilde{\partial}^\nu V) W g \otimes W_1 f_1 \otimes ...) = \sum_{r_1, ..., r_{l+q}} i^{l+q}$$
$$\Big[\partial^\nu_x \big(\Delta^F_{\varphi_1,\cdot}(x - x_{r_1}) ... \Delta^F_{\varphi_l,\cdot}(x - x_{r_l})\big) \Delta^F_{\varphi_{m+1},\cdot}(x - x_{r_{l+1}}) ... \Delta^F_{\varphi_q,\cdot}(x - x_{r_{l+q-m}}) \cdot$$
$$: T(...)(x_1, ..., x_n) \prod_{k=l+1}^m \partial^{a_k} \phi_k(x) \cdot \prod_{k=q+1}^p \partial^{a_k} \phi_k(x) : +$$
$$\Delta^F_{\varphi_1,\cdot}(x - x_{r_1}) ... \Delta^F_{\varphi_l,\cdot}(x - x_{r_l}) \Delta^F_{\varphi_{m+1},\cdot}(x - x_{r_{l+1}}) ... \Delta^F_{\varphi_q,\cdot}(x - x_{r_{l+q-m}}) \cdot$$
$$: T(...)(x_1, ..., x_n) \partial^\nu_x \big(\partial^{a_k} \prod_{k=l+1}^m \phi_k(x)\big) \cdot \prod_{k=q+1}^p \partial^{a_k} \phi_k(x) :\Big] + ..., \tag{38}$$

where the double dots simply mean that the $\phi_k(x)$, $k = l+1, ..., m, q+1, ..., p$ are not contracted. (Note that normal ordering is defined for monomials only, not for polynomials.) With (38) we obtain the following heuristic result for the

[11] On the heuristic level of the Feyman rules this can be understood as follows (for simplicity we assume $|a| = 1$): one shifts the derivative ∂ from the difference (34) of Feynman propagators $\delta^\mu \sim \partial\delta(x-y)$ to $V(x)$, however the (time-ordered) contractions of the legs of V are already performed, i.e. ∂V must be an external derivative. Thereby the term $VW(\partial g)f$ is the boundary term.

[12] In this calculation the indices k of φ_k and ϕ_k have nothing to do with the ones introduced in sect. 2.1.

difference (32)

$$\sum_{r_1,...,r_{l+q}} i^{l+q} \sum_{t=1}^{l} \Delta^F_{\varphi_1,\cdot}(x-x_{r_1})...\delta^\nu_{\varphi_t,\cdot}(x-x_{r_t})...\Delta^F_{\varphi_l,\cdot}(x-x_{r_l})\Delta^F_{\varphi_{m+1},\cdot}(x-x_{r_{l+1}})...$$

$$...\Delta^F_{\varphi_q,\cdot}(x-x_{r_{l+q-m}}) : T(...)(x_1,...,x_n) \prod_{k=l+1}^{m} \partial^{a_k}\phi_k(x) \cdot \prod_{k=q+1}^{p} \partial^{a_k}\phi_k(x) : +... . \tag{39}$$

We now require that this structure is maintained in the process of renormalization:

$$\begin{aligned} (\tilde{\mathbf{N}}) \qquad & T_{n+1}((\tilde{\partial}^\nu V)Wg \otimes W_1 f_1 \otimes ... \otimes W_n f_n) = \\ & T_{n+1}((\partial^\nu V)Wg \otimes W_1 f_1 \otimes ... \otimes W_n f_n) \\ & + i \sum_{m=1}^{n} \sum_{\chi,\psi \in \tilde{\mathcal{G}}} (\pm) T_n\Big(\Delta^\nu_{\chi,\psi}(\frac{\partial V}{\partial \chi}Wg, \frac{\partial W_m}{\partial \psi}f_m) \\ & \otimes W_1 f_1 \otimes ...\hat{m}... \otimes W_n f_n\Big) \end{aligned}$$

where $V, W, W_1, ..., W_n \in \tilde{\mathcal{P}}_0$, the sign $(\pm)$ comes from permutations of Fermi operators and $\hat{m}$ means that the corresponding factor is omitted. We now assume that $(\tilde{\mathbf{N}})$ holds true to lower orders $\leq n$. Then, due to causal factorization of time ordered products, we conclude that the condition $(\tilde{\mathbf{N}})$ is satisfied for $\text{supp}\,(g \otimes f_1 \otimes ... \otimes f_n) \cap D_{n+1} = \emptyset$. Hence $(\tilde{\mathbf{N}})$ is in fact a normalization condition. It can be satisfied by taking $(\tilde{\mathbf{N}})$ as the *definition* of the normalization of $T_{n+1}((\tilde{\partial}^\nu V)Wg \otimes W_1 f_1 \otimes ... \otimes W_n f_n)$. There is only one non-trivial step in this procedure: the compatibility with **(N3)**. This is shown in sect. 3.1.

In models with anomalies, i.e. terms which violate the MWI (see the next subsection), the normalization condition $(\tilde{\mathbf{N}})$ will be modified: in order that (31) holds true these anomalies must be taken into account in the difference (32), they give an additional contribution to the r.h.s. of $(\tilde{\mathbf{N}})$ (cf. sect. 5).

In particular the normalization condition $(\tilde{\mathbf{N}})$ implies

$$\Delta^F_{\tilde{\partial}^a\varphi_j,\tilde{\partial}^b\varphi_l} = (-1)^{|b|}\partial^a\partial^b\Delta^F_{\varphi_j,\varphi_l}, \quad \varphi_j,\varphi_l \in \mathcal{G}, \tag{40}$$

and hence

$$\delta^\mu_{\tilde{\partial}^a\varphi_j,\tilde{\partial}^b\varphi_l} = (-1)^{|b|}\partial^a\partial^b\delta^\mu_{\varphi_j,\varphi_l}, \quad \varphi_j,\varphi_l \in \mathcal{G}. \tag{41}$$

By repeated application of $(\tilde{\mathbf{N}})$ and $\langle\Omega, T_1(Uh)\Omega\rangle = 0$ for $\tilde{\mathcal{P}}_0 \ni U \neq \lambda\mathbf{1}$, $\lambda \in \mathbb{C}$, one finds

$$\begin{aligned} &\langle\Omega, T_2\Big((\prod_{k=1}^{r}\tilde{\partial}^{a^{(k)}}\varphi_{j_k})g \otimes (\prod_{k=1}^{r}\tilde{\partial}^{b^{(k)}}\varphi_{l_k})f\Big)\Omega\rangle = \\ &\langle\Omega, T_2\Big((\prod_{k=1}^{r}\partial^{a^{(k)}}\varphi_{j_k})g \otimes (\prod_{k=1}^{r}\partial^{b^{(k)}}\varphi_{l_k})f\Big)\Omega\rangle \end{aligned} \tag{42}$$

for $r > 1$, where $\varphi_m \in \mathcal{G} \quad \forall m,\ a^{(k)} \equiv (a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, a_3^{(k)})$ and similar for $b^{(k)}$.

2.4 The master Ward identity

The MWI is an explicit formula for the difference

$$\partial_x^\nu T(V,W_1,...,W_n)(x,x_1,...,x_n)-T(\partial^\nu V,W_1,...,W_n)(x,x_1,...,x_n), \qquad (43)$$

where $V,W_1,...,W_n\in\mathcal{P}_0$. It may be regarded as the postulate that the recursive definition ($\tilde{\mathbf{N}}$) reproduces, in the case $W=1$ and $V,W_1,...,W_n\in\mathcal{P}_0$, the direct definition (31) (see the Remark below). However, this is a very technical and indirect way to the MWI. We found it by the following, intuitive procedure: the result of the Feynman rules for the difference (43) is obtained from (39) by choosing $\varphi_k\in\mathcal{G}$, $\forall k$, and putting $W=1$ (i.e. $p=q=m$). The MWI requires that renormalization is done in such a way that this heuristic result is essentially preserved:

$$(\mathbf{N}) \qquad \begin{aligned} &-T_{n+1}(V\partial^\nu g\otimes W_1f_1\otimes...\otimes W_nf_n)=\\ &T_{n+1}((\partial^\nu V)g\otimes W_1f_1\otimes...\otimes W_nf_n)\\ &+i\sum_{m=1}^n\sum_{\chi,\psi\in\mathcal{G}}(\pm)T_n\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}g,\frac{\partial W_m}{\partial\psi}f_m\big)\\ &\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)\end{aligned} \qquad (44)$$

where $V,W_1,...,W_n\in\mathcal{P}_0$ (not $\in\tilde{\mathcal{P}}_0$), the sign $(\pm)$ is due to permutations of Fermi operators and $\hat{m}$ means that the corresponding factor is omitted. We recall that Δ^μ contains external derivatives. To give the correct formula for the difference (43) one needs the external derivative or an equivalent formalism (for a latter see [9] and the Remark at the end of sect. 2.1).

Remark: Instead of requiring ($\tilde{\mathbf{N}}$) and (**N**), one can take (31) and ($\tilde{\mathbf{N}}$) as the primary normalization conditions, because the latter two imply (**N**). This alternative and more compact formulation is the straightforward way to formulate the quantum MWI [9], when departing from classical field theory. However, the advantage of the present procedure is that it explicitly distinguishes the 'weak' normalization condition ($\tilde{\mathbf{N}}$) (which only *defines* the normalization of the time ordered products with external derivatives) from the 'hard' one (**N**) (which expresses deep symmetries). This distinction plays an important role in our (incomplete) proof of the MWI (sect. 3).

We now assume that **(N)** holds true to lower orders $\leq n$. Then, due to causal factorization of time ordered products, we conclude that the condition **(N)** is satisfied for $\mathrm{supp}\,(g\otimes f_1\otimes...\otimes f_n)\cap D_{n+1}=\emptyset$. Hence **(N)** is in fact a normalization condition. The compatibility with **(N0)-(N2)** is trivial and the compatibility with **(N3)** is proved in sect. 3.2. The hard question is whether **(N)** can be satisfied by choosing suitable normalizations (which are compatible with the other normalization conditions). The answer depends on the model. We will see that the MWI implies that there is no axial anomaly and no trace anomaly of the energy momentum tensor. Hence it must be impossible to fulfil

the MWI in these cases. Generally we call any term that violates the MWI (and cannot be removed by an admissible, finite renormalization of the T-products) an *anomaly*.

If there is at most one contraction between V and $W_1, ..., W_n$ (i.e. we have $l = 0$ or $l = 1$ and of course $p = q = m$ in (39)) the expression (39) is well-defined and (re)normalization can be done such that (39) gives the contribution of these diagrams to the difference (43). In other words one can fulfil the MWI **(N)** for these 'tree-like' diagrams. The anomalies must come from 'loop-like' diagrams. In sect. 5 we give a more general formulation of the MWI which takes anomalies into account.

3 Steps towards a proof of the master Ward identity

We have to show that there exists a normalization of the T-products which satisfies ($\tilde{\mathbf{N}}$), (**N**) and also **(N0)-(N3)**). The compatibility of ($\tilde{\mathbf{N}}$) and (**N**) with **(N0)-(N2)** is obvious, but the compatibility with (**N3**) requires some work which is done in the next two subsections. The proof of ($\tilde{\mathbf{N}}$) is then easily completed (sect. 3.1).

But a general proof of (**N**) is impossible, since it is well-known that there exist anomalies in certain models. If solely massive fields appear and if the scaling degrees (29) of the individual C-number distributions appearing in **(N)** are not relatively lowered[13], we can give a constructive proof of **(N)** (sect. 3.3). More precisely we show that the so-called 'central solution' of Epstein and Glaser, which is a distinguished extension $t^{(c)} \in \mathcal{D}'(\mathbb{R}^k)$ of $t_0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$, satisfies **(N)** in this case.

3.1 Proof of ($\tilde{\mathbf{N}}$)

The nontrivial part in the proof of ($\tilde{\mathbf{N}}$) is the compatibility with **(N3)**. The keys to show this and the compatibility of **(N)** with **(N3)** (see sect. 3.2) are the following two Lemmas:

Lemma 1: Let $V \in \tilde{\mathcal{P}}_0$, $\varphi, \psi \in \tilde{\mathcal{G}}$ and $f, h \in \mathcal{D}(\mathbb{R}^4)$. Then the following

[13] We explain what we mean by this expression for the example of the identity $\partial_\nu t_1^\nu = t_2 \quad (t_1, t_2 \in \mathcal{D}'(\mathbb{R}^k))$. According to (28) we naively expect $\mathrm{sd}(t_1) + 1 = \mathrm{sd}(t_2)$. We say that the scaling degree of t_1 (or t_2 resp.) is relatively lowered if $\mathrm{sd}(t_1) < \mathrm{sd}(t_2) - 1$ (or $\mathrm{sd}(t_2) < \mathrm{sd}(t_1) + 1$ resp.).

A relative pre-factor m^a ($m =$ mass), $a > 0$, indicates a relatively lowered scaling degree. Let $\partial_\nu t_1^\nu = m^a t_2$ and we assume that t_1 and t_2 contain no global factor m^b ($b \in \mathbb{R} \setminus \{0\}$). Then, for dimensional reasons, the scaling degree of t_2 is relatively lowered: $\mathrm{sd}(t_2) = \mathrm{sd}(t_1) + 1 - a$.

identities hold true within $\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)$:

$$f \sum_{\varphi \in \tilde{\mathcal{G}}} \frac{\partial(\partial^a V)}{\partial \varphi} \Delta_{\varphi,\psi} \star h = f \sum_{\varphi \in \tilde{\mathcal{G}}} \sum_{0 \le b \le a} \frac{a!}{b!(a-b)!} \partial^b \frac{\partial V}{\partial \varphi} \partial^{(a-b)} \Delta_{\varphi,\psi} \star h \tag{45}$$

$$f \sum_{\varphi \in \tilde{\mathcal{G}}} \frac{\partial(\tilde{\partial}^a V)}{\partial \varphi} \Delta_{\varphi,\psi} \star h = f \sum_{\varphi \in \tilde{\mathcal{G}}} \sum_{0 \le b \le a} \frac{a!}{b!(a-b)!} \tilde{\partial}^b \frac{\partial V}{\partial \varphi} \partial^{(a-b)} \Delta_{\varphi,\psi} \star h, \tag{46}$$

where again $a! \equiv \prod_\mu a_\mu!$.
Proof: We first prove (45) for $|a| = 1$, i.e.

$$f \sum_{\varphi \in \tilde{\mathcal{G}}} \frac{\partial(\partial^\mu V)}{\partial \varphi} \Delta_{\varphi,\psi} \star h = f \sum_{\varphi \in \tilde{\mathcal{G}}} [\partial^\mu \frac{\partial V}{\partial \varphi} \Delta_{\varphi,\psi} \star h + \frac{\partial V}{\partial \varphi} \partial^\mu \Delta_{\varphi,\psi} \star h]. \tag{47}$$

It suffices to consider the case in which V is a monomial. The proof goes by induction on the degree of this monomial. The case $V = 1$ is trivial. Let $V = \chi W$, $\chi \in \tilde{\mathcal{G}}$, $W \in \tilde{\mathcal{P}}_0$. By assumption W satisfies (47). Inserting now $V = \chi W$ into (47) and using this assumption most terms cancel and it remains to show

$$f \sum_{\varphi \in \tilde{\mathcal{G}}} \frac{\partial(\partial^\mu \chi)}{\partial \varphi} W \Delta_{\varphi,\psi} \star h = f \sum_{\varphi \in \tilde{\mathcal{G}}} \frac{\partial \chi}{\partial \varphi} W \partial^\mu \Delta_{\varphi,\psi} \star h. \tag{48}$$

The l.h.s. is equal to $fW\Delta_{\partial^\mu \chi,\psi} \star h$ and the r.h.s. to $fW\partial^\mu \Delta_{\chi,\psi} \star h$. Obviously these two expressions agree.

To prove (45) for arbitrary $|a|$ we proceed by induction on $|a|$:

$$\begin{aligned} &f \sum_{\varphi \in \tilde{\mathcal{G}}} \frac{\partial(\partial^\mu \partial^a V)}{\partial \varphi} \Delta_{\varphi,\psi} \star h = \\ &f \sum_{\varphi \in \tilde{\mathcal{G}}} [(\partial^\mu \frac{\partial(\partial^a V)}{\partial \varphi}) \Delta_{\varphi,\psi} \star h + \frac{\partial(\partial^a V)}{\partial \varphi} \partial^\mu \Delta_{\varphi,\psi} \star h] = \\ &f \sum_{\varphi \in \tilde{\mathcal{G}}} \sum_{0 \le b \le a} \frac{a!}{b!(a-b)!} [(\partial^\mu \partial^b \frac{\partial V}{\partial \varphi}) \partial^{(a-b)} \Delta_{\varphi,\psi} \star h + (\partial^b \frac{\partial V}{\partial \varphi}) \partial^\mu \partial^{(a-b)} \Delta_{\varphi,\psi} \star h] = \\ &f \sum_{\varphi \in \tilde{\mathcal{G}}} \sum_{0 \le b \le (a+e_\mu)} \frac{(a+e_\mu)!}{b!(a+e_\mu-b)!} (\partial^b \frac{\partial V}{\partial \varphi}) \partial^{(a+e_\mu-b)} \Delta_{\varphi,\psi} \star h, \end{aligned} \tag{49}$$

where $e_\mu = (0, ..., 1, ..., 0)$ with 1 at the μ-th position. First we have used (47) (with V replaced by $\partial^a V$) and in the second equality sign we have inserted (45) (which is the inductive assumption) and $\partial^\mu \otimes \mathbf{1}$ applied to (45) (cf. (9)).

The proof of the second identity (46) is completely similar. One simply has to replace the internal derivatives $\partial^a \otimes \mathbf{1}$ by external ones $\tilde{\partial}^a \otimes \mathbf{1}$. In particular the validity of the equation corresponding to (48) relies on $\Delta_{\tilde{\partial}^\mu \chi,\psi} = \partial^\mu \Delta_{\chi,\psi}$. $\square$

By means of Lemma 1 we will prove

Lemma 2: Let $V,W\in\tilde{\mathcal{P}}_0,\ \chi,\psi,\kappa\in\tilde{\mathcal{G}}$ and $f,g,h\in\mathcal{D}(\mathbb{R}^4)$. Then

$$\sum_{\varphi\in\tilde{\mathcal{G}}}\frac{\partial\Delta^\mu_{\chi,\psi}(Vg,Wf)}{\partial\varphi}\Delta_{\varphi,\kappa}\star h=$$

$$\sum_{\varphi\in\tilde{\mathcal{G}}}[\Delta^\mu_{\chi,\psi}(\frac{\partial V}{\partial\varphi}g\Delta_{\varphi,\kappa}\star h,Wf)+\Delta^\mu_{\chi,\psi}(Vg,\frac{\partial W}{\partial\varphi}f\Delta_{\varphi,\kappa}\star h)]. \tag{50}$$

Proof: Using the explicit form (36) for Δ^μ the l.h.s. of (50) is equal to

$$\sum_a\sum_{\varphi\in\tilde{\mathcal{G}}}C^\mu_{\chi,\psi;a}(-1)^{|a|}\sum_{0\le b\le a}\frac{a!}{b!(a-b)!}[\frac{\partial(\tilde{\partial}^{(a-b)}V)}{\partial\varphi}W(\partial^b g)f$$
$$+(\tilde{\partial}^{(a-b)}V)\frac{\partial W}{\partial\varphi}(\partial^b g)f]\Delta_{\varphi,\kappa}\star h. \tag{51}$$

Again by means of (36) the r.h.s. of (50) can be written as

$$\sum_a\sum_{\varphi\in\tilde{\mathcal{G}}}C^\mu_{\chi,\psi;a}(-1)^{|a|}\sum_{0\le b\le a}\frac{a!}{b!(a-b)!}[\sum_{0\le c\le b}\frac{b!}{c!(b-c)!}$$
$$(\tilde{\partial}^{(a-b)}\frac{\partial V}{\partial\varphi})W(\partial^c g)(\partial^{(b-c)}\Delta_{\varphi,\kappa}\star h)f+(\tilde{\partial}^{(a-b)}V)\frac{\partial W}{\partial\varphi}(\partial^b g)f(\Delta_{\varphi,\kappa}\star h)]. \tag{52}$$

Due to (46) the expressions (51) and (52) agree. □

We now come to the proof of ($\tilde{\mathbf{N}}$), i.e. we show that there exists a normalization of the T-products which satisfies **(N0)**, **(N1)**, **(N2)**, **(N3)** and ($\tilde{\mathbf{N}}$). Let the T-products fulfil the first four of these normalization conditions to all orders. In a double inductive procedure we assume that ($\tilde{\mathbf{N}}$) holds to lower orders $\le n$ and for all T-products to order $n+1$ of sub-polynomials. More precisely, the second induction goes (for each fixed n) with respect to the 'polynomial degree' d which is the sum of the degrees of the polynomials $V_1,...,V_n\in\tilde{\mathcal{P}}_0$ in $T_n(V_1g_1\otimes...\otimes V_ng_n)$: $d\stackrel{\text{def}}{=}|V_1|+...+|V_n|$. Note $|\partial^aV|=|V|=|\tilde{\partial}^aV|$. By using **(N3)** we want to show that the commutators of the l.h.s. and of the r.h.s. of ($\tilde{\mathbf{N}}$) with $T_1(\kappa h)$ agree. The commutator of the l.h.s. is equal to

$$i\sum_{\varphi\in\tilde{\mathcal{G}}}\Big[T_{n+1}\Big([\frac{\partial(\tilde{\partial}^\nu V)}{\partial\varphi}W+(\tilde{\partial}^\nu V)\frac{\partial W}{\partial\varphi}]g(\Delta_{\varphi,\kappa}\star h)\otimes W_1f_1\otimes...\Big) \tag{53}$$

$$+\sum_j T_{n+1}\Big((\tilde{\partial}^\nu V)Wg\otimes W_1f_1\otimes...\otimes\frac{\partial W_j}{\partial\varphi}f_j(\Delta_{\varphi,\kappa}\star h)\otimes...\Big)\Big]. \tag{54}$$

To compute the commutator of the r.h.s. of ($\tilde{\mathbf{N}}$) with $T_1(\kappa h)$ we use again **(N3)**

and in addition Lemma 2. We obtain

$$i\sum_{\varphi\in\tilde{\mathcal{G}}}\Big[T_{n+1}\Big([\frac{\partial(\partial^\nu V)}{\partial\varphi}W+(\partial^\nu V)\frac{\partial W}{\partial\varphi}]g(\Delta_{\varphi,\kappa}\star h)\otimes W_1f_1\otimes ...\Big) \tag{55}$$

$$+\sum_j T_{n+1}\Big(\Big[(\partial^\nu V)Wg\otimes W_1f_1\otimes ...\otimes\frac{\partial W_j}{\partial\varphi}f_j(\Delta_{\varphi,\kappa}\star h)\otimes ...\Big) \tag{56}$$

$$+i\sum_{m=1}^n\sum_{\chi,\psi\in\tilde{\mathcal{G}}}(\pm)T_n\Big(\big[\Delta^\nu_{\chi,\psi}\big([\frac{\partial^2 V}{\partial\varphi\partial\chi}W+\frac{\partial V}{\partial\chi}\frac{\partial W}{\partial\varphi}]g(\Delta_{\varphi,\kappa}\star h),\frac{\partial W_m}{\partial\psi}f_m\big) \tag{57}$$

$$+\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}Wg,\frac{\partial^2 W_m}{\partial\varphi\partial\psi}f_m(\Delta_{\varphi,\kappa}\star h)\big)\big]\otimes W_1f_1\otimes ...\hat{m}...\Big) \tag{58}$$

$$+i\sum_{m,j\,(m\neq j)}\sum_{\chi,\psi\in\tilde{\mathcal{G}}}(\pm)T_n\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}Wg,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes ...$$

$$...\hat{m}...\otimes\frac{\partial W_j}{\partial\varphi}f_j(\Delta_{\varphi,\kappa}\star h)\otimes ...\Big)\Big] \tag{59}$$

Due to ($\tilde{\mathbf{N}}$) for subpolynomials we have the following equations:
(second term in (53))=(second term in (55))+ (second term in (57))
and (54)=(56)+(58)+(59).
To get the equality of (53)+(54) and (55)+(56)+ (57)+(58)+(59) it remains to show:
(first term in (53))=(first term in (55))+ (first term in (57)). $(*)$
To verify this we insert (47) with $\partial^\mu\otimes\mathbf{1}$ replaced by $\tilde{\partial}^\mu\otimes\mathbf{1}$ into the first term in (53) and the original (47) into the first term in (55). The remaining terms in $(*)$ cancel by means of ($\tilde{\mathbf{N}}$) for subpolynomials.

From the just now proved result we conclude that ($\tilde{\mathbf{N}}$) can be violated by a C-number only:

$$\begin{aligned}&T_{n+1}((\tilde{\partial}^\nu V)Wg\otimes W_1f_1\otimes ...)-T_{n+1}((\partial^\nu V)Wg\otimes W_1f_1\otimes ...)\\ &-i\sum_{m=1}^n\sum_{\chi,\psi\in\tilde{\mathcal{G}}}(\pm)T_n\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}Wg,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes ...\hat{m}...\Big)=\\ &\langle\Omega|T_{n+1}((\tilde{\partial}^\nu V)Wg\otimes W_1f_1\otimes ...)|\Omega\rangle-\langle\Omega|T_{n+1}((\partial^\nu V)Wg\otimes W_1f_1\otimes ...)|\Omega\rangle\\ &-i\sum_{m=1}^n\sum_{\chi,\psi\in\tilde{\mathcal{G}}}(\pm)\langle\Omega|T_n\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}Wg,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes ...\hat{m}...\Big)|\Omega\rangle\\ &\stackrel{\mathrm{def}}{=}:\tilde{a}(g,f_1,...,f_n).\end{aligned} \tag{60}$$

Due to causal factorization of the T-products and the validity of ($\tilde{\mathbf{N}}$) to lower orders $\leq n$, the possible violation $\tilde{a}(g,f_1,...,f_n)$ of ($\tilde{\mathbf{N}}$) must be local

$$\tilde{a}(g,f_1,...,f_n)=\int dx\,dx_1...dx_n\sum_{|a|=0}^{\omega}C_a\partial^a\delta(x_1-x,...,x_n-x)g(x)f_1(x_1)...f_n(x_n), \tag{61}$$

with unknown constants C_a and

$$\omega \stackrel{\text{def}}{=} \mathrm{sd}\big(\langle\Omega|T_{n+1}((\tilde{\partial}^\nu V)W, W_1, ..., W_n)|\Omega\rangle\big) - 4n. \tag{62}$$

After the finite renormalization

$$\langle\Omega|T_{n+1}((\tilde{\partial}^\nu V)Wg\otimes W_1 f_1\otimes ...)|\Omega\rangle \to$$
$$\langle\Omega|T_{n+1}((\tilde{\partial}^\nu V)Wg\otimes W_1 f_1\otimes ...)|\Omega\rangle - \tilde{a}(g, f_1, ...) \tag{63}$$

($\tilde{\mathbf{N}}$) holds true. By construction (in particular (62)) this renormalization respects **(N0)**. From the definition (60) of $\tilde{a}(g, f_1, ...)$ we see that (63) maintains **(N1)**, **(N2)** and the permutation symmetry of $\langle\Omega|T_{n+1}((\tilde{\partial}^\nu V)W, W_1, ...)|\Omega\rangle$. However, in general (63) violates **(N3)**, namely in the cases in which $T_{n+1}((\tilde{\partial}^\nu V)W, W_1, ...)$ appears on the r.h.s. of **(N3)**.[14] So we everywhere repair **(N3)** by a chain of finite renormalizations of T-products of order $n+1$ with polynomial degree $d > |V| + |W| + |W_1| + ... + |W_n|$.[15] It is obvious that this can be done such that **(N0)**, **(N1)** and **(N2)** are preserved. The validity of ($\tilde{\mathbf{N}}$) up to order $n+1$ and polynomial degree $|V| + |W| + |W_1| + ... + |W_n|$ is not touched by these renormalizations. So the inductive step is finished. □

In other words, the compatibility of the renormalization (63) with **(N3)** follows from the following general observation: **(N3)** determines the operator-valued map T_n completely in terms of the $\mathbb{C}$-valued map $\langle\Omega|T_n(\cdot)|\Omega\rangle : \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)^{\otimes n} \to \mathbb{C}$. However, **(N3)** does not give any relation among the vacuum expectation values of the T-products, they may be arbitrarily given. Hence, renormalizations of the vacuum expectation values of the T-products are not in conflict with **(N3)**. We will use this second way of argumentation in the following.

3.2 Compatibility of the master Ward identity with (N3)

We start with T-products which fulfil **(N0)**, **(N1)**, **(N2)**, **(N3)** and ($\tilde{\mathbf{N}}$) to all orders. We use the same double induction as in the preceding subsection: we assume that (**N**) holds to lower orders $\leq n$ and for T_{n+1} restricted to the elements of $\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)^{\otimes n+1}$ with a lower polynomial degree.

By means of **(N3)** we are going to prove that the commutators of the l.h.s. and of the r.h.s. of (**N**) with $T_1(\kappa h)$ are equal. For the l.h.s. it results

$$-i\sum_{\varphi\in\mathcal{G}}\Big[T_{n+1}\big(\frac{\partial V}{\partial\varphi}\partial^\nu g(\Delta_{\varphi,\kappa}\star h)\otimes W_1 f_1\otimes ...\big) \tag{64}$$

$$+\sum_{l=1}^{n} T_{n+1}\big(V\partial^\nu g\otimes W_1 f_1\otimes ...\otimes\frac{\partial W_l}{\partial\varphi}f_l(\Delta_{\varphi,\kappa}\star h)\otimes ...\big)\Big]. \tag{65}$$

[14] The cases of **(N3)** in which $T_{n+1}((\tilde{\partial}^\nu V)W, W_1, ...)$ appears on the l.h.s. remain true, because only the C-number part of $T_{n+1}((\tilde{\partial}^\nu V)W, W_1, ...)$ gets changed.

[15] The vacuum expectation values of these T-products remain unchanged; solely the operator parts get renormalized.

By using again **(N3)** and in addition Lemma 2 we compute the commutator of the r.h.s. and obtain

$$i\sum_{\varphi\in\mathcal{G}}\Big\{T_{n+1}\big(\frac{\partial(\partial^\nu V)}{\partial\varphi}g(\Delta_{\varphi,\kappa}\star h)\otimes W_1f_1\otimes ...\big) \tag{66}$$

$$+\sum_{l=1}^{n}T_{n+1}\big((\partial^\nu V)g\otimes W_1f_1\otimes ...\otimes\frac{\partial W_l}{\partial\varphi}f_l(\Delta_{\varphi,\kappa}\star h)\otimes ...\big) \tag{67}$$

$$+i\sum_{m=1}^{n}\sum_{\chi,\psi\in\mathcal{G}}(\pm)\Big(T_n\big(\big[\Delta^\nu_{\chi,\psi}\big(\frac{\partial^2 V}{\partial\varphi\partial\chi}g(\Delta_{\varphi,\kappa}\star h),\frac{\partial W_m}{\partial\psi}f_m\big) \tag{68}$$

$$+\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}g,\frac{\partial^2 W_m}{\partial\varphi\partial\psi}f_m(\Delta_{\varphi,\kappa}\star h)\big)\big]\otimes W_1f_1\otimes ...\hat{m}...\big) \tag{69}$$

$$+\sum_{l\ (l\neq m)}T_n\big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}g,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes ...$$

$$...\hat{m}...\otimes\frac{\partial W_l}{\partial\varphi}f_l(\Delta_{\varphi,\kappa}\star h)\otimes ...\big)\Big)\Big\}. \tag{70}$$

The validity of (**N**) for sub-polynomials implies
(65)=(67)+(69)+(70).
It remains to prove
(64)=(66)+(68). $\quad (**)$
After inserting (47) into (66) this equation $(**)$ takes the form of (**N**) for some sub-polynomials, which holds true by the inductive assumption. □

As in the case of ($\tilde{\mathbf{N}}$) (60)-(62) we conclude that the operator identity **(N)** can be violated by a local C-number $a(g,f_1,...,f_n)$ only:

$$a(g,f_1,...,f_n)\stackrel{\mathrm{def}}{=}\langle\Omega|T_{n+1}(V\partial^\nu g\otimes W_1f_1\otimes ...)|\Omega\rangle+$$
$$\langle\Omega|T_{n+1}((\partial^\nu V)g\otimes W_1f_1\otimes ...)|\Omega\rangle$$
$$+i\sum_{m=1}^{n}\sum_{\chi,\psi\in\mathcal{G}}(\pm)\langle\Omega|T_n\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}g,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes ...\hat{m}...\Big)|\Omega\rangle. \tag{71}$$

The aim is now to remove $a(g,f_1,...,f_n)$ by finite renormalizations of the vacuum expectation values $\langle\Omega|T(...)|\Omega\rangle$ on the r.h.s.. Such renormalizations are not in conflict with **(N3)** (see the end of the preceding subsection). So we have proved the compatibility of **N** with **(N3)**.

We discuss the possibilities to remove $a(g,f_1,...,f_n)$:

(A) The finite renormalization

$$\langle\Omega|T_{n+1}((\partial^\nu V)g\otimes W_1f_1\otimes ...)|\Omega\rangle\rightarrow$$
$$\langle\Omega|T_{n+1}((\partial^\nu V)g\otimes W_1f_1\otimes ...)|\Omega\rangle-a(g,f_1,...), \tag{72}$$

does this job and is compatible with **(N0)-(N3)**, ($\tilde{\mathbf{N}}$) and permutation symmetry. However, this procedure works only if $\partial^\nu V\neq 0$ and if $\mathrm{sd}\big(\langle\Omega|T_{n+1}(\partial^\nu V,W_1,...,W_n)|\Omega\rangle\big)=$

$\mathrm{sd}\big(\langle\Omega|T_{n+1}(V,W_1,...,W_n)|\Omega\rangle\big)+1$. In case that the latter does not hold one gets in conflict with **(N0)**. This happens e.g. for the axial and pseudo-scalar triangle-diagram, see (90). In many important applications of the MWI V corresponds to a conserved current (i.e. $\partial^\nu V=0$), for example $V=\bar\psi\gamma\psi$, or if V is the free ghost current (cf. sect. 4.2 for both) or the free BRST-current (sect. 4.4).

(B) If (72) does not work one tries to satisfy **(N)** by renormalizing also $\langle\Omega|T_{n+1}(V,W_1,...)|\Omega\rangle$ and eventually $\langle\Omega|T_n\Big(\Delta^\nu_{\chi,\psi}(...),W_1...\Big)|\Omega\rangle$. This method does not ensure success. In detail one proceeds in the following way:

(B1) $a(g,f_1,...)$ has the form (61) with $\omega\stackrel{\mathrm{def}}{=}\mathrm{sd}\big(\langle\Omega|T_{n+1}(V,W_1,...,W_n)|\Omega\rangle\big)+1-4n$. Using symmetry properties (e.g. Poincare covariance, permutation symmetries) of the r.h.s. of (71) the constants C_a (we use the notation of (61)) can be strongly restricted.

(B2) One works out the freedoms of normalization (30) of $\langle\Omega|T_{n+1}(V,W_1,...)|\Omega\rangle$, $\langle\Omega|T_{n+1}((\partial^\nu V),W_1...)|\Omega\rangle$ and eventual $\langle\Omega|T_n\Big(\Delta^\nu(...),W_1...\Big)|\Omega\rangle$ (the second is only available if $\partial^\nu V\neq 0$) which respect **(N0)-(N2)**, ($\tilde{\mathbf{N}}$) and permutation symmetry. Renormalizations of $\langle\Omega|T_n\Big(\Delta^\nu(...),W_1...\Big)|\Omega\rangle$ are also restricted by the validity of the inductive assumption for **(N)**.

(B3) One then tries to remove the remaining $a(g,f_1,...)$ by using the freedoms which result from step (2).

Because the restricted $a(g,f_1,...)$ (step(1)) and the free normalization polynomials (step (2)) depend strongly on $(V,W_1,...,W_n)$, one has to treat each combination $(V,W_1,...,W_n)$ separately and this gives quite a lot of work. This method was used in [11] to prove 'perturbative gauge invariance' (which is equation (148) with $j_1=...=j_n=0$) for $SU(N)$-Yang-Mills theories. To restrict $a(g,f_1,...)$ sufficiently a weak assumption about the infrared behaviour was necessary. (However, if this assumption would not hold, the Green's functions would not exist.)

3.3 Proof of the master Ward identity for solely massive fields and not relatively lowered scaling degree

We return to the end of sect. 2.2 and set

$$\omega(t_0)\stackrel{\mathrm{def}}{=}\mathrm{sd}(t_0)-4(n-1). \tag{73}$$

A possible extension $t\in\mathcal{D}'(\mathbb{R}^{4(n-1)})$ of $t_0\in\mathcal{D}'(\mathbb{R}^{4(n-1)}\setminus\{0\})$ which respects **(N0)** is given by (cf. [5],[28])

$$(t^{(w)},h)\stackrel{\mathrm{def}}{=}(t_0,h^{(w)}),\qquad \forall h\in\mathcal{D}(\mathbb{R}^{4(n-1)}) \tag{74}$$

where

$$h^{(w)}(x)\stackrel{\mathrm{def}}{=}h(x)-w(x)\sum_{|a|=0}^{\omega(t_0)}\frac{x^a}{a!}(\partial^a h)(0),\qquad w\in\mathcal{D}(\mathbb{R}^{4(n-1)}), \tag{75}$$

and there exists a neighbourhood $\mathcal{U}$ of $0 \in \mathbb{R}^{4(n-1)}$ with $w|_{\mathcal{U}} \equiv 1$. A change of w alters the normalization of $t^{(w)}$. For $\omega(t_0) < 0$ we have $h^{(w)} = h$ in agreement with the fact that the extension is unique in that case. Because there is no Lorentz invariant $w \in \mathcal{D}(\mathbb{R}^{4(n-1)})$, the extension $t^{(w)}$ is not Lorentz covariant and one has to perform a finite renormalization (30) to restore this symmetry (see the second paper of [11] and [6]). To avoid this one is tempted to choose $w \equiv 1$. But $h^{(w\equiv 1)}$ is not a test-function. However, if all fields are massive the infrared behavior is harmless and Epstein and Glaser [18] have shown that one may indeed choose $w \equiv 1$ in this case. The extension $t^{(c)} \stackrel{\mathrm{def}}{=} t^{(w\equiv 1)}$ is called 'central solution' (or better 'central extension' in our framework) and it was pointed out that it preserves nearly all symmetries [18], [13], [32] [16].

We are now going to show that the central extensions fulfil the MWI provided the scaling degree is not relatively lowered for the individual, contributing C-number distributions (a precise explanation of the latter expression is given

[16]Epstein and Glaser have proved that one may choose $w \equiv 1$ for the method of distribution splitting. In this footnote we show how their result applies to our extension procedure (74). From Epstein and Glaser [18] we know $t_0 \in \mathcal{S}'(\mathbb{R}^{4(n-1)} \setminus \{0\})$ and hence $t^{(w)} \in \mathcal{S}'(\mathbb{R}^{4(n-1)})$, so we may use Fourier transformation. Epstein and Glaser have proved that in the massive case the Fourier transformation $\hat{t}^{(w)}(p)$ (and therefore any extension (30)) is analytic in a neighbourhood of $p = 0$. Then they define the central extension $t^{(c)}$ by

$$\partial^a \hat{t}^{(c)}(0) = 0 \qquad \forall |a| \leq \omega(t_0). \tag{76}$$

The Fourier transformation of $t^{(w)}$ reads [28]

$$\hat{t}^{(w)}(p) = \hat{t}_0(p) - \sum_{|a|=0}^{\omega(t_0)} \frac{p^a}{a!} \partial^a (\widehat{t_0 w})(0). \tag{77}$$

Note $\widehat{t_0 w} = (2\pi)^{-\frac{n}{2}} \hat{t}_0 \star \hat{w} \in \mathcal{C}^\infty$, i.e. $\partial^a (\widehat{t_0 w})(0)$ exists. Using the definition (76) of the central extension we now find

$$\hat{t}^{(c)}(p) = \hat{t}^{(w)}(p) - \sum_{|a|=0}^{\omega(t_0)} \frac{p^a}{a!} (\partial^a \hat{t}^{(w)})(0) = \hat{t}_0(p) - \sum_{|a|=0}^{\omega(t_0)} \frac{p^a}{a!} (\partial^a \hat{t}_0)(0). \tag{78}$$

We see that we may set $w \equiv 1$ in (77) and hence also in (74), and that this choice is the central extension.

below in (87), (88)). We define $\bar{t}^{\nu(c)}$, $t^{(c)}$ and $t^{\nu(c)}_{b;\chi,\psi}$ to be central extensions:

$$\int dx\, dx_1...dx_n\, \bar{t}^{\nu(c)}(x_1-x,...,x_n-x)g(x)f_1(x_1)...f_n(x_n) \stackrel{\text{def}}{=}$$
$$\langle\Omega|T_{n+1}((\partial^\nu V)g\otimes W_1f_1\otimes...\otimes W_nf_n)|\Omega\rangle^{(c)} \tag{79}$$
$$\int dx\, dx_1...dx_n\, t^{(c)}(x_1-x,...,x_n-x)g(x)f_1(x_1)...f_n(x_n) \stackrel{\text{def}}{=}$$
$$\langle\Omega|T_{n+1}(Vg\otimes W_1f_1\otimes...\otimes W_nf_n)|\Omega\rangle^{(c)} \tag{80}$$
$$\sum_b\int dx\, dx_1...dx_n\, t^{\nu(c)}_{b;\chi,\psi}(x_1-x_m,...\hat{m}...,x_n-x_m)\cdot$$
$$(\partial^b\delta)(x_m-x)g(x)f_1(x_1)...f_n(x_n) \stackrel{\text{def}}{=}$$
$$\langle\Omega|T_n\Big(\Delta^\nu_{\chi,\psi}(\frac{\partial V}{\partial\chi}g,\frac{\partial W_m}{\partial\psi}f_m)\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)|\Omega\rangle^{(c)}, \tag{81}$$

where we have taken the definition (36) of Δ^μ into account[17]. The corresponding non-extended distributions are

$$\bar{t}^\nu_0,\quad t_0\in\mathcal{D}'(\mathbb{R}^{4n}\setminus\{0\})\quad\text{and}\quad t^\nu_{b;\chi,\psi;0}\in\mathcal{D}'(\mathbb{R}^{4(n-1)}\setminus\{0\}). \tag{83}$$

In the preceding subsection we have learnt that the validity of **(N3)** reduces the proof of the MWI to the vacuum sector (see (71)). So we only have to show

$$-\partial^\nu t^{(c)}(y_1,...,y_n)=\bar{t}^{\nu(c)}(y_1,...,y_n)$$
$$+i\sum_{m=1}^n\sum_{\chi,\psi\in\mathcal{G}}(\pm)t^{\nu(c)}_{b;\chi,\psi}(y_1-y_m,...\hat{m}...,y_n-y_m)\partial^b\delta(y_m), \tag{84}$$

where

$$\partial^\nu\stackrel{\text{def}}{=}\partial^\nu_1+...+\partial^\nu_n. \tag{85}$$

By causal factorization and induction we know that this equation is fulfilled by the corresponding non-extended distributions (83). Setting $y\equiv(y_1,...,y_n)$ we

[17]From (36) we see that $\langle\Omega|T_n\Big(\Delta^\mu(...)\otimes...\hat{m}...\Big)|\Omega\rangle^{(c)}$ is of the form

$$\sum_b\tilde{t}_b(f_1\otimes...\otimes(\partial^bg)f_m\otimes...\otimes f_n)=$$
$$\sum_b\int dx\,dx_1...dx_n\,t_b(x_1-x_m,...\hat{m}...,x_n-x_m)(\partial^b\delta)(x_m-x)g(x)f_1(x_1)...f_n(x_n), \tag{82}$$

$\tilde{t}_b\in\mathcal{D}'(\mathbb{R}^{4n})$, $t_b\in\mathcal{D}'(\mathbb{R}^{4(n-1)})$.

obtain

$$-\Big(\partial^\nu t^{(c)}(y), h(y)\Big) = \Big(t_0(y), [\partial^\nu h(y) - \sum_{|a|=0}^{\omega(t_0)} \frac{y^a}{a!}(\partial^a\partial^\nu h)(0)]\Big) =$$

$$\Big(t_0(y), \partial^\nu[h(y) - \sum_{|a|=0}^{\omega(t_0)+1} \frac{y^a}{a!}(\partial^a h)(0)]\Big) =$$

$$\Big(\bar{t}_0^\nu(y), [h(y) - \sum_{|a|=0}^{\omega(t_0)+1} \frac{y^a}{a!}(\partial^a h)(0)]\Big)$$

$$+i\sum_{m=1}^{n}\sum_{\chi,\psi\in\mathcal{G}}(\pm)\Big(t^\nu_{b;\chi,\psi;0}(y_1-y_m,...\hat{m}...,y_n-y_m)\partial^b\delta(y_m), [h(y) - \sum_{|a|=0}^{\omega(t_0)+1} \frac{y^a}{a!}(\partial^a h)(0)]\Big). \tag{86}$$

If the scaling degree is not relatively lowered, more precisely if

$$\omega(\bar{t}_0^\nu) = \omega(t_0)+1 \tag{87}$$

and

$$\omega(t^\nu_{b;\chi,\psi;0}) = \omega(t_0)+1-|b|, \quad \forall b, \quad \forall \chi,\psi\in\mathcal{G}, \tag{88}$$

then the terms in the final expression of (86) are the central extensions. For the $\bar{t}_0^\nu$-term this is obvious. To verify this statement for the $t^\nu_{b;\chi,\psi;0}$-terms (we omit the indices ν,χ,ψ in the following) it suffices to consider the term $m=n$ and test-functions of the form $h(y_1,...,y_n) = h_1(z_1,...,z_{n-1})h_2(z_n)$ where

$$z \equiv (z_1,...,z_n) = (y_1-y_n,...,y_{n-1}-y_n,y_n) \stackrel{\mathrm{def}}{=}: Ay, \qquad A\in SL(n,\mathbb{R}).$$

Then we have

$$h(y) - \sum_{|a|=0}^{\omega(t_{b;0})+|b|} \frac{y^a}{a!}(\partial^a h)(0) =$$

$$(h_1\otimes h_2)(z) - \sum_{|a|=0}^{\omega(t_{b;0})+|b|} \frac{y^a}{a!}((A^T\partial)^a(h_1\otimes h_2))(0) =$$

$$(h_1\otimes h_2)(z) - \sum_{|a|=0}^{\omega(t_{b;0})+|b|} \frac{z^a}{a!}(\partial^a(h_1\otimes h_2))(0)$$

where A^T denotes the transposed matrix and we have used $y^a\cdot(A^T\partial)^a = (Ay)^a\cdot\partial^a$. We set $a=(\bar{a},a_n)$ and $z\equiv(\bar{z},z_n)$. Then the last term in (86)

can be transformed in the following way:

$$\Big(t_{b;0}(y_1-y_n,...,y_{n-1}-y_n)\partial^b\delta(y_n),[h(y)-\sum_{|a|=0}^{\omega(t_{b;0})+|b|}\frac{y^a}{a!}(\partial^a h)(0)]\Big)_y=$$

$$\Big(t_{b;0}(\bar z)\partial^b\delta(z_n),[h_1(\bar z)h_2(z_n)-\sum_{|\bar a|+|a_n|=0}^{\omega(t_{b;0})+|b|}\frac{\bar z^{\bar a}z_n^{a_n}}{\bar a!a_n!}(\partial^{\bar a}h_1)(0)(\partial^{a_n}h_2)(0)]\Big)_z=$$

$$(-1)^{|b|}(\partial^b h_2)(0)\Big(t_{b;0}(\bar z),[h_1(\bar z)-\sum_{|\bar a|=0}^{\omega(t_{b;0})}\frac{\bar z^{\bar a}}{\bar a!}(\partial^{\bar a}h_1)(0)]\Big)_{\bar z}=$$

$$\Big(t_b^{(c)}(\bar z)\partial^b\delta(z_n),h_1(\bar z)h_2(z_n)\Big)_z=$$

$$\Big(t_b^{(c)}(y_1-y_n,...,y_{n-1}-y_n)\partial^b\delta(y_n),h(y)\Big)_y. \tag{89}$$

Summing up we find the assertion (84) if (87) and (88) hold true, otherwise we have over-subtracted extensions. Note that this proof works also for $\bar t_0^\nu=0$ and hence $\bar t^{\nu(c)}=0$. Obviously this method fails for extensions $t^{(w)}$ (74) with $w\in\mathcal{D}(\mathbb{R}^{4(n-1)})$, because additional terms $\sim\partial^\nu w$ appear in (86).[18] □

In case of the axial anomaly we set $j_A^\mu\stackrel{\mathrm{def}}{=}\overline{\psi}\gamma^\mu\gamma^5\psi$, $j^\mu\stackrel{\mathrm{def}}{=}\overline{\psi}\gamma^\mu\psi$ and $j_\pi\stackrel{\mathrm{def}}{=}i\overline{\psi}\gamma^5\psi$, and have

$$\begin{aligned}t^{\mu\lambda\tau(c)}(x,x_1,x_2)&=\langle\Omega|T_3(j_A^\mu,j^\lambda,j^\tau)(x,x_1,x_2)|\Omega\rangle^{(c)},\\ \bar t^{\nu\mu\lambda\tau(c)}(x,x_1,x_2)&=2mg^{\mu\nu}\langle\Omega|T_3(j_\pi,j^\lambda,j^\tau)(x,x_1,x_2)|\Omega\rangle^{(c)}\end{aligned}\tag{90}$$

for the AVV-triangle diagram. The corresponding distributions for the AAA-triangle are obtained by replacing j^λ,j^τ by j_A^λ,j_A^τ. All $t_{b;...}$-distributions vanish. One finds $\omega(t^{(c)})=1$ and $\omega(\bar t^{(c)})=0<\omega(t^{(c)})+1$.[19] Hence, the present proof (86) does not apply.

4 Applications of the master Ward identity

The main success of the MWI are its many, important and far-reaching consequences.

4.1 Field equation

Let us consider the pair (φ,χ) of symbols (corresponding to massive or massless free fields which fulfill the Klein-Gordon or wave equation) that is studied in

[18] For the method of distribution splitting the central solution in momentum space can be obtained by a dispersion integral [18],[13],[32]. In [13] this dispersion integral has been used to prove gauge invariance of QED. The present proof (86) is a kind of x-space version of that procedure, which yields a more general result. In addition, it has the advantage that it is not necessary to treat the cases of different external legs individually.

[19] According to power counting one expects $\omega(\bar t^{(c)})=1$, but the $(\omega=1)$-terms are proportional to the spinor trace $\mathrm{tr}(\gamma^5 p_{1\mu}\gamma^\mu\gamma^\lambda p_{2\nu}\gamma^\nu\gamma^\tau p_{3\rho}\gamma^\rho)=0$.

appendix A[20] and let $W_1,...,W_n\in\mathcal{P}_0$. We assume that $W_1,...,W_n$ contain only zeroth and first (internal) derivatives of χ. By applying twice the MWI and using the explicit expressions (294)-(298) for δ^μ we obtain

$$\begin{gathered}
T_{n+1}(\varphi(\square+m^2)g\otimes W_1f_1\otimes...\otimes W_nf_n)=\\
-T_{n+1}((\partial_\mu\varphi)\partial^\mu g\otimes W_1f_1\otimes...\otimes W_nf_n)+m^2T_{n+1}(\varphi g\otimes W_1f_1\otimes...\otimes W_nf_n)\\
-i\sum_{m=1}^n\sum_{\psi\in\mathcal{G}}(\pm)T_n\Big(\Delta_{\mu\,\varphi,\psi}\big(\partial^\mu g,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)=\\
i\sum_{m=1}^n\sum_{\psi\in\mathcal{G}}(\pm)T_n\Big(\Delta^\mu_{\partial_\mu\varphi,\psi}\big(g,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)\\
-i\sum_{m=1}^n(\pm)CT_n\Big(\frac{\partial W_m}{\partial(\partial^\mu\chi)}(\partial^\mu g)f_m\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)=\\
i\sum_{m=1}^n(\pm)T_n\Big(\frac{\partial W_m}{\partial\chi}gf_m\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)\\
+i\sum_{m=1}^n(\pm)T_n\Big(\frac{\partial W_m}{\partial(\partial^\mu\chi)}(\partial^\mu g)f_m\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big).
\end{gathered}\tag{91}$$

This is the normalization condition **(N4)** of [7] and [3]. It is equivalent to

$$\begin{gathered}
T_{n+1}(\varphi g\otimes W_1f_1\otimes...\otimes W_nf_n)=\\
i\sum_{l=1}^n\sum_{\psi\in\mathcal{G}}T_n(W_1f_1\otimes...\otimes\frac{\partial W_l}{\partial\psi}f_l\Delta^F_{\psi,\varphi}\star g\otimes...\otimes W_nf_n)+...\,,
\end{gathered}\tag{92}$$

where the dots stand for the terms in which φg is not contracted. We see from this formula (92) that the normalization condition **(N4)** can always be satisfied without getting in conflict with **(N0)-(N3)**, even if anomalies are present. Note that the final result (on the r.h.s. in (4.1)) is independent from the normalization constant C which appears in the intermediate formula. This must be so, because the Feynman propagators $\Delta^F_{\psi,\varphi}$ in (92) do not contain this constant.

Generalizing Bogoliubov's idea [4] we define the interacting field $\Lambda_{g\mathcal{L}}$ belonging to $\Lambda\in\mathcal{D}(\mathbb{R}^4,\tilde{\mathcal{P}}_0)$ and to the interaction $\mathcal{L}\in\mathcal{P}_0$ in terms of the T-products by

$$\begin{gathered}
\Lambda_{g\mathcal{L}}\stackrel{\mathrm{def}}{=}S(g\mathcal{L})^{-1}\frac{d}{id\lambda}|_{\lambda=0}S(g\mathcal{L}+\lambda\Lambda)=\\
\sum_{n=0}^\infty\frac{i^n}{n!}R_{n+1}((\mathcal{L}g)^{\otimes n};\Lambda)=T_1(\Lambda)+\mathcal{O}(g),
\end{gathered}\tag{93}$$

[20] For simplicity we choose $\epsilon=1$ in (286).

where the 'totally retarded products' $R_{n+1}(...)$ (also called 'R-product') are defined by

$$R_{n+1}(\Lambda_1 \otimes ... \otimes \Lambda_n; \Lambda) \stackrel{\mathrm{def}}{=} \sum_{I \subset \{1,...,n\}} (-1)^{|I|} \bar{T}(\otimes_{l \in I} \Lambda_l) T((\otimes_{j \in I^c} \Lambda_j) \otimes \Lambda) \quad (94)$$

and we have used (18) and (19). Similarly to the S-matrix (18), the interacting fields are formal power series. In the particular case $\Lambda = Wf$, $W \in \mathcal{P}_0$, $f \in \mathcal{D}(\mathbb{R}^4)$ we write $W_{g\mathcal{L}}(f)$ instead of $(Wf)_{g\mathcal{L}}$. Following [7] the condition (4.1) can easily be translated into an identity for $R_{n+1}(W_1 f_1 \otimes ... \otimes W_n f_n; \varphi(\square + m^2) f)$. The latter implies the field equation

$$(\square + m^2)\varphi_{g\mathcal{L}} = -g\Big(\frac{\partial \mathcal{L}}{\partial \chi}\Big)_{g\mathcal{L}} + \partial^\mu [g\Big(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \chi)}\Big)_{g\mathcal{L}}], \quad (95)$$

where g is a test function.

The calculation (4.1) can be carried over to external derivatives by using $(\tilde{\mathbf{N}})$ instead of **(N)**. More precisely let $W, W_1, ..., W_n \in \mathcal{P}_0$ and let us assume that $W_1, ..., W_n$ contain only zeroth and first (internal) derivatives of χ. Then we obtain

$$\begin{aligned} &T_{n+1}(((\tilde{\square} + m^2)\varphi) W g \otimes W_1 f_1 \otimes ... \otimes W_n f_n) = \\ &i \sum_{m=1}^n \sum_{\psi \in \mathcal{G}} (\pm) T_n \Big(\Delta^\mu_{\partial_\mu \varphi, \psi} (Wg, \frac{\partial W_m}{\partial \psi} f_m) \otimes W_1 f_1 \otimes ...\hat{m}... \otimes W_n f_n \Big) \\ &+ i \sum_{m=1}^n \sum_{\psi \in \mathcal{G}} (\pm) T_n \Big(\Delta^\mu_{\tilde{\partial}_\mu \varphi, \psi} (Wg, \frac{\partial W_m}{\partial \psi} f_m) \otimes W_1 f_1 \otimes ...\hat{m}... \otimes W_n f_n \Big) = \\ &i \sum_{m=1}^n (\pm) T_n \Big(W \frac{\partial W_m}{\partial \chi} g f_m \otimes W_1 f_1 \otimes ...\hat{m}... \otimes W_n f_n \Big) \\ &+ i \sum_{m=1}^n (\pm) T_n \Big(W \frac{\partial W_m}{\partial(\partial_\mu \chi)} (\partial_\mu g) f_m \otimes W_1 f_1 \otimes ...\hat{m}... \otimes W_n f_n \Big) \\ &+ i \sum_{m=1}^n (\pm) T_n \Big((\tilde{\partial}_\mu W) \frac{\partial W_m}{\partial(\partial_\mu \chi)} g f_m \otimes W_1 f_1 \otimes ...\hat{m}... \otimes W_n f_n \Big). \quad (96) \end{aligned}$$

In the first equation we have twice applied $(\tilde{\mathbf{N}})$ and in the second step we have inserted the explicit formulas (294)-(298) and (41) for δ^μ as well as the definition (36) of Δ^μ. In the special case that no derivatives of χ are present, the last two terms on the r.h.s. vanish.

4.2 Charge- and ghost-number conservation

We consider massive or massless spinors $\psi, \overline{\psi} \in \mathcal{P}_0$ fulfilling the Dirac equation and in particular the matter current $j_\mu \stackrel{\mathrm{def}}{=} \overline{\psi}\gamma_\mu \psi$ (which is conserved). We assume $W_1, ..., W_n \in \mathcal{P}_0$ and that no derivatives of ψ and $\overline{\psi}$ are present. **Charge**

conservation is expressed by the following Ward identity **(N5) (charge)** which is an immediate consequence of the master Ward identity **(N)**

$$
\begin{aligned}
&-T_{n+1}(j_\mu\partial^\mu g\otimes W_1f_1\otimes...\otimes W_nf_n)=\\
&i\sum_{m=1}^n\Big[(\pm)T_n\Big(\Delta^\mu_{\gamma_\mu\psi,\overline{\psi}}\big(\frac{\partial j_\mu}{\partial(\gamma_\mu\psi)}g,\frac{\partial W_m}{\partial\overline{\psi}}f_m\big)\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)\\
&+(\pm)T_n\Big(\Delta^\mu_{\overline{\psi}\gamma_\mu,\psi}\big(\frac{\partial j_\mu}{\partial(\overline{\psi}\gamma_\mu)}g,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)\Big]=\\
&\sum_{m=1}^n T_n\Big(W_1f_1\otimes...\otimes\big(\overline{\psi}\frac{\partial W_m}{\partial\overline{\psi}}-\psi\frac{\partial W_m}{\partial\psi}\big)gf_m\otimes...\otimes W_nf_n\Big).
\end{aligned}
\tag{97}
$$

In the second step we have used the formulas (302)-(303) for δ^μ. Each monomial W is an eigenvector of the operator $(\overline{\psi}\frac{\partial}{\partial\overline{\psi}}-\psi\frac{\partial}{\partial\psi})$ with eigenvalue: (number of $\overline{\psi}$ in W) minus (number of ψ in W), which we call 'spinor charge'. That this Ward identity can be satisfied by choosing suitable normalizations which are compatible with **(N0)-(N4)** has been proved in [7] for the case that $W_1,...,W_n$ are sub-monomials of the QED-interaction $\mathcal{L}=A^\mu\overline{\psi}\gamma_\mu\psi$.

We turn to models which contain pairs $(\tilde{u}_a,\,u_a)$ of massive or massless, scalar, but **fermionic ghost fields**, e.g. non-Abelian gauge theories (see appendix A for the anti-commutators and Feynman propagators of the free ghost fields $\tilde{u}_a,\,u_a\in\mathcal{P}_0$.) The free ghost current

$$
k^\mu=i\sum_a[u_a\partial^\mu\tilde{u}_a-\partial^\mu u_a\tilde{u}_a]
\tag{98}
$$

is conserved, because u_a, $\tilde{u}_a$ satisfy the Klein-Gordon or wave equation. Let $W_1,...,W_n\in\mathcal{P}_0$ and we assume that only zeroth and first (internal) derivatives of u_a and $\tilde{u}_a$ appear in $W_1,...,W_n$. Similarly to (97) the MWI **(N)** implies the following Ward identity **(N5) (ghost)**:

$$
\begin{aligned}
&-T_{n+1}(k_\mu\partial^\mu h\otimes W_1f_1\otimes...\otimes W_nf_n)=\\
&\sum_{m=1}^n T_n\Big(W_1f_1\otimes...\otimes\Big[\Big(u_a\frac{\partial W_m}{\partial u_a}-C_{u_a}\partial^\mu u_a\frac{\partial W_m}{\partial(\partial^\mu u_a)}\\
&-\tilde{u}_a\frac{\partial W_m}{\partial\tilde{u}_a}+C_{u_a}\partial^\mu\tilde{u}_a\frac{\partial W_m}{\partial(\partial^\mu\tilde{u}_a)}\Big)hf_m\\
&+(1+C_{u_a})\Big((\tilde{\partial}^\mu u_a)\frac{\partial W_m}{\partial(\partial^\mu u_a)}hf_m+u_a\frac{\partial W_m}{\partial(\partial^\mu u_a)}(\partial^\mu h)f_m\\
&-(\tilde{\partial}^\mu\tilde{u}_a)\frac{\partial W_m}{\partial(\partial^\mu\tilde{u}_a)}hf_m-\tilde{u}_a\frac{\partial W_m}{\partial(\partial^\mu\tilde{u}_a)}(\partial^\mu h)f_m\Big)\Big]\otimes...\otimes W_nf_n\Big),
\end{aligned}
\tag{99}
$$

where the normalization constant C appearing in (295), (298) is specified by a lower index u_a. Every monomial W is an eigenvector of the operator

$$
\Theta_g\stackrel{\mathrm{def}}{=}u_a\frac{\partial}{\partial u_a}+(\partial^\mu u_a)\frac{\partial}{\partial(\partial^\mu u_a)}-\tilde{u}_a\frac{\partial}{\partial\tilde{u}_a}-(\partial^\mu\tilde{u}_a)\frac{\partial}{\partial(\partial^\mu\tilde{u}_a)}
\tag{100}
$$

and the eigenvalue is the ghost number $g(W)$:

$$\Theta_g W = g(W)W, \qquad g(W) \in \mathbb{Z}. \tag{101}$$

The identity (99) expresses **ghost number conservation** correctly if and only if

$$C_{u_a} = -1, \qquad \forall a. \tag{102}$$

With this normalization **(N5) (ghost)** takes the form

$$-T_{n+1}(k_\mu \partial^\mu h \otimes W_1 f_1 \otimes ... \otimes W_n f_n) = \\ \sum_{m=1}^{n} g(W_m) T_n\Big(W_1 f_1 \otimes ... \otimes W_m f_m h \otimes ... \otimes W_n f_n\Big) \tag{103}$$

for monomials $W_1, ..., W_m \in \mathcal{P}_0$. That the normalization condition **(N5) (ghost)** (with $C_{u_a} = -1$) has common solutions with **(N0)-(N4)** has been proved in [3] by using the method of [7] appendix B. (A slight restriction on $W_1, ..., W_n$ is used in that proof).

Remark: The (free) ghost charge Q_g is defined by

$$Q_g \stackrel{\mathrm{def}}{=} \int_{x^0=\mathrm{const.}} d^3x \, T_1(k^0(x)). \tag{104}$$

(N5) (ghost) implies the identity

$$[Q_g, T_n\Big(W_1 f_1 \otimes ... \otimes W_n f_n\Big)] = \Big(\sum_{m=1}^{n} g(W_m)\Big) T_n\Big(W_1 f_1 \otimes ... \otimes W_n f_n\Big) \tag{105}$$

as can be seen by a suitable choice of the test-function h in (103). For the details of this conclusion as well as for the existence of Q_g see the corresponding procedure (120)-(123) for the free BRST-current.

4.3 Non-Abelian matter currents

The aim of this subsection is to derive the identity (5) from the MWI. Let

$$j_a^\mu \stackrel{\mathrm{def}}{=} \overline{\psi}_\alpha \gamma^\mu \frac{(\lambda_a)_{\alpha\beta}}{2} \psi_\beta \tag{106}$$

(we use matrix notation for the spinor structur) and

$$[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c, \tag{107}$$

where $(f_{abc})_{a,b,c}$ are the structure constants of some Lie algebra. We assume that the masses of the spinor fields are colour independent

$$(i\gamma_\mu \partial^\mu - m)\psi_\alpha = 0, \qquad \forall \alpha, \tag{108}$$

which implies

$$\partial_\mu j^\mu_a = 0. \tag{109}$$

We denote by $(A_a)_a$ the gauge fields and by $(u_a, \tilde{u}_a)_a$ the corresponding fermionic ghost fields, and consider an interaction of the form

$$\mathcal{L} = j^\mu_a A_{a\mu} + \mathcal{L}_1(A, u, \tilde{u}), \tag{110}$$

where $\mathcal{L}_1(A, u, \tilde{u})$ is a polynomial in the symbols $A, u, \tilde{u}$ and internal derivatives thereof. QCD fits in this framework: the quark fields ψ_α are in the fundamental representation of $SU(3)$.

To apply the MWI we need

$$i\sum_{\chi,\varphi} \Delta^\mu_{\chi,\varphi}\Big(\frac{\partial j_{a\mu}}{\partial\chi} f, \frac{\partial j^\nu_b}{\partial\varphi} h\Big) = \frac{fh}{4}\overline{\psi}\gamma^\nu[\lambda_a, \lambda_b]\psi = ifhf_{abc}j^\nu_c \tag{111}$$

((302) and (303) are used), which gives

$$i\sum_{\chi,\varphi} \Delta^\mu_{\chi,\varphi}\Big(\frac{\partial j_{a\mu}}{\partial\chi} f, \frac{\partial \mathcal{L}}{\partial\varphi} g\Big) = ifgf_{abc}A_{b\nu}j^\nu_c \tag{112}$$

by contracting with $A_{b\nu}$. So the MWI for $T(g\mathcal{L}\otimes ... \otimes g\mathcal{L}\otimes j^\mu_a\partial_\mu f)$ implies

$$-R_{n+1}((g\mathcal{L})^{\otimes n}; j^\mu_a\partial_\mu f) = inR_{n+1}((g\mathcal{L})^{\otimes(n-1)}; f_{abc}A_{b\nu}j^\nu_c fg), \tag{113}$$

and hence

$$j^\mu_{a\,g\mathcal{L}}(\partial_\mu f) = (f_{abc}A_{b\nu}j^\nu_c)_{g\mathcal{L}}(fg), \tag{114}$$

which corresponds to the covariant conservation of the interacting classical current.

To formulate (5) we need the time-ordered product $T_{g\mathcal{L}}(W_1f_1\otimes ...\otimes W_mf_m)$ of the interacting fields $W_{1\,g\mathcal{L}}(f_1), ..., W_{m\,g\mathcal{L}}(f_m)$, which is defined by generalizing (93) (cf. [4], [18])

$$\begin{aligned} &T_{g\mathcal{L}}(W_1f_1\otimes ... \otimes W_mf_m) \stackrel{\mathrm{def}}{=} \\ &S(g\mathcal{L})\frac{d^m}{i^m d\lambda_1...d\lambda_m}|_{\lambda_1=...=\lambda_m=0} S(g\mathcal{L}+\sum_{l=1}^m \lambda_l W_l f_l) = \\ &\sum_{n=0}^\infty \frac{i^n}{n!} R_{n,m}((g\mathcal{L})^{\otimes n}; W_1f_1\otimes ... \otimes W_mf_m) \end{aligned} \tag{115}$$

with[21]

$$\begin{aligned} &R_{n,m}(g_1V_1\otimes ... \otimes g_nV_n; W_1f_1\otimes ...\otimes W_mf_m) \stackrel{\mathrm{def}}{=} \\ &\sum_{I\subset\{1,...,n\}} (-1)^{|I|}\bar{T}(\otimes_{l\in I}g_lV_l)T((\otimes_{j\in I^c}g_jV_j)\otimes(\otimes_{k=1}^m f_kW_k)). \end{aligned} \tag{116}$$

[21] The connection to the notation (94) reads: $R_{n,1} \equiv R_{n+1}$.

By using (109), (111) and (112) the master Ward identity yields

$$-R_{n,2}((g\mathcal{L})^{\otimes n};j_a^\mu\partial_\mu f\otimes j_b^\nu h)=R_{n,1}((g\mathcal{L})^{\otimes n};if_{abc}j_c^\nu fh)$$
$$+inR_{n-1,2}((g\mathcal{L})^{\otimes(n-1)};f_{acd}A_{c\tau}j_d^\tau fg\otimes j_b^\nu h) \qquad (117)$$

which gives

$$-T_{g\mathcal{L}}(j_a^\mu\partial_\mu f\otimes j_b^\nu h)=if_{abc}j^\nu_{c\,g\mathcal{L}}(fh)-T_{g\mathcal{L}}(f_{acd}A_{c\tau}j_d^\tau fg\otimes j_b^\nu h). \qquad (118)$$

Due to (114) this is the formulation of (5) in the framework of causal perturbation theory. In the simple case that the gauge fields A_a are external fields (which implies $\mathcal{L}_1(A,u,\tilde{u})\equiv 0$) and the spinor fields are massive ($m>0$), the proof of sect. 3.3 applies, i.e. the central extensions fulfil the MWI. (Note that no factor m appears in (113) and (117), which indicates that the scaling degree is not lowered.)

4.4 The master BRST-identity

We consider free gauge fields A_a^μ, $a=1,...,N$, with mass $m_a\geq 0$ in Feynman gauge and the corresponding free ghost fields $\tilde{u}_a$, u_a with the same mass m_a. For each fixed value of a and μ the field A_a^μ is quantized as a real scalar field satisfying the Klein-Gordon or wave equation, i.e. in the formalism of appendix A we set $\varphi=A_a^\mu=\chi$, $\epsilon=1$. The free ghost fields fulfil the same algebraic relations as in sect. 4.2 and in appendix A. For each massive gauge field A_a^μ, $m_a>0$, we introduce a free, real scalar field ϕ_a with the same mass m_a, which is quantized with a minus sign in the commutator, i.e. we have $\varphi=\phi_a=\chi,\epsilon=-1$ in the formalism of appendix A. (For the Fock space representation of these free fields see e.g. [33].) There is no obstacle to include spinor fields in our treatment of BRST-symmetry (sects. 4.4 and 4.5), see [7], the last paper of of [11], [14] and [20].

The free BRST-current (cf. [23],[14])

$$j^\mu\stackrel{\mathrm{def}}{=}\sum_a[(\partial_\tau A_a^\tau+m_a\phi_a)\partial^\mu u_a-\partial^\mu(\partial_\tau A_a^\tau+m_a\phi_a)u_a] \qquad (119)$$

is conserved, because $\partial_\tau A_a^\tau$, u_a and ϕ_a fulfill the Klein-Gordon equation with the same mass m_a. We will see that the corresponding charge

$$Q_0\stackrel{\mathrm{def}}{=}\int_{x^0=\mathrm{const.}}d^3x\,T_1(j^0)(x), \qquad (120)$$

is the generator of the BRST-transformation of the free fields and Wick monomials. Q_0 is nilpotent,

$$2Q_0^2=[Q_0,Q_0]_+=0, \qquad (121)$$

because $[(\partial_\tau A_a^\tau+m_a\phi_a),(\partial_\rho A_b^\rho+m_b\phi_b)]=0$. Without the scalar fields ϕ_a the charge Q_0 would not be nilpotent, if some gauge fields are massive. So, a main

purpose of the the scalar fields ϕ_a is to restore the nilpotency of Q_0. (For a rigorous definition of Q_0, with 4-dimensional smearing with a test function and taking a suitable limit, see [7] where a method of Requardt [31] is used.)

To obtain the master BRST-identity (i.e. the (anti)commutator of Q_0 with arbitrary T-products (2)) we compute

$$T_{n+1}(j_\mu\partial^\mu g\otimes W_1f_1\otimes ...\otimes W_nf_n) \tag{122}$$

by means of the MWI **(N)**. Thereby we assume that $W_1,...,W_n$ have an even or odd ghost number (no mixture). From this result we shall get $[Q_0,T(W_1,...,W_n)]_\mp$ in the following way: let $\mathcal{O}$ be an open double cone with supp $f_j\subset\mathcal{O}$, $\forall j=1,...,n$. Following [7] (appendix B) we choose g to be equal to 1 on a neighbourhood of $\overline{\mathcal{O}}$ and decompose $\partial^\mu g=b^\mu-a^\mu$ such that supp $a^\mu\cap(\overline{V}_-+\mathcal{O})=\emptyset$ and supp $b^\mu\cap(\overline{V}_++\mathcal{O})=\emptyset$. Then we apply causal factorization of the T-products:

$$\begin{gathered}-T_{n+1}(j_\mu(\partial^\mu g)\otimes W_1f_1\otimes ...\otimes W_nf_n)=\\ T_1(j_\mu a^\mu)T_n(W_1f_1\otimes ...\otimes W_nf_n)\mp T_n(W_1f_1\otimes ...\otimes W_nf_n)T_1(j_\mu b^\mu)=\\ [T_1(j_\mu a^\mu),T_n(W_1f_1\otimes ...\otimes W_nf_n)]_\mp\mp T_n(W_1f_1\otimes ...\otimes W_nf_n)T_1(j_\mu\partial^\mu g).\end{gathered} \tag{123}$$

The last term on the r.h.s. vanishes because of $\partial^\mu j_\mu=0$. Since $T_n(W_1f_1\otimes ...\otimes W_nf_n)$ is localized in $\mathcal{O}$, we may vary a^μ in the spatial complement of $\overline{\mathcal{O}}$ without affecting $[T_1(j_\mu a^\mu),T_n(W_1f_1\otimes ...)]_\mp$. In this way and by using $\partial^\mu j_\mu=0$ we find

$$[T_1(j_\mu a^\mu),T_n(W_1f_1\otimes ...\otimes W_nf_n)]_\mp=[Q_0,T_n(W_1f_1\otimes ...\otimes W_nf_n)]_\mp \tag{124}$$

(see [7], appendix B for details of this conclusion).

We start the computation of (122) with the simplest case: $n=1$. We assume that the symbols in W carry at most a first (internal) derivative (no higher derivatives) and give the calculation in detail

$$-T_2(j_\mu(\partial^\mu g)\otimes Wf)=i\sum_{\chi,\psi\in\mathcal{G}}T_1\Big(\Delta^\mu_{\chi,\psi}\big(\frac{\partial j_\mu}{\partial\chi}g,\frac{\partial W}{\partial\psi}f\big)\Big). \tag{125}$$

The explicit results for the Δ^μ with a non-vanishing contribution are listed in appendix B. Thereby C_{A_a} (C_{1A_a} resp.), C_{ϕ_a} and C_{u_a} mean the normalization constants $C(C_1$ resp.) in the cases $\varphi=A^\mu_a=\chi,\varphi=\phi_a=\chi$ and $\varphi=\tilde{u}_a,\chi=u_a$. In the present context they may depend on the colour index a. Inserting (305)-(316) into (125) we obtain

$$\begin{aligned}-T_2(j_\mu(\partial^\mu g)\otimes Wf)=iT_1\Big(\Big[&(\partial^\mu u_a)\frac{\partial W}{\partial A^\mu_a}+(\partial^\sigma\partial^\mu u_a)\frac{\partial W}{\partial(\partial^\sigma A^\mu_a)}\\ &-(\partial_\tau A^\tau_a+m_a\phi_a)\frac{\partial W}{\partial\tilde{u}_a}-(\partial_\nu(\partial_\tau A^\tau_a+m_a\phi_a))\frac{\partial W}{\partial(\partial_\nu\tilde{u}_a)}\\ &+m_au_a\frac{\partial W}{\partial\phi_a}+m_a(\partial_\mu u_a)\frac{\partial W}{\partial(\partial_\mu\phi_a)}\Big]gf\\ &+\Big[...\Big](\partial^\nu g)f+\Big[...\Big](\partial^\nu\partial^\sigma g)f+\Big[...\Big](\square g)f\Big)\end{aligned} \tag{126}$$

by means of $T_1((\tilde{\partial}^a V)Wg)=T_1((\partial^a V)Wg)$. (The terms which are not written out depend on the normalization constants $C_{A_a}, C_{\phi_a}, C_{u_a}$ and C_{1A_a}.) On the r.h.s. of (126) we use $gf=f$ and $(\partial^a g)f=0, \forall |a|\geq 1$. We end up with

$$[Q_0, T_1(Wf)]_{\mp}=iT_1\Big([(\partial^\mu u_a)\frac{\partial W}{\partial A_a^\mu}+(\partial^\sigma\partial^\mu u_a)\frac{\partial W}{\partial(\partial^\sigma A_a^\mu)}$$
$$-(\partial_\tau A_a^\tau+m_a\phi_a)\frac{\partial W}{\partial \tilde{u}_a}-(\partial_\nu(\partial_\tau A_a^\tau+m_a\phi_a))\frac{\partial W}{\partial(\partial_\nu \tilde{u}_a)}+m_a u_a\frac{\partial W}{\partial \phi_a}$$
$$+m_a(\partial_\mu u_a)\frac{\partial W}{\partial(\partial_\mu\phi_a)}]f\Big)\stackrel{\mathrm{def}}{=}T_1(s_0(W)f) \tag{127}$$

where we have the anti-commutator iff W has an odd ghost number. We point out that the result is independent from the normalization constants $C_{A_a}, C_{\phi_a}, C_{u_a}$ and C_{1A_a}. The result (127) is the well-known free BRST-transformation of a Wick polynomial $W\to s_0(W)$ (cf. [11]) which we have obtained here with quite a lot of calculations. Note that in our framework s_0 is a derivation $s_0:\mathcal{P}_0\to\mathcal{P}_0$.

However, the advantage of the present method is that it can be used to compute commutators of Q_0 with T-products of higher orders. For $n=2$ in (122) we obtain

$$-T_3(j_\mu(\partial^\mu g)\otimes W_1f_1\otimes W_2f_2)=i\sum_{\chi,\psi\in\mathcal{G}}\Big[T_2\Big(\Delta^\mu_{\chi,\psi}(\frac{\partial j_\mu}{\partial\chi}g,\frac{\partial W_1}{\partial\psi}f_1)\otimes W_2f_2\Big)$$
$$+(\pm)[(W_1,f_1)\leftrightarrow(W_2,f_2)]\Big] \tag{128}$$

where $(\pm)$ is still a sign coming from permutations of Fermi operators. We insert the expressions (305)-(316) for the various Δ^μ. For given f_1, f_2 we then choose g as in (123), hence $gf_j=f_j$ and $(\partial^a g)f_j=0, \forall|a|\geq 1$. It results

$$[Q_0, T_2(W_1f_1\otimes W_2f_2)]_{\mp}=i\Big[T_2\Big(\Big[(\frac{1}{4}(\partial^\mu u_a)+\frac{3}{4}(\tilde{\partial}^\mu u_a))\frac{\partial W_1}{\partial A_a^\mu}$$
$$+[(C_{A_a}+\frac{1}{2}+2C_{1A_a})(\tilde{\partial}^\mu\partial_\mu u_a)-(\frac{1}{2}+2C_{1A_a})\tilde{\square}u_a+C_{A_a}m_a^2u_a]\frac{\partial W_1}{\partial(\partial_\nu A_a^\nu)}$$
$$+[-C_{1A_a}(\tilde{\partial}^\sigma\partial^\nu u_a)+(\frac{1}{2}+2C_{1A_a})(\tilde{\partial}^\nu\partial^\sigma u_a)+(\frac{1}{2}-C_{1A_a})(\tilde{\partial}^\nu\tilde{\partial}^\sigma u_a)]\frac{\partial W_1}{\partial(\partial^\sigma A_a^\nu)}$$
$$-(\partial_\tau A_a^\tau+m_a\phi_a)\frac{\partial W_1}{\partial\tilde{u}_a}+[-(1+C_{u_a})(\tilde{\partial}_\nu(\partial_\tau A_a^\tau+m_a\phi_a)$$
$$+C_{u_a}(\partial_\nu(\partial_\tau A_a^\tau+m_a\phi_a))]\frac{\partial W_1}{\partial(\partial_\nu\tilde{u}_a)}$$
$$+m_au_a\frac{\partial W_1}{\partial\phi_a}+m_a[(1+C_{\phi_a})\tilde{\partial}_\nu u_a-C_{\phi_a}\partial_\nu u_a]\frac{\partial W_1}{\partial(\partial_\nu\phi_a)}\Big]f_1\otimes W_2f_2\Big)$$
$$+(\pm)[(W_1,f_1)\leftrightarrow(W_2,f_2)]\Big]. \tag{129}$$

To simplify this expression we insert the value $C_{u_a}=-1$ which is required from ghost number conservation (99)-(102). By means of $(\tilde{\mathbf{N}})$ we replace the external

derivatives by internal ones

$$[Q_0, T_2(W_1 f_1 \otimes W_2 f_2)]_\mp = \Big[T_2\Big(s_0(W_1) f_1 \otimes W_2 f_2\Big)$$
$$+T_1\Big(G^{(1)}(W_1 f_1, W_2 f_2)\Big)\Big] + (\pm)\Big[(W_1, f_1) \leftrightarrow (W_2, f_2)\Big], \tag{130}$$

where

$$G^{(1)}(W_1 f_1, W_2 f_2) \stackrel{\text{def}}{=} -\sum_{\psi \in \mathcal{G}} \Big[\frac{3}{4}\Delta^\mu_{u_a,\psi}(\frac{\partial W_1}{\partial A^\mu_a} f_1, \frac{\partial W_2}{\partial \psi} f_2)$$
$$+C_{A_a}\Delta^\mu_{\partial_\mu u_a,\psi}(\frac{\partial W_1}{\partial(\partial_\nu A^\nu_a)} f_1, \frac{\partial W_2}{\partial \psi} f_2)$$
$$-(\frac{1}{2} + 2C_{1A_a})\Delta^\mu_{\tilde{\partial}_\mu u_a,\psi}(\frac{\partial W_1}{\partial(\partial_\nu A^\nu_a)} f_1, \frac{\partial W_2}{\partial \psi} f_2)$$
$$+(\frac{1}{2} - 2C_{1A_a})\Delta^\sigma_{\partial^\nu u_a,\psi}(\frac{\partial W_1}{\partial(\partial^\sigma A^\nu_a)} f_1, \frac{\partial W_2}{\partial \psi} f_2)$$
$$+(\frac{1}{2} + 2C_{1A_a})\Delta^\nu_{\partial^\sigma u_a,\psi}(\frac{\partial W_1}{\partial(\partial^\sigma A^\nu_a)} f_1, \frac{\partial W_2}{\partial \psi} f_2)$$
$$+(\frac{1}{2} - C_{1A_a})\Delta^\nu_{\tilde{\partial}^\sigma u_a,\psi}(\frac{\partial W_1}{\partial(\partial^\sigma A^\nu_a)} f_1, \frac{\partial W_2}{\partial \psi} f_2)$$
$$+m_a(1 + C_{\phi_a})\Delta^\nu_{u_a,\psi}(\frac{\partial W_1}{\partial(\partial^\nu \phi_a)} f_1, \frac{\partial W_2}{\partial \psi} f_2)\Big]. \tag{131}$$

Note that $G^{(1)}(\cdot,\cdot)$ is not invariant with respect to the exchange of the two arguments. Now we assume that $s_0(W_j)$ is a divergence, i.e. that there exists a (Lorentz) vector $(W'_{j\nu})_{\nu=0,\dots,3}$, $W'_{j\nu} \in \mathcal{P}_0$ with

$$s_0(W_j) = i\partial^\nu W'_{j\nu}, \quad j = 1, 2. \tag{132}$$

By means of the MWI **(N)** we shift this derivative to the test-function

$$[Q_0, T_2(W_1 f_1 \otimes W_2 f_2)]_\mp = \Big[-iT_2\Big(W'_{1\nu}\partial^\nu f_1 \otimes W_2 f_2\Big)$$
$$+T_1\Big(G((W_1, W'_1) f_1, W_2 f_2)\Big)\Big] + (\pm)\Big[(W_1, W'_1, f_1) \leftrightarrow (W_2, W'_2, f_2)\Big], \tag{133}$$

where

$$G((W_1, W'_1) f_1, W_2 f_2) = G^{(1)}(W_1 f_1, W_2 f_2) + G^{(2)}(W'_1 f_1, W_2 f_2), \tag{134}$$

with

$$G^{(2)}(W'_1 f_1, W_2 f_2) \stackrel{\text{def}}{=} \sum_{\chi,\psi \in \mathcal{G}} \Delta^\nu_{\chi,\psi}(\frac{\partial W'_{1\nu}}{\partial \chi} f_1, \frac{\partial W_2}{\partial \psi} f_2). \tag{135}$$

If we only know that $s_0(W_1)$ is a divergence, our final result reads

$$[Q_0, T_2(W_1 f_1 \otimes W_2 f_2)]_\mp =$$
$$-iT_2\Big(W'_{1\nu}\partial^\nu f_1 \otimes W_2 f_2\Big) + T_1(G((W_1, W'_{1\nu}) f_1, W_2 f_2))$$
$$+(\pm)\Big[T_2(s_0(W_2) f_2 \otimes W_1 f_1) + T_1(G^{(1)}(W_2 f_2, W_1 f_1))\Big] \tag{136}$$

instead of (133).

The $(n=2)$-calculation generalizes to higher orders $n\geq 2$ in a straightforward way: let $W_1,...,W_k,V_1,...,V_{n-k}\in\mathcal{P}_0$ with $s_0(W_j)=i\partial^\nu W'_{j\nu}$, $\forall j=1,...,k$ and $f_j,h_i\in\mathcal{D}(\mathbb{R}^4)$. For simplicity we assume that each polynomial $W_1,...,W_k,V_1,...,V_{n-k}$ has an even ghost number, otherwise some additional, obvious signs appear in the following formula. Setting $m\stackrel{\mathrm{def}}{=}n-k$ we obtain

$$\begin{aligned}
&[Q_0,T_n(W_1f_1\otimes...\otimes W_kf_k\otimes V_1h_1\otimes...\otimes V_mh_m)]=\\
&=-i\sum_{l=1}^{k}T_n(W_1f_1\otimes...\otimes W'_{l\nu}\partial^\nu f_l\otimes...\otimes W_kf_k\otimes V_1h_1\otimes...)\\
&\quad+\sum_{l=1}^{m}T_n(W_1f_1\otimes...\otimes V_1h_1\otimes...\otimes s_0(V_l)h_l\otimes...\otimes V_mh_m)\\
&+\sum_{l,r=1\ (l\neq r)}^{k}T_{n-1}(G((W_l,W'_l)f_l,W_rf_r)\otimes W_1f_1\otimes...\hat{l}...\hat{r}...\otimes W_kf_k\otimes V_1h_1\otimes...)\\
&\quad+\sum_{l=1}^{k}\sum_{r=1}^{m}[T_{n-1}(G((W_l,W'_l)f_l,V_rh_r)\otimes W_1f_1\otimes...\hat{l}...\otimes V_1h_1\otimes...\hat{r}...)\\
&\qquad+T_{n-1}(G^{(1)}(V_rh_r,W_lf_l)\otimes W_1f_1\otimes...\hat{l}...\otimes V_1h_1\otimes...\hat{r}...)]\\
&+\sum_{l,r=1\ (l\neq r)}^{m}T_{n-1}(G^{(1)}(V_lh_l,V_rh_r)\otimes W_1f_1\otimes...\otimes V_1h_1\otimes...\hat{l}...\hat{r}...\otimes V_mh_m),
\end{aligned}\tag{137}$$

where $\hat{l}$ or $\hat{r}$ means that the corresponding factor is omitted. We call this equation the **'master BRST-identity'**. It is a consequence of the master Ward identity. Hence, the master BRST-identity is also a normalization condition.

So far we have not spoken about the interaction $\mathcal{L}\equiv\mathcal{L}_0$; the master BRST-identity is a condition on T-products of arbitrary factors. Now we require that the interaction is s_0-invariant in some sense. The requirement $s_0\mathcal{L}_0=0$ is too restrictive, it is not satisfied for physically relevant models. So we impose the weaker condition that $s_0\mathcal{L}_0$ is a divergence:

$$s_0\mathcal{L}_0=i\partial^\nu\mathcal{L}_{1\nu}.\tag{138}$$

The requirements that

(a) the master BRST-identity becomes particularly simple, and

(b) can be satisfied to all orders

for T-products involving the interaction are good criterions (among others) to restrict $\mathcal{L}_0$ further. We will make (a) explicit by the formula

$$G((\mathcal{L}_0,\mathcal{L}_1)f,\mathcal{L}_0g)+(f\leftrightarrow g)=0.\tag{139}$$

(b) means that anomalies (in the master BRST-identity) may not occur or must cancel. It is a hard job to work this out. For example it is well-known that in

weak interactions the axial anomalies cancel only if the numbers of generations for leptons and quarks agree.

For an interaction $\mathcal{L}$ fulfilling (138) and (139) the validity of the master BRST-identity for $[Q_0, T_n(\mathcal{L}, ..., \mathcal{L})]$ $\forall n \in \mathbb{N}$ implies

$$[Q_0, S] = 0, \tag{140}$$

where S is the S-matrix in the adiabatic limit,

$$S \stackrel{\mathrm{def}}{=} \lim_{g\to 1} S(g\mathcal{L}), \tag{141}$$

provided this limit exists [19]. Hence, S induces a well-defined operator on the physical Hilbert space $\mathcal{H}_{\mathrm{phys}} = \frac{\ker Q_0}{\mathrm{ran}\, Q_0}$, which is unitary if $\mathcal{L}_0^* = \mathcal{L}_0$ and **(N2)** is satisfied [19], [14], [15], [20].

Having determined the interaction by using (138), (139) and other (quite obvious) requirements, we will show that the validity of the master BRST-identity and of the ghost number conservation **(N5) (ghost)** suffices for a local construction of observables in non-Abelian gauge theories. This is a generalization of the corresponding construction for QED in [7]. In particular we will obtain an explicit formula for the computation of the nonlinear term in the BRST-transformation of an arbitrary interacting field.

4.5 Local construction of observables in gauge theories

For massive gauge fields the procedure is more involved. So we first treat massless gauge fields and afterwards give the modifications for the massive case.

4.5.1 Massless gauge fields: determination of the interaction

Since we are considering solely massless fields ($m_a = 0$, $\forall a$), the scalar fields ϕ_a are superfluous. So we set $\phi_a \equiv 0$, $\forall a$.

First we **determine the interaction** $\mathcal{L}_0$ by the following requirements (cf. [36], [17], [20], [15] and [33]):

(A) There exist $\mathcal{L}_j \in (\mathcal{P}_0)^{\oplus j}$, $j = 0, 1, ..., M$ which satisfy the ladder equations

$$s_0 \mathcal{L}_j^{\mu_1 \dots \mu_j} = i\partial_{\mu_{j+1}} \mathcal{L}_{j+1}^{\mu_1 \dots \mu_j \mu_{j+1}}, \quad j = 0, 1, ..., M-1, \quad s_0 \mathcal{L}_M = 0 \tag{142}$$

(B) $\mathcal{L}_j$ is a polynomial in the gauge field A_a^μ and in the fermionic ghost fields $u_a, \tilde{u}_a$, $a = 1, ..., N$, and internal derivatives of these symbols; each monomial has at least three factors.

(C) $\mathcal{L}_j$ has UV-dimension ≤ 4.

(D) $\mathcal{L}_j$ is a Lorentz tensor of rank j.

(E) $\mathcal{L}_j$ has ghost number j:

$$g(\mathcal{L}_j) = j \tag{143}$$

(cf. (101)). Thereby we take into account that s_0 increases the ghost number by 1. We conclude that the ladder (142) stops at $M \leq 3$ for trilinear terms.

(F) unitarity (for $\mathcal{L}_0$ only): $\mathcal{L}_0^* = \mathcal{L}_0$.

Following [36] we make the most general ansatz for $\mathcal{L}_j,\ j = 0,1,2,3$ which satisfies (B)-(F) and insert it into (142). The calculation excludes quadrilinear terms in $\mathcal{L}_0$. Using $F^{\nu\mu} \stackrel{\text{def}}{=} \partial^\nu A^\mu - \partial^\mu A^\nu$ the most general solution for $\mathcal{L}_0$ reads

$$\mathcal{L}_0 = g_0 f_{abc}[\frac{1}{2} A_{a\mu} A_{b\nu} F_c^{\nu\mu} - u_a \partial^\mu \tilde{u}_b A_{c\mu}] - i s_0 K_1 + \partial_\nu K_2^\nu, \tag{144}$$

where f_{abc} must be totally antisymmetric and $g_0 \in \mathbb{R}$ is a constant. This implies that the colour index takes at least $N \geq 3$ values. The K_j are trilinear polynomials with ghost number $(j-2)$. We assume that the colour tensor in $K_j, j = 1,2$ is also totally antisymmetric (i.e. $K_j = h^j_{abc} \varphi_a^{(1)} \varphi_b^{(2)} \varphi_c^{(3)}$ with a totally antisymmetric h^j). Then one finds

$$K_1 = g_0 h^1_{abc} u_a \tilde{u}_b \tilde{u}_c, \qquad K_2^\mu = g_0 h^2_{abc} A_a^\mu u_b \tilde{u}_c. \tag{145}$$

The most general solutions for $\mathcal{L}_j, j \geq 1$ contain trilinear terms only. Assuming again that solely totally antisymmetric colour tensors appear, we obtain

$$\begin{aligned}
\mathcal{L}_1^\nu &= g_0 f_{abc}[A_{a\mu} u_b F_c^{\nu\mu} - \frac{1}{2} u_a u_b \partial^\nu \tilde{u}_c] - i s_0 K_2^\nu + g_0 h^3_{abc} \partial_\mu (u_a A_b^\nu A_c^\mu), \\
\mathcal{L}_2^{\nu\mu} &= g_0 \frac{1}{2} f_{abc} u_a u_b F_c^{\nu\mu} - i g_0 s_0 (h^3_{abc} u_a A_b^\nu A_c^\mu) \\
&\quad + g_0 h^4_{abc}[u_a u_b \partial^\nu A_c^\mu + 2 u_a \partial^\nu u_b A_c^\mu - g^{\nu\mu}(u_a u_b \partial_\lambda A_c^\lambda - 2 u_a \partial_\lambda u_b A_c^\lambda)] \\
\mathcal{L}_3^{\nu\mu\lambda} &= g_0 h^4_{abc}[g^{\nu\mu} u_a u_b \partial^\lambda u_c + g^{\lambda\nu} u_a u_b \partial^\mu u_c],
\end{aligned} \tag{146}$$

where h^3, h^4 are totally antisymmetric. Note that the divergence ∂_μ of the h^4-term in $\mathcal{L}_2^{\nu\mu}$ vanishes. To simplify the formulas we choose

$$h^j_{abc} = 0, \quad \forall j = 1,2,3,4. \tag{147}$$

$h^4 = 0$ is equivalent to $\mathcal{L}_2^{\mu\nu} = -\mathcal{L}_2^{\nu\mu}$ and also to $\mathcal{L}_3 = 0$.

The requirements (A)-(F) used so far do not involve T-products, they are of first order perturbation theory. We now restrict $\mathcal{L}_0$ further by (139), which can be interpreted as a requirement for second order tree diagrams, see (151). More precisely, we will work with the generalization of (139) to the ladder (142): in order that the master BRST-identity (137) implies the important equation

('generalized perturbative gauge invariance'[22])

$$[Q_0, T_n(\mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n)]_\mp =$$
$$-i \sum_{l=1}^{n} (-1)^{j_1+...+j_{l-1}} T_n(\mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}^\nu_{j_l+1} \partial_\nu f_l \otimes ... \otimes \mathcal{L}_{j_n} f_n), \tag{148}$$

we require

$$G((\mathcal{L}_j, \mathcal{L}_{j+1})f, \mathcal{L}_k g) + (-1)^{jk} G((\mathcal{L}_k, \mathcal{L}_{k+1})g, \mathcal{L}_j f) = 0, \qquad j,k = 0,1,2,3, \tag{149}$$

$\forall f, g \in \mathcal{D}(\mathbb{R}^4)$, where we set $\mathcal{L}_4 \equiv 0$. Or, with the simplification (147) (which will always be used in the following), j and k run only through the values $j,k = 0,1,2$. In the present case of solely massless fields this requirement can be fulfilled. It restricts the interaction $\mathcal{L}_0$ further and determines the normalization constant C_{A_a}. Namely, using the simplification (147), one finds by explicit calculation that the requirement (149) holds true if and only if

$$C_{A_a} = -\frac{1}{2}, \qquad \forall a, \tag{150}$$

and the f_{abc} fulfil the *Jacobi identity* [36].[23] Hence, the f_{abc} are structure constants of some Lie algebra. The total antisymmetry of f_{abc} implies that this Lie algebra is isomorphic to a direct sum of Abelian and simple compact Lie algebras, see e.g. [33]. We point out, that the Lie algebraic structure is not put in, it is a consequence of our requirements.

[22]In [3] this identity is called the normalization condition **(N6)**, in [15] it is called 'generalized (free perturbative operator) gauge invariance'. The importance and usefulness of this identity has also been pointed out in the earlier paper [17]. The particular case $j_1 = ... = j_n = 0$ is the 'perturbative gauge invariance' (or 'free perturbative operator gauge invariance') which has been proved in [11] for $SU(N)$-Yang-Mills theories. In [16] it has been shown that this perturbative gauge invariance implies the usual Slavnov-Taylor identities.

[23]The Jacobi identity and (150) are required even from the particular case $G((\mathcal{L}_0, \mathcal{L}_1)f, \mathcal{L}_0 g) + (f \leftrightarrow g) = 0$. This was demonstrated in [36] by reversing the calculation in [11]. The computation of the l.h.s. of (149) is lengthy. The straightforward way uses the definitions (131) and (135) of $G^{(1)}$ and $G^{(2)}$. To shorten the calculation one may choose $C_{1A_a} = 0 = C_{1u_a}$, because the terms $\sim C_{1A_a}, C_{1u_a}$ must drop out. This follows from the fact that $\mathcal{L}_0$ (144), $\mathcal{L}_1$ and $\mathcal{L}_2$ (146) do not contain symbols with second or higher derivatives and, hence, the r.h.s. in

$$T_1\Big(G((\mathcal{L}_j, \mathcal{L}_{j+1})f, \mathcal{L}_k g) + (-1)^{jk}\{(j,f) \leftrightarrow (k,g)\}\Big) =$$
$$[Q_0, T_2(\mathcal{L}_j f \otimes \mathcal{L}_k g)]|_{4-\mathrm{legs}} + i\Big(T_2(\mathcal{L}^\nu_{j+1}\partial_\nu f \otimes \mathcal{L}_k g) + (-1)^{jk}\{(j,f) \leftrightarrow (k,g)\}\Big)|_{4-\mathrm{legs}} \tag{151}$$

(cf. (133), $...|_{4-\mathrm{legs}}$ expresses that we mean the terms with 4 free field operators only) does not contain the constants C_{1A_a} and C_{1u_a} (according to the definition(291)). However, even with this simplification, it seems to be faster to compute the r.h.s. of (151) (by using the techniques of [11], [14]), instead of the straightforward computation of the l.h.s.. Note that the derivation of (151) uses the MWI for tree diagrams only and, hence, (151) holds surely true.

Remarks: (1) If we do not use the simplification (147), but assume that K_1 and K_2 are $\sim f_{abc}$,

$$K_1 = -i\beta_1 g_0 f_{abc} u_a \tilde{u}_b \tilde{u}_c, \quad K_2^\mu = \beta_2 g_0 f_{abc} A_a^\mu u_b \tilde{u}_c, \quad \beta_1, \beta_2 \in \mathbb{R} \tag{152}$$

(instead of (145)), we obtain

$$(\beta_1, \beta_2) \in \{(0,0),\, (12,1),\, (-12,0),\, (1,1)\} \tag{153}$$

from the particular case $(j,k) = (0,0)$ of the requirement (149) (additionally to (150) and the Jacobi identity), where $C_{u_a} = -1$ (102) is used. Note that the first two solutions in (153), and also the latter two, are obtained from each other by replacing u by $\gamma\tilde{u}$ and $\tilde{u}$ by $(-1\gamma u)$ in $\mathcal{L}_0$, $\gamma \in i\mathbb{R} \setminus \{0\}$ arbitrary.

(2) By using **N4** (92) we obtain for the interacting F-field ($F^{\mu\nu} \stackrel{\mathrm{def}}{=} \partial^\mu A^\nu - \partial^\nu A^\mu$)

$$F^{\mu\nu}_{a\, g\mathcal{L}}(x) = \partial^\mu A^\nu_{a\, g\mathcal{L}}(x) - \partial^\nu A^\mu_{a\, g\mathcal{L}}(x) - 2C_{A_a} g_0 g(x) f_{abc} (A^\mu_b A^\nu_c)_{g\mathcal{L}}(x). \tag{154}$$

We see that the nonlinear term is due to the non-vanishing of C_{A_a} and that it agrees with the usual nonlinear term iff $C_{A_a} = -\frac{1}{2}$, in agreement with (150).

4.5.2 Massless gauge fields: local construction of observables

In [7] a general **local construction of observables in gauge theories and of the physical Hilbert space** (in which the observables are faithfully represented) is given. This construction relies on some assumptions which can be fulfilled in QED [7]. We are now going to generalize the latter result to the class of interactions we have selected in the preceeding subsection, which includes non-Abelian gauge theories. Thereby we assume that ghost number conservation **(N5) (ghost)** and certain cases of the master BRST-identity (137) are satisfied.

As in [7] we start with the *local* algebra of interacting gauge and ghost fields (93)

$$\mathcal{F}(\mathcal{O}) \stackrel{\mathrm{def}}{=} \bigvee \{W_{g\mathcal{L}}(f) \mid f \in \mathcal{D}(\mathcal{O}),\, W = A^\mu,\, u,\, \tilde{u}, ...\} \tag{155}$$

(the dots stand for polynomials in A^μ, u and $\tilde{u}$), where $\mathcal{O}$ is a double cone and $g(x) = 1$, $\forall x \in \mathcal{O}$. In [5] the crucial observation has been made that a change of the switching function g outside of $\mathcal{O}$, transforms all interacting fields $\in \mathcal{F}(\mathcal{O})$ by the same unitary transformation[24]. Therefore, the algebraic properties of $\mathcal{F}(\mathcal{O})$ are independent of the adiabatic limit $g(x) \to 1$, $\forall x$. Hence, we may avoid this limit, which saves us from infrared divergences. It seems that a consistent perturbative construction of massless non-Abelian gauge theories can be done only *locally*, i.e. without performing the adiabatic limit, due to the confinement.

The field algebra $\mathcal{F}(\mathcal{O})$ contains unphysical fields. The central problem in gauge theories is to eliminate the latter, i.e. to select the observables, and, in

[24] An alternative proof of this fact is given in the second paper of [8].

a second step, to construct (physical) states on the algebra of observables. We proceed as in [7]: roughly speaking we will define the observables to be the BRST-invariant fields. Thereby, we will define the BRST-transformation $\tilde{s}$ as the ($\mathbb{Z}_2$-graded) commutator with the (modified) interacting BRST-charge $Q_{g\mathcal{L}}$ of Kugo and Ojima [22]. But in contrast to this reference we do not perform the adiabatic limit. The latter causes the complication that $Q_{g\mathcal{L}}$ does not agree with Q_0; the main difference between the present formalism and [22] is that $Q_{g\mathcal{L}}$ is a non-trivial formal power series, i.e. there are non-vanishing coefficients to orders $n \geq 1$ (cf. [7], [15]).

The current belonging to $Q_{\mathcal{L}}$ is the interacting BRST-current $\tilde{j}^\mu_{\mathcal{L}}$. From our experience made in QED [7] we know that $\tilde{j}^\mu_{\mathcal{L}}$ should have the following properties:
(a) to zeroth order it agrees with the free BRST current j^μ (119),
(b) it is conserved up to (first) derivatives of the coupling 'constant' g: $\partial_\mu \tilde{j}^\mu_{\mathcal{L}}(x) \sim (\partial g)(x)$.
Unfortunately, (b) does not hold true for the interacting field $j_{g\mathcal{L}}$ (93)-(94) where $j_{g\mathcal{L}}$ is constructed in terms of T-products satisfying the MWI **(N)**: from (125)-(126) we get

$$T_2(j^\mu \partial_\mu f \otimes \mathcal{L}_0 g) = -iT_1(\mathcal{L}_1^\nu f \partial_\nu g) + iT_1(\mathcal{M}^\nu (\partial_\nu f) g), \tag{156}$$

where

$$\mathcal{M}^\nu \stackrel{\mathrm{def}}{=} -\mathcal{L}_1^\nu + (3C_{1A_a} + \frac{1}{2})\partial_\mu u_a \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A_{a\nu})} + \frac{3}{4} u_a \frac{\partial \mathcal{L}_0}{\partial A_{a\nu}}. \tag{157}$$

However, the wanted conservation property can be achieved by a change of the normalization of $T_{n+1}(j^\mu, \mathcal{L}_{j_1}, ..., \mathcal{L}_{j_n})$: motivated by

$$s_0(k^\mu) = j^\mu \tag{158}$$

(where k^μ is the (free) ghost current (98)) and generalized perturbative gauge invariance (148) we define

$$\begin{aligned}\tilde{T}_{n+1}(j^\mu f \otimes \mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n) &\stackrel{\mathrm{def}}{=} [Q_0, T_{n+1}(k^\mu f \otimes \mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n)]_\mp \\ +i \sum_{l=1}^n (-1)^{j_1+...+j_{l-1}} T_{n+1}(k^\mu f &\otimes \mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}^\nu_{j_l+1} \partial_\nu f_l \otimes ... \otimes \mathcal{L}_{j_n} f_n).\end{aligned} \tag{159}$$

By means of (158) and (148) we find that $\tilde{T}_{n+1}(j^\mu, \mathcal{L}_{j_1}, ..., \mathcal{L}_{j_n})$ factorizes causally (14), e.g. $\tilde{T}_{n+1}(j, \mathcal{L}_{...}, ...) = \tilde{T}_{l+1}(j, \mathcal{L}_{...}, ...)T_{n-l}(\mathcal{L}_{...}, ...)$. In addition it is symmetrical in $\mathcal{L}_{j_1}, ..., \mathcal{L}_{j_n}$ and fulfills the normalization conditions **(N1), (N2)** and **(N0)**. Hence, $T_{n+1}(j^\mu, \mathcal{L}_{j_1}, ...) \to \tilde{T}_{n+1}(j^\mu, \mathcal{L}_{j_1}, ...)$ is an admissible finite renormalization of T-products (solely the extension to D_{n+1} is changed), which however violates **(N3)** and the MWI **(N)**. To compute the divergence of (159) with respect to j^μ we first apply **(N5)(ghost)** (103) to all terms on the r.h.s. (where we use $g(\mathcal{L}_j) = j$) and afterwards generalized gauge invariance (148). In

this way we obtain

$$\tilde{T}_{n+1}(j^\mu \partial_\mu f \otimes \mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n) =$$
$$i \sum_{l=1}^{n} (-1)^{j_1+...+j_{l-1}} \Big(T_n(\mathcal{L}_{j_1} f_1 \otimes ...... \otimes \mathcal{L}^\nu_{j_l+1} f \partial_\nu f_l \otimes ... \otimes \mathcal{L}_{j_n} f_n)$$
$$-j_l T_n(\mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}^\nu_{j_l+1} (\partial_\nu f) f_l \otimes ... \otimes \mathcal{L}_{j_n} f_n) \Big). \tag{160}$$

Conversely, this current conservation identity (160) implies the generalized perturbative gauge invariance (148) by proceeding similarly to (123). We denote by $\tilde{R}_{n+1}(\mathcal{L}, ..., \mathcal{L}; j)$ (where $\mathcal{L} \equiv \mathcal{L}_0$) the R-product (94) which is constructed in terms of $T_k(\mathcal{L}, ..., \mathcal{L})$ and $\tilde{T}_{k+1}(j, \mathcal{L}, ..., \mathcal{L})$, $1 \leq k \leq n$. Then, the identity (160) implies

$$\tilde{R}_{n+1}((\mathcal{L}g)^{\otimes n}; j^\mu \partial_\mu f) = in R_n((\mathcal{L}g)^{\otimes n-1}; \mathcal{L}^\nu_1 f \partial_\nu g). \tag{161}$$

Analogously to (93) we define

$$\tilde{j}^\mu_{g\mathcal{L}}(f) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{i^n}{n!} \tilde{R}_{n+1}((\mathcal{L}g)^{\otimes n}; j^\mu f), \tag{162}$$

and this interacting BRST-current has the wanted conservation property

$$\tilde{j}^\mu_{g\mathcal{L}}(\partial_\mu f) = -\mathcal{L}^\nu_{1\, g\mathcal{L}}(f \partial_\nu g). \tag{163}$$

The difference $(\tilde{j}^\mu_{g\mathcal{L}} - j^\mu_{g\mathcal{L}})$ (where $j^\mu_{g\mathcal{L}}$ still denotes the interacting field constructed in terms of T-products satisfying also the MWI and **(N3)**) is immediately obtained by applying the master BRST-identity:

$$\tilde{j}^\mu_{g\mathcal{L}}(f) - j^\mu_{g\mathcal{L}}(f) = i(G^{(1)}(k^\mu f, \mathcal{L}g))_{g\mathcal{L}} + i(G((\mathcal{L}, \mathcal{L}_1)g, k^\mu f))_{g\mathcal{L}}. \tag{164}$$

The interacting BRST-charge operator is now defined by

$$Q_{g\mathcal{L}} \stackrel{\text{def}}{=} \int d^4x\, h^\mu(x) \tilde{j}_{\mu\, g\mathcal{L}}(x), \tag{165}$$

where $\mathcal{L} \equiv \mathcal{L}_0$ and h^μ is a suitable test function (see [7] and subsection 4.5.4). $Q_{g\mathcal{L}}$ is a formal power series and the construction is such that the relations

$$Q_{g\mathcal{L}} = Q^*_{g\mathcal{L}}, \qquad (Q_{g\mathcal{L}})_0 = Q_0, \tag{166}$$

hold true (where $(...)_0$ means the zeroth order) and that $Q_{g\mathcal{L}}$ is nilpotent

$$(Q_{g\mathcal{L}})^2 = 0. \tag{167}$$

The latter property is proved in subsection 4.5.4 by using current conservation (163) and generalized gauge invariance (148). We point out that the conservation of the BRST-current (163) and the construction of the nilpotent

BRST-charge (165)-(167) use the master BRST-identity for $T_n(\mathcal{L}_0,...,\mathcal{L}_0)$ and $T_n(\mathcal{L}_1,\mathcal{L}_0,...,\mathcal{L}_0)$ ($\forall n\in\mathbb{N}$) only.

The BRST-transformation $\tilde{s}$ of the interacting fields $W_{g\mathcal{L}}(f)$, $f\in\mathcal{D}(\mathcal{O})$, is then defined by the commutator with $Q_{g\mathcal{L}}$ (or anti-commutator if W has an odd ghost number)

$$\tilde{s}(W_{g\mathcal{L}}(f))\stackrel{\mathrm{def}}{=}[Q_{g\mathcal{L}},W_{g\mathcal{L}}(f)]_{\mp},\qquad f\in\mathcal{D}(\mathcal{O}).\tag{168}$$

We extend $\tilde{s}$ to a graded derivation $\mathcal{F}(\mathcal{O})\to\mathcal{F}(\mathcal{O})$. The local observables are selected by the definition [7]

$$\mathcal{A}(\mathcal{O})\stackrel{\mathrm{def}}{=}\frac{\ker\tilde{s}}{\mathrm{ran}\,\tilde{s}}.\tag{169}$$

We are looking for states on $\mathcal{A}(\mathcal{O})$ which take values in $\tilde{\mathbb{C}}$ (by which we mean the formal power series with coefficients in $\mathbb{C}$), because the elements of $\mathcal{A}(\mathcal{O})$ are formal power series. Thereby, we call $a\in\tilde{\mathbb{C}}$ positive, if there exists $b\in\tilde{\mathbb{C}}$ with $a=b^*b$, where $*$ means complex conjugation. The positivity of our states is understood in this sense. In [7] it is shown that $\mathcal{A}(\mathcal{O})$ can be naturally represented on the cohomology of $Q_{g\mathcal{L}}$

$$\mathcal{H}_{\mathrm{phys}}\stackrel{\mathrm{def}}{=}\frac{\ker Q_{g\mathcal{L}}}{\mathrm{ran}\,Q_{g\mathcal{L}}}\tag{170}$$

and that the induced inner product on $\mathcal{H}_{\mathrm{phys}}$ is positive definite. Hence $\mathcal{H}_{\mathrm{phys}}$ is a pre Hilbert space and we interpret the elements of $\mathcal{H}_{\mathrm{phys}}$ as physical states. Positive definiteness of the inner product on $\mathcal{H}_{\mathrm{phys}}$ is proved in two steps: first in zeroth order by explicit determination of $\frac{\ker Q_0}{\mathrm{ran}\,Q_0}$. Then we have proved that positivity is stable under perturbations [7] ('deformation stability of BRST-quantization').

The master BRST-identity (137) yields also an explicit formula for the **BRST-transformation of the interacting fields** $\in\mathcal{F}(\mathcal{O})$, by which we will see that the definition (168) agrees with the usual BRST-transformation. (In addition this ensures the existence of non-trivial observables.) For this purpose we note that the master BRST-identity and the requirement (149) imply the following relation

$$\begin{gathered}[Q_0,T_{n+1}(\mathcal{L}_0f_1\otimes...\otimes\mathcal{L}_0f_n\otimes Wf)]_{\mp}=\\ -i\sum_{l=1}^{n}T_{n+1}(\mathcal{L}_0f_1\otimes...\otimes\mathcal{L}_1^{\nu}\partial_\nu f_l\otimes...\otimes\mathcal{L}_0f_n\otimes Wf)\\ -iT_{n+1}(\mathcal{L}_0f_1\otimes...\otimes\mathcal{L}_0f_n\otimes W'_\nu\partial^\nu f)\\ +\sum_{k=1}^{n}T_n\Big([G((\mathcal{L}_0,\mathcal{L}_1)f_k,Wf)+G((W,W')f,\mathcal{L}_0f_k)]\otimes\mathcal{L}_0f_1\otimes...\hat{k}...\otimes\mathcal{L}_0f_n\Big),\end{gathered}\tag{171}$$

where we have assumed $s_0(W) = i\partial^\nu W'_\nu$. Equation (171) translates into an identity for the R-products (94)

$$\begin{aligned}[]
&[Q_0, R_{n+1}(\mathcal{L}_0 f_1 \otimes ... \otimes \mathcal{L}_0 f_n; Wf)]_\mp = \\
&-i\sum_{l=1}^{n} R_{n+1}(\mathcal{L}_0 f_1 \otimes ... \otimes \mathcal{L}_1^\nu \partial_\nu f_l \otimes ... \otimes \mathcal{L}_0 f_n; Wf) \\
&-iR_{n+1}(\mathcal{L}_0 f_1 \otimes ... \otimes \mathcal{L}_0 f_n; W'_\nu \partial^\nu f) \\
&+\sum_{k=1}^{n} R_n\Big(\mathcal{L}_0 f_1 \otimes ...\hat{k}... \otimes \mathcal{L}_0 f_n; [G((\mathcal{L}_0, \mathcal{L}_1) f_k, Wf) + G((W, W')f, \mathcal{L}_0 f_k)]\Big).
\end{aligned} \tag{172}$$

In subsection 4.5.4 it will be shown that this identity implies the BRST-transformation formula

$$\begin{aligned}
\tilde{s}(W_{g\mathcal{L}}(f)) = [Q_{g\mathcal{L}}, W_{g\mathcal{L}}(f)]_\mp = -iW'^{\nu}_{g\mathcal{L}}(\partial_\nu f) \\
+i\big(G((\mathcal{L}, \mathcal{L}_1)g, Wf) + G((W, W')f, \mathcal{L}g)\big)_{g\mathcal{L}}, \qquad f \in \mathcal{D}(\mathcal{O}),
\end{aligned} \tag{173}$$

where $\mathcal{L} \equiv \mathcal{L}_0$. The term in the second line is the nonlinear part of the BRST-transformation. In case that W is a single symbol we find that $[G((\mathcal{L}, \mathcal{L}_1)g, Wf) + G((W, W')f, \mathcal{L}g)]$ is quadratic in the symbols (because $\mathcal{L}$ and $\mathcal{L}_1$ are trilinear), in agreement with the usual BRST-transformation. (To prove (173) it will be shown that the terms $[(Q_{g\mathcal{L}} - Q_0), W_{g\mathcal{L}}(f)]$ cancel out with the terms $-i\sum_n \frac{i^n}{n!} \sum_{l=1}^n R_{n+1}(\mathcal{L}_0 g \otimes ... \otimes \mathcal{L}_1^\nu \partial_\nu g \otimes ...; Wf)$.)

If we do not assume that $s_0 W$ is a divergence we end up with

$$\tilde{s}(W_{g\mathcal{L}}(f)) = (s_0 W)_{g\mathcal{L}}(f) + i\big(G^{(1)}(Wf, \mathcal{L}g) + G((\mathcal{L}, \mathcal{L}_1)g, Wf)\big)_{g\mathcal{L}}, \tag{174}$$

instead of (173). We choose $W = k^\mu$, compare with (164) and find

$$\{Q_{g\mathcal{L}}, k^\mu_{g\mathcal{L}}\} = \tilde{j}^\mu_{g\mathcal{L}}. \tag{175}$$

Introducing the interacting ghost charge

$$Q^u_{g\mathcal{L}} \stackrel{\text{def}}{=} \int d^4x \, h^\mu(x) k_{\mu\, g\mathcal{L}}(x), \tag{176}$$

where $h^\mu \in \mathcal{D}(\mathbb{R}^4)$ is chosen in precisely the same way as in $Q_{g\mathcal{L}}$ (see sect. 4.5.4), it results

$$\{Q_{g\mathcal{L}}, Q^u_{g\mathcal{L}}\} = Q_{g\mathcal{L}} \tag{177}$$

as in [22].

Examples: In most of the following examples the computation of $G^{(1)}$ and $G^{(2)}$ gives less work than it seems, because only very few terms contribute. We use the values $C_{u_a} = -1$ (102) and $C_{A_a} = -\frac{1}{2}$ (150) without further mentioning it.

(1) BRST-transformation of $A^\mu_{a\,g\mathcal{L}}(h)$:

$$G^{(1)}(\mathcal{L}_0 g, A^\mu_a h) = 0, \quad G^{(1)}(A^\mu_a h, \mathcal{L}_0 g) = -\frac{3}{4} g_0 f_{abc} A^\mu_b u_c h g,$$
$$G^{(2)}(\mathcal{L}_1 g, A^\mu_a h) = \frac{3}{4} g_0 f_{abc} A^\mu_b u_c g h, \quad G^{(2)}(g^{\mu\nu} u_a h, \mathcal{L}_0 g) = g_0 f_{abc} A^\mu_b u_c h g. \tag{178}$$

Therefore,

$$\tilde{s}(A^\mu_{a\,g\mathcal{L}}(h)) = -i u_{a\,g\mathcal{L}}(\partial^\mu h) + i g_0 (f_{abc} A^\mu_b u_c)_{g\mathcal{L}}(gh). \tag{179}$$

Taking $g|_{\mathrm{supp}\,h} = 1$ into account the last term takes the usual form $i g_0 (f_{abc} A^\mu_b u_c)_{g\mathcal{L}}(h)$. We see that $G^{(1)}(A^\mu_a h, \mathcal{L}_0 g) \neq 0$ gives a non-vanishing contribution to the non-linear term in (179). This shows that the distinction of internal and external derivatives and in particular the appearance of the external derivative in the definition of Δ^μ (36) is crucial to obtain the correct BRST-transformation.

(2) BRST-transformation of $u_{a\,g\mathcal{L}}(h)$:

$$G^{(1)}(\mathcal{L}_0 g, u_a h) = 0, \quad G^{(1)}(u_a h, \mathcal{L}_0 g) = 0,$$
$$G^{(2)}(\mathcal{L}_1 g, u_a h) = -\frac{1}{2} g_0 f_{abc} u_b u_c g h, \quad G(0h, \mathcal{L}_0 g) = 0. \tag{180}$$

Hence,

$$\tilde{s}(u_{a\,g\mathcal{L}}(h)) = -\frac{i}{2} g_0 (f_{abc} u_b u_c)_{g\mathcal{L}}(h). \tag{181}$$

(3) BRST-transformation of $\tilde{u}_{a\,g\mathcal{L}}(h)$:

$$G^{(1)}(\mathcal{L}_0 g, \tilde{u}_a h) = 0, \quad G^{(1)}(\tilde{u}_a h, \mathcal{L}_0 g) = 0,$$
$$G^{(2)}(\mathcal{L}_1 g, \tilde{u}_a h) = 0, \quad G^{(2)}(-\partial_\nu A^\nu_a h, \mathcal{L}_0 g) = 0, \tag{182}$$

where we have used $g^{\nu\mu} \frac{\partial \mathcal{L}_0}{\partial(\partial^\nu A^\mu_a)} = 0$. So we obtain

$$\tilde{s}(\tilde{u}_{a\,g\mathcal{L}}(h)) = i A^\nu_{a\,g\mathcal{L}}(\partial_\nu h). \tag{183}$$

(4) BRST-transformation of $F^{\mu\nu}_{a\,g\mathcal{L}}(h)$:

$$G^{(1)}(\mathcal{L}_0 g, F^{\mu\nu}_a h) = 0,$$
$$G^{(1)}(F^{\mu\nu}_a h, \mathcal{L}_0 g) = (\frac{1}{2} + 3C_{1A_a}) g_0 f_{abc} A^\nu_b u_c (\partial^\mu h) g - (\mu \leftrightarrow \nu),$$
$$G^{(2)}(\mathcal{L}_1 g, F^{\mu\nu}_a h) = (\frac{1}{2} + 3C_{1A_a}) g_0 f_{abc} [\tilde{\partial}^\mu (A^\nu_b u_c) g h + A^\nu_b u_c (\partial^\mu g) h]$$
$$-(\mu \leftrightarrow \nu) + g_0 f_{abc} F^{\mu\nu}_b u_c g h, \qquad G^{(2)}(0h, \mathcal{L}_0 g) = 0. \tag{184}$$

Now we apply the relation (31) (which holds obviously also for the R-products; cf. the remark at the end of sect. 2.4)

$$R_{n+1}(\mathcal{L}_0 g \otimes ...; f_{abc} \tilde{\partial}^\mu (A^\nu_b u_c) g h) = -R_{n+1}(\mathcal{L}_0 g \otimes ...; f_{abc} A^\nu_b u_c \partial^\mu (gh)). \tag{185}$$

Hence, inserting these formulas into (173), the terms $\sim (\frac{1}{2} + 3C_{1A_a})$ cancel and it remains

$$\tilde{s}(F^{\mu\nu}_{a\,g\mathcal{L}}(h)) = ig_0(f_{abc}F^{\mu\nu}_b u_c)_{g\mathcal{L}}(h). \tag{186}$$

(5) BRST-transformation of $(\sum_a F^{\mu\nu}_a F^{\rho\tau}_a)_{g\mathcal{L}}(h)$ (we do not write $\sum_a$ but always perform this sum):

$$\begin{aligned}
&G^{(1)}(\mathcal{L}_0 g, F^{\mu\nu}_a F^{\rho\tau}_a h) = 0,\\
&G^{(1)}(F^{\mu\nu}_a F^{\rho\tau}_a h, \mathcal{L}_0 g) = \{(\frac{1}{2} + 3C_{1A_a})g_0 f_{abc}[F^{\rho\tau}_a A^\nu_b u_c(\partial^\mu h)g + (\tilde{\partial}^\mu F^{\rho\tau}_a)A^\nu_b u_c hg]\\
&\qquad -(\mu \leftrightarrow \nu)\} + \{(\mu,\nu) \leftrightarrow (\rho,\tau)\},\\
&G^{(2)}(\mathcal{L}_1 g, F^{\mu\nu}_a F^{\rho\tau}_a h) = \{(\frac{1}{2} + 3C_{1A_a})g_0 f_{abc}[F^{\rho\tau}_a \tilde{\partial}^\mu(A^\nu_b u_c)gh + F^{\rho\tau}_a A^\nu_b u_c(\partial^\mu g)h]\\
&-(\mu \leftrightarrow \nu) + g_0 f_{abc}F^{\rho\tau}_a F^{\mu\nu}_b u_c gh\} + \{(\mu,\nu) \leftrightarrow (\rho,\tau)\}, \qquad G^{(2)}(0h, \mathcal{L}_0 g) = 0.
\end{aligned} \tag{187}$$

The term $\sim FFugh$ in $G^{(2)}(\mathcal{L}_1 g, FFh)$ drops out because f_{abc} is totally antisymmetric. Inserting these formulas into (173), the terms $\sim (\frac{1}{2} + 3C_{1A_a})$ cancel again due to (31). So we obtain

$$\tilde{s}((F^{\mu\nu}_a F^{\rho\tau}_a)_{g\mathcal{L}}(h)) = 0 \tag{188}$$

and, hence, the corresponding equivalence class (cf. (169)) is a non-trivial observable.

(6) Due to the requirement $G((\mathcal{L}_j, \mathcal{L}_{j+1})f, \mathcal{L}_k g) + (-1)^{jk}\{(j,f) \leftrightarrow (k,g)\} = 0$ (149) we can easily write down the BRST-transformation of $\mathcal{L}_{j\,g\mathcal{L}},\ j = 0,1,2$:

$$\begin{aligned}
\tilde{s}(\mathcal{L}_{0\,g\mathcal{L}}(h)) &= -i\mathcal{L}^\nu_{1\,g\mathcal{L}}(\partial_\nu h),\\
\tilde{s}(\mathcal{L}^\nu_{1\,g\mathcal{L}}(h)) &= -i\mathcal{L}^{\nu\mu}_{2\,g\mathcal{L}}(\partial_\mu h),\\
\tilde{s}(\mathcal{L}^{\nu\mu}_{2\,g\mathcal{L}}(h)) &= 0.
\end{aligned} \tag{189}$$

Remark: Having determined the interaction $\mathcal{L}_0$ we can explicitly write down the interacting field equations by means of (95): with the simplification (147) they read

$$\begin{aligned}
\square A_{\mu\,a\,g\mathcal{L}} &= g_0 f_{abc}\partial^\nu[g(A_{\mu\,b}A_{\nu\,c})_{g\mathcal{L}}] -\\
&-g_0 g f_{abc}(A^\nu_b F_{\nu\mu\,c})_{g\mathcal{L}} + g_0 g f_{abc}(u_b \partial_\mu \tilde{u}_c)_{g\mathcal{L}}, &(190)\\
\square u_{a\,g\mathcal{L}} &= -g_0 f_{abc}\partial_\mu[g(A^\mu_b u_c)_{g\mathcal{L}}], &(191)\\
\square \tilde{u}_{a\,g\mathcal{L}} &= -g_0 g f_{abc}(A^\mu_b \partial_\mu \tilde{u}_c)_{g\mathcal{L}}. &(192)
\end{aligned}$$

They hold true everywhere, g needs not to be constant. In the classical limit $\hbar \to 0$ interacting fields factorize, $(VW)_{g\mathcal{L}}(x) = V_{g\mathcal{L}}(x)W_{g\mathcal{L}}(x)$ (see [9]), and hence (190)-(192) go over into the usual Yang-Mills equations.

4.5.3 Massive gauge fields

To simplify the notations we consider the most simple non-Abelian model, namely three massive gauge fields, $m_a > 0$, $a = 1, 2, 3$ and no massless fields. However, as far as anomalies are absent, our method applies also to general models with arbitrary numbers of massive and massless gauge fields and spinor fields. We will find the well-known result that with the fields $A_a^\mu, u_a, \tilde{u}_a$ and ϕ_a $(a = 1, 2, 3)$ only, a consistent construction of the model is impossible, more precisely generalized perturbative gauge invariance (148) for second order tree diagrams cannot be satisfied [14]. We will solve this problem in the usual way: besides the scalar fields $(\phi_a)_{a=1,2,3}$, we introduce an additional real (free) scalar field H, the 'Higgs field', with arbitrary mass $m_H \geq 0$, which is quantized according to

$$(\square + m_H^2)H = 0, \qquad H^* = H, \qquad \Delta_{H,H} = -D_{m_H} \tag{193}$$

and H commutes with all other free fields.

To determine the interaction $\mathcal{L}_0$ we require the same properties (A)-(F) as in the massless case. The only modification is that $\mathcal{L}_j$ is now a polynomial in A_a^μ, u_a, $\tilde{u}_a, \phi_a, a = 1, 2, 3$ and H and internal derivatives of these symbols (again we solely admit monomials which have at least three factors). Proceeding as in the massless case we find the following particular solution of (A)-(F):

$$\begin{aligned}
\mathcal{L}_0 &= g_0\{f_{abc}[A_{a\,\mu}A_{b\,\nu}\partial^\nu A_c^\mu - u_a\partial^\mu\tilde{u}_b A_{c\,\mu}] \\
&+ d_{abc}A_a^\mu\phi_b\partial_\mu\phi_c + e_{abc}A_a^\mu A_{b\,\mu}\phi_c + h_{abc}\tilde{u}_a u_b\phi_c \\
&+ l_{ab}[\frac{1}{m_b}(-HA_a^\mu\partial_\mu\phi_b + (\partial_\mu H)A_a^\mu\phi_b) + A_a^\mu A_{b\,\mu}H \\
&- H\tilde{u}_a u_b - \frac{m_H^2}{2m_a m_b}H\phi_a\phi_b] + pH^3 + tH^4\},
\end{aligned} \tag{194}$$

$$\begin{aligned}
\mathcal{L}_1^\nu &= g_0\{f_{abc}[A_{a\mu}u_b F_c^{\nu\mu} - \frac{1}{2}u_a u_b\partial^\nu\tilde{u}_c] + 2e_{abc}u_a A_b^\nu\phi_c \\
&+ d_{abc}[u_a\phi_b\partial^\nu\phi_c + m_c A_a^\nu\phi_b u_c] \\
&+ l_{ab}[\frac{1}{m_b}u_a(\phi_b\partial^\nu H - H\partial^\nu\phi_b) + A_a^\nu u_b H]\},
\end{aligned} \tag{195}$$

$$\mathcal{L}_2^{\nu\mu} = g_0\frac{f_{abc}}{2}u_a u_b F_c^{\nu\mu}, \tag{196}$$

$$\mathcal{L}_3 = 0, \tag{197}$$

where $f_{abc} \in \mathbb{R}$ is totally antisymmetric, $l_{ab} \in \mathbb{R}$ is symmetric and

$$\begin{aligned}
d_{abc} &= f_{abc}\frac{m_b^2 + m_c^2 - m_a^2}{2m_b m_c}, \quad e_{abc} = f_{abc}\frac{m_b^2 - m_a^2}{2m_c}, \\
h_{abc} &= f_{abc}\frac{m_a^2 + m_c^2 - m_b^2}{2m_c}, \qquad p, t \in \mathbb{R}.
\end{aligned} \tag{198}$$

The most general solution for $\mathcal{L}_0$ differs from the particular solution (194) by a coboundary $-is_0K_1$ and a divergence $\partial_\nu K_2^\nu$ as in (144). In addition one has the freedom to add terms with vanishing divergence to $\mathcal{L}_1$ and $\mathcal{L}_2$ as in (146). It is a peculiarity of the present model, that the total antisymmetry of f_{abc} implies the Jacobi identity, so that we obtain

$$f_{abc} = \epsilon_{abc} \,(= \text{ structure constant of } SU(2)) \tag{199}$$

by absorbing a constant factor in g_0. So far we could set $l_{ab} = 0,\ p = 0,\ t = 0$, in other words the Higgs field H is not needed to satisfy (A)-(F).

Now we come to an interesting complication of the massive case: the requirement $G((\mathcal{L}_j, \mathcal{L}_{j+1})f, \mathcal{L}_k g) + (-1)^{jk}\{(j,f) \leftrightarrow (k,g)\} = 0$ (149) *cannot be satisfied!* To save generalized perturbative gauge invariance (148) we require instead the following weaker condition: there exist $N_{j,k} \in \mathcal{P}_0,\ j,k = 0,1,2$ such that

$$\begin{aligned} &G((\mathcal{L}_j, \mathcal{L}_{j+1})f, \mathcal{L}_k g) + (-1)^{jk} G((\mathcal{L}_k, \mathcal{L}_{k+1})g, \mathcal{L}_j f) + s_0(N_{j,k})fg \\ &= -i[N^\nu_{j+1,k}(\partial_\nu f)g + (-1)^{jk} N^\nu_{j,k+1} f(\partial_\nu g)], \quad j,k \in \{0,1,2\},\ \forall f,g \in \mathcal{D}(\mathbb{R}^4), \end{aligned} \tag{200}$$

where $N_{3,k} = 0 = N_{k,3},\ k = 0,1,2$, and that the finite renormalization

$$T_2(\mathcal{L}_j f \otimes \mathcal{L}_k g) \to T_2^N(\mathcal{L}_j f \otimes \mathcal{L}_k g) \stackrel{\text{def}}{=} T_2(\mathcal{L}_j f \otimes \mathcal{L}_k g) + T_1(N_{j,k} fg) \tag{201}$$

maintains the permutation symmetry of T_2 and the normalization conditions **(N1), (N2), (N0)**, and preserves the ghost number: $[Q_g, T_2^N(\mathcal{L}_j f \otimes \mathcal{L}_k g)] = (j+k)T_2^N(\mathcal{L}_j f \otimes \mathcal{L}_k g)$ where we take $g(\mathcal{L}_j) = j$ (143) into account (cf. (105)). This requirement can only be satisfied for $t = 0$ in (194). Hence, the G-terms in (200) are 4-legs terms and, therefore, we may restrict the $N_{j,k}$ to be 4-legs terms, too. In other words the $N_{j,k}$ are sums of monomials of degree four in $A_a^\mu, u_a, \tilde{u}_a, \phi_a,\ a = 1,2,3$ and in H (without any derivative), which are Lorentz tensors of rank $(j+k)$ and satisfy

$$g(N_{j,k}) = j + k, \qquad N^*_{j,k} = -N_{j,k}, \qquad N_{j,k} = (-1)^{jk} N_{k,j}. \tag{202}$$

These properties imply

$$N_{2,1} = 0 = N_{1,2}, \qquad N_{2,2} = 0, \qquad N^{\nu\mu}_{1,1} = -N^{\mu\nu}_{1,1}. \tag{203}$$

If (200)-(201) is satisfied we indeed obtain

$$\begin{aligned} [Q_0, T_2^N(\mathcal{L}_{j_1} f_1 \otimes \mathcal{L}_{j_2} f_2)]_\mp = -i\Big(& T_2^N(\mathcal{L}^\nu_{j_1+1} \partial_\nu f_1 \otimes \mathcal{L}_{j_2} f_2) \\ &+ (-1)^{j_1} T_2^N(\mathcal{L}_{j_1} f_1 \otimes \mathcal{L}^\nu_{j_2+1} \partial_\nu f_2)\Big), \end{aligned} \tag{204}$$

by using the master BRST-identity (133).

Turning to arbitrary orders $n \geq 3$ we look for a sequence of T-products $(T_n^N)_{n\in\mathbb{N}}$ (in the sense of sect. 2.2)

- which satisfies the normalization conditions **(N1), (N2)** and **(N0)**,

- which agrees as far as possible with the given sequence $(T_n)_{n\in\mathbb{N}}$ that satisfies all normalization conditions (also **(N3)**, $(\tilde{\mathbf{N}})$ and the MWI **(N)**),

- and for which $T_2^N(\mathcal{L}_j,\mathcal{L}_k)$ is connected with $T_2(\mathcal{L}_j,\mathcal{L}_k)$ by (201) for all j,k.

For this purpose let $\mathcal{B}=\{\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2,B_1,B_2,...\}$ be a (vector space) basis of $\tilde{\mathcal{P}}_0$. Due to causality (14) the renormalization terms $T_1(N_{j,k}fg)$ in (201) propagate to higher orders. More precisely we define

$$T^N_{l+n}(B_{k_1}g_1\otimes...\otimes B_{k_l}g_l\otimes\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}_{j_n}f_n)\stackrel{\mathrm{def}}{=}\sum_{m=0}^{[\frac{n}{2}]}\sum_{\pi\in\mathcal{S}_n}\frac{1}{2^m m!(n-2m)!}$$
$$\eta^\pi(j_1,...,j_n)T_{l+n-m}(B_{k_1}g_1\otimes...\otimes B_{k_l}g_l\otimes N_{j_{\pi1},j_{\pi2}}f_{\pi1}f_{\pi2}\otimes...$$
$$...\otimes N_{j_{\pi(2m-1)},j_{\pi2m}}f_{\pi(2m-1)}f_{\pi2m}\otimes\mathcal{L}_{j_{\pi(2m+1)}}f_{\pi(2m+1)}\otimes...\otimes\mathcal{L}_{j_{\pi n}}f_{\pi n}),\quad(205)$$

where $B_{k_1},...,B_{k_l}\in\mathcal{B}\setminus\{\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2\}$ and $\eta^\pi(j_1,...,j_n)$ is the sign coming from the permutation of Fermi-operators in $(\mathcal{L}_{j_1},...,\mathcal{L}_{j_n})\to(\mathcal{L}_{j_{\pi1}},...,\mathcal{L}_{j_{\pi n}})$. We extend this definition to $\mathcal{D}(\mathbb{R}^4,\tilde{\mathcal{P}}_0)^{\otimes(l+n)}$ by requiring linearity and (permutation) symmetry. Obviously this $(T_n^N)_{n\in\mathbb{N}}$ solves our requirements. The formula (205) is a particular (simple) case of Theorem 3.1 in [30], which is a precise formulation of a formula given in [4]. For later purpose we mention that the T^N-products (205) satisfy **(N5)(ghost)**, because the T-products do so. In particular we have

$$T^N_{n+1}(k^\mu\partial_\mu g\otimes\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}_{j_n}f_n)=$$
$$\sum_{l=1}^n j_l T^N(\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}_{j_l}f_lg\otimes...\otimes\mathcal{L}_{j_n}f_n).\quad(206)$$

But in general the T^N-products violate **(N3)** and the MWI **(N)**.

To obtain generalized perturbative gauge invariance to orders $n\geq 3$ we additionally require

$$G((\mathcal{L}_j,\mathcal{L}_{j+1})f,N_{k,l}g)+G^{(1)}(N_{k,l}g,\mathcal{L}_jf)=0,\quad j,k,l\in\{0,1,2\},\quad\forall f,g\in\mathcal{D}(\mathbb{R}^4),\quad(207)$$

and

$$G^{(1)}(N_{k,l}f,N_{r,s}g)=0,\qquad k,l,r,s\in\{0,1,2\},\quad\forall f,g\in\mathcal{D}(\mathbb{R}^4).\quad(208)$$

Then, applying the master BRST-identity (137) to $[Q_0,T_n^N(\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}_{j_n}f_n)]_\mp$ and taking (200), (207) and (208) into account, we find the wanted (modified) generalized perturbative gauge invariance[25]

$$[Q_0,T_n^N(\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}_{j_n}f_n)]_\mp=$$
$$-i\sum_{l=1}^n(-1)^{j_1+...+j_{l-1}}T_n^N(\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}^\nu_{j_l+1}\partial_\nu f_l\otimes...\otimes\mathcal{L}_{j_n}f_n).\quad(209)$$

[25] For $j_1=...=j_n=0$ this is the formulation of pertubative gauge invariance for massive fields in [14], [34] and [20].

The particular case $j = k = l = 0$ of the requirements (200)-(201) and (207) has been worked out for general models in [14], [34] and [20]. We specialize these results to the present model:

(1) The second order requirement (200)-(201) is very restrictive: its restriction to $j = k = 0$ is satisfied if and only if the following relations (a)-(e) hold:
(a) the masses agree[26]

$$m \stackrel{\text{def}}{=} m_1 = m_2 = m_3. \tag{210}$$

(b) f_{abc} satisfies the Jacobi identity. (In our simple model this is already known (199), but in the general case the Jacobi identity is obtained only at this stage here, similar to the massless case.)
(c) The H-coupling parameters take the values

$$l_{ab} = \frac{\kappa m}{2}\delta_{ab}, \qquad t = 0, \tag{211}$$

where $\kappa \in \{-1, 1\}$ is an undetermined sign, and p is still free. In particular we see that the Higgs field (or another enlargement/modification of the model) is indispensable.
(d) The constants C_{A_a}, C_{ϕ_a} and C_H have the values[27]

$$C_{A_a} = -\frac{1}{2}, \quad C_{\phi_a} = -1, \quad \forall a, \qquad C_H = -1. \tag{212}$$

(e) The polynomial $N_{0,0}$ reads

$$N_{0,0} = ig_0^2\{\frac{m_H^2}{16m^2}[(\sum_{a=1}^{3}\phi_a^2)^2 + 2H^2\sum_{a=1}^{3}\phi_a^2] + \lambda H^4\}\delta(x_1 - x_2), \tag{213}$$

where $\lambda \in \mathbb{R}$ is a constant which is undetermined so far.

(2) The third oder requirement (207) fixes the remaining free parameters p and λ (which are the parameters of the H-self-couplings): the particular case $j = k = l = 0$ of (207) holds true if and only if

$$p = \frac{m_H^2}{4m}, \qquad \lambda = -\frac{m_H^2}{16m^2}. \tag{214}$$

The C_A, C_u, C_ϕ- and C_H-terms in $T_2^N(\mathcal{L}_0, \mathcal{L}_0)$ and $N_{0,0}$ correspond to the *quadrilinear terms in the interaction Lagrangian of the conventional theory* (the latter are also of order g_0^2). With this identification our resulting interaction agrees precisely with the $SU(2)$ Higgs-Kibble model, which is usually obtained by the Higgs mechanism. Here we have derived it in a completely different way (cf. [14], [34], [20] and [15]).

[26]For general models (210) is replaced by more complicated mass relations, see [14], [34], [20].

[27]We recall that $C_{u_a} = -1$ has already been used in the derivation of the master BRST-identity.

By inserting the explicit expressions (213) into the definition (131) of $G^{(1)}(.,.)$ we verify that the fourth order requirement (208) holds true for $k=l=r=s=0$. We strongly presume that the requirements (200)-(201), (207) and (208) can be fulfilled for all values of j,k,l,r,s.[28] In the following we assume that this conjecture holds true and that the master BRST-identity is fulfilled. In particular we will use the modified generalized perturbative gauge invariance (209).

From the time ordered products $(T^N_n)_n$ we obtain the corresponding anti-chronological products $(\bar{T}^N_n)_n$ by (20). In terms of the T^N- and $\bar{T}^N$-products we construct the totally retarded products $(R^N_{n+1})_n$ (94). The generating functional of the latter is the interacting field $\Lambda^N_{g\mathcal{L}}$ or $W^N_{g\mathcal{L}}(f)$ (93). Similarly to the original R-products, the R^N-products have retarded support

$$\mathrm{supp}\, R^N_{n+1}(...)(x_1,...,x_n;x)\subset\{(y_1,...,y_n,y)|y_l\in y+\bar{V}_-,\forall l\}. \tag{215}$$

The proof of this support property uses only the causal or anti-causal factorization of the T^N- and $\bar{T}^N$-products (see [18]). The replacement $\Lambda_{g\mathcal{L}}\to\Lambda^N_{g\mathcal{L}}(f)$ is a finite renormalization of the interacting field.

The field equation for $\varphi^N_{g\mathcal{L}}$ (where $\varphi\in\mathcal{P}_0$ corresponds to a free field without any derivative) differs from the one of $\varphi_{g\mathcal{L}}$: instead of (95) the master Ward identity implies

$$\begin{aligned}(\square+m^2)\varphi^N_{g\mathcal{L}}(x)&=-g(x)\Big(\frac{\partial\mathcal{L}}{\partial\chi}\Big)^N_{g\mathcal{L}}(x)\\ -\frac{1}{2}(g(x))^2\Big(\frac{\partial N_{0,0}}{\partial\chi}\Big)^N_{g\mathcal{L}}(x)&+\partial^\mu_x[g(x)\Big(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\chi)}\Big)^N_{g\mathcal{L}}(x)],\end{aligned} \tag{216}$$

where we use that $N_{0,0}$ contains no derivatives. The additional term corresponds to the contribution to the Euler-Lagrange equation of that quadrilinear terms (in the conventional Lagrangian) which belong to $N_{0,0}$. The contributions of the quadrilinear terms belonging to the C_A, C_u, C_ϕ- and C_H-terms are contained in the first term on the r.h.s. of (216). For example the contribution of the $g_0^2A^4$-coupling (which belongs to the C_A-terms) is contained in the g_0AF-term in (190), because $F_{g\mathcal{L}}$ (154) has a nonlinear term $\sim g_0(AA)_{g\mathcal{L}}$. The latter is indeed $\sim C_A$ in our framework.

The construction of the interacting BRST-current $\tilde{j}_{g\mathcal{L}}$ (162) must be modified correspondingly. We define

$$\begin{aligned}\tilde{T}^N_{n+1}(j^\mu f\otimes\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}_{j_n}f_n)&\stackrel{\mathrm{def}}{=}[Q_0,T^N_{n+1}(k^\mu f\otimes\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}_{j_n}f_n)]_\mp\\ +i\sum_{l=1}^n(-1)^{j_1+...+j_{l-1}}T^N_{n+1}(k^\mu f&\otimes\mathcal{L}_{j_1}f_1\otimes...\otimes\mathcal{L}^\nu_{j_l+1}\partial_\nu f_l\otimes...\otimes\mathcal{L}_{j_n}f_n).\end{aligned} \tag{217}$$

$\tilde{T}^N(j,\mathcal{L}_{j_1},...)$ has the same properties as $\tilde{T}(j,\mathcal{L}_{j_1},...)$ (159), in particular it satisfies causality (14) and the normalization condition **(N0)**. Hence, $T(j,\mathcal{L}_{j_1},...)\to$

[28] Additionally we expect that these requirements determine $N_{1,0},N_{1,1}$ and $N_{2,0}$ uniquely, similarly to $N_{0,0}$.

$\tilde{T}^N(j, \mathcal{L}_{j_1}, ...)$ is a change of normalization. The divergence identity (160) holds true also for $(\tilde{T}^N_{n+1}, T^N_n)$, because $T^N(k^\mu, \mathcal{L}_{j_1}, ...)$ fulfills **(N5)(ghost)** (206) and $T^N(\mathcal{L}_{j_1}, ...)$ satisfies generalized perturbative gauge invariance (209). Let $\tilde{R}^N_{n+1}(\mathcal{L}, ..., \mathcal{L}; j)$ be the R-product (94) which is constructed in terms of $T^N_k(\mathcal{L}, ..., \mathcal{L})$ and $\tilde{T}^N_{k+1}(j, \mathcal{L}, ..., \mathcal{L})$, $1 \leq k \leq n$. Then we define $\tilde{j}^{N\,\mu}_{g\mathcal{L}}(f)$ similarly to (162), replacing $\tilde{R}_{n+1}(...)$ by $\tilde{R}^N_{n+1}(...)$. Analogously to (163) we then find that this interacting BRST-current is conserved up to terms $\sim \partial g$, more precisely

$$\tilde{j}^{N\,\mu}_{g\mathcal{L}}(\partial_\mu f) = -\mathcal{L}^{N\,\nu}_{1\,g\mathcal{L}}(f \partial_\nu g). \tag{218}$$

The interacting BRST-charge is now defined by

$$Q^N_{g\mathcal{L}} \stackrel{\mathrm{def}}{=} \int d^4x\, h^\mu(x) \tilde{j}^N_{\mu\, g\mathcal{L}}(x) \tag{219}$$

instead of (165). As in the massless case the construction can be done such that $Q^N_{g\mathcal{L}}$ fulfills (166) and is nilpotent (167) (see subsection 4.5.4).

We turn to the BRST-transformation of the interacting fields $\in \mathcal{F}(\mathcal{O})$ (155). Let $s_0(W) = i\partial^\nu W'_\nu$. The formula (171) is violated by the non-vanishing of $G((\mathcal{L}_0, \mathcal{L}_1)f, \mathcal{L}_0 g) + (f \leftrightarrow g)$. It must be modified:

$$\begin{aligned}
&[Q_0, T^N_{n+1}(Wf \otimes \mathcal{L}_0 f_1 \otimes ... \otimes \mathcal{L}_0 f_n)] = \\
&\mp i \sum_{l=1}^n T^N_{n+1}(Wf \otimes \mathcal{L}_0 f_1 \otimes ... \otimes \mathcal{L}^\nu_1 \partial_\nu f_l \otimes ... \otimes \mathcal{L}_0 f_n) \\
&-iT^N_{n+1}(W'_\nu \partial^\nu f \otimes \mathcal{L}_0 f_1 \otimes ... \otimes \mathcal{L}_0 f_n) \\
&+\sum_{k=1}^n T^N_n\Big([G((\mathcal{L}_0, \mathcal{L}_1)f_k, Wf) + G((W, W')f, \mathcal{L}_0 f_k)] \otimes \mathcal{L}_0 f_1 \otimes ...\hat{k}... \otimes \mathcal{L}_0 f_n\Big),
\end{aligned} \tag{220}$$

for $W \notin [\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2\}]$ (the [...]-bracket denotes the linear span) and, for simplicity, we assume that $W' \in \mathcal{P}_0$ does not contain any derivative of ϕ_a or H. This assumption ensures

$$G((W, W')f, N_{0,0}g) = 0, \qquad \forall f, g \in \mathcal{D}(\mathbb{R}^4), \tag{221}$$

as can be seen by inserting the explicit expression (213) for $N_{0,0}$ into the definition (134) of $G(.,.)$. Moreover, one finds

$$G^{(1)}(N_{0,0}g, Wf) = 0, \qquad \forall f, g \in \mathcal{D}(\mathbb{R}^4), \tag{222}$$

by inserting (213) into the definition (131) of $G^{(1)}$. Analogously to the derivation of (209), the proof of (220) is a straightforward application of the master BRST-identity to $[Q_0, T^N_{n+1}(Wf \otimes \mathcal{L}_0 f_1 \otimes ...)]$, which uses (200), (207), (208), (221) and (222). As in the massless case (220) translates into a similar formula for the R-products, which yields the BRST-transformation (see subsection 4.5.4)

$$\begin{aligned}
&\tilde{s}(W^N_{g\mathcal{L}}(f)) \stackrel{\mathrm{def}}{=} [Q^N_{g\mathcal{L}}, W^N_{g\mathcal{L}}(f)]_\mp = -iW'^{\nu\,N}_{g\mathcal{L}}(\partial_\nu f) \\
&+i\big(G((\mathcal{L}, \mathcal{L}_1)g, Wf) + G((W, W')f, \mathcal{L}g)\big)^N_{g\mathcal{L}}, \quad f \in \mathcal{D}(\mathcal{O}).
\end{aligned} \tag{223}$$

For $W \in [\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2\}]$ the G-term in (220) is absent ((220) is replaced by (209)) and, hence, there is no nonlinear term in the BRST-transformation (223), as in the massless case (189).

4.5.4 The interacting BRST-charge

In this subsection we recall the details of the construction of the interacting BRST-charge from [7], and prove the properties (166) and the nilpotency. Finally we show that the identity (172) ((220) resp.) implies the BRST-transformation formula (173) ((223) resp.) for the interacting fields. We deal with massive gauge fields, however, the massless case is included: in all formulas it is allowed to set $m = 0$, $\phi \equiv 0$ and $H \equiv 0$ (which replaces T^N by T etc.).

For simplicity we assume that the double cone $\mathcal{O}$ (155) is centered at the origin. Let r be the diameter of $\mathcal{O}$. The question is, how to choose g and h^μ such that $Q^N_{g\mathcal{L}}$ (219) satisfies (166) and is nilpotent. From the requirement $(Q^N_{g\mathcal{L}})_0 = Q_0$ and the fact that Q_0 annihilates the Fock vacuum, we conclude that h^μ must have *unbounded* support, due to the Reeh-Schlieder theorem. At least for massless fields, it is hard to avoid a volume divergence in $\int d^4x\, h^\mu(x) \tilde{j}^N_{\mu\, g\mathcal{L}}(x)$. To get rid of this problem we proceed as in [7]: we embed our double cone $\mathcal{O}$ isometrically into the cylinder $\mathbb{R} \times C_L$ (the first factor denotes the time axis), where C_L is a cube of length L, $\quad L \gg r$, with metallic boundary conditions[29] for the free gauge fields A_a, and Diriclet boundary conditions for the free ghost fields u_a, $\tilde{u}_a$ and the free bosonic scalar fields ϕ_a and H. (It turns out that the corresponding interacting fields fulfil the same boundary conditions, see appendix A of [7]). If we choose the compactification length L big enough, the physical properties of the local algebra $\mathcal{F}(\mathcal{O})$ are unchanged.

The inductive construction of the T-products and, hence, of the interacting fields is also not changed by the compactification [5]. We assume the switching function g to fulfil

$$g(x) = 1 \quad \forall x \in \mathcal{O} \cup \{(x_0, \vec{x}) | \, |x_0| < \epsilon\} \quad (r \gg \epsilon > 0) \tag{224}$$

on $\mathbb{R} \times C_L$ and to have compact support in timelike directions. Now we choose

$$h^\mu(x) \stackrel{\mathrm{def}}{=} \delta^\mu_0 h(x_0), \quad \text{where} \quad h \in \mathcal{D}([-\epsilon, \epsilon], \mathbb{R}), \quad \int dx_0\, h(x_0) = 1. \tag{225}$$

Then

$$Q^N_{g\mathcal{L}} \stackrel{\mathrm{def}}{=} \int dx_0\, h(x_0) \int_{C_L} d^3x\, \tilde{j}^N_{0\, g\mathcal{L}}(x) \tag{226}$$

is well-defined, because $(x_0, \vec{x}) \to h(x_0)$ is an admissible test function on $\mathbb{R} \times C_L$. $(Q^N_{g\mathcal{L}})_0 = Q_0$ holds true, since $(\tilde{j}^N_{\mu\, g\mathcal{L}})_0 = j_\mu$ is conserved. From the conservation of the interacting current, $\partial^\mu \tilde{j}^N_{\mu\, g\mathcal{L}}(x) = 0$ for $x \in [-\epsilon, \epsilon] \times C_L$, we conclude that

[29]This means that for every index a the pullback of the 2-form F_a vanishes at the boundary. Or, we require for every free gauge field A_a, $a = 1, ..., N$, the same boundary conditions as for the electromagnetic potential A in [7], appendix A.

$Q^N_{g\mathcal{L}}$ is independent from the choice of h. By **(N2)** and by the fact that h and g are real-valued we obtain $Q^N_{g\mathcal{L}}{}^* = Q^N_{g\mathcal{L}}$.

It remains to prove the nilpotency. For this purpose we first show

$$Q^N_{g\mathcal{L}} = Q_0 + \mathcal{L}^{N\,\nu}_{1\,g\mathcal{L}}(H\partial_\nu g) \tag{227}$$

where

$$H(x) \equiv \tilde{H}(x_0) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{x_0} dt\,[-h(t) + h(t-a)] \tag{228}$$

and $a \in \mathbb{R}$ is such that the support of $(x_0, \vec{x}) \to h(x_0 - a)$ is earlier than the support of g:

$$x_0 < y_0, \quad \forall x_0 \in \mathrm{supp}\, h(\cdot - a) \ \wedge\ \forall y_0 \text{ with } \exists \vec{y} \in C_L \text{ with } (y_0, \vec{y}) \in \mathrm{supp}\, g. \tag{229}$$

In particular we will need

$$H(y)\partial g(y) = \partial g(y), \qquad \forall y \in ((\mathrm{supp}\, H \cup \mathcal{O}) + \bar{V}_-), \tag{230}$$

$$\mathcal{O} \cap (\mathrm{supp}\,(H\partial g) + \bar{V}_-) = \emptyset \tag{231}$$

and

$$\partial H(y)\partial g(y) = 0. \tag{232}$$

Proof of (227): From our definitions we immediately obtain

$$Q^N_{g\mathcal{L}} = \int_{\mathbb{R}\times C_L} d^4x\, \tilde{j}^{N\,\mu}_{g\mathcal{L}}(x)[-\partial_\mu H(x) + g_{\mu 0} h(x_0 - a)]. \tag{233}$$

Due to (229) and the retarded support of the R-products (215) we have

$$\mathrm{supp}\,(\tilde{j}^N_{\mu\, g\mathcal{L}} - j_\mu) \cap \mathrm{supp}\,((x_0, \vec{x}) \to h(x_0 - a)) = \emptyset. \tag{234}$$

So the contribution of $g_{\mu 0} h(x_0 - a)$ to (233) is Q_0, and by inserting (218) into the $\partial_\mu H$-term we obtain the assertion (227). $\square$

The formula (227) manifestly shows that $Q^N_{g\mathcal{L}}$ converges to Q_0 in the adiabatic limit $g(x) \to 1$, $\forall x$, provided this limit exists. For pure massive theories this limit exists indeed in the strong sense [19]. So, in the adiabatic limit of a pure massive gauge theory, one has the simplification that the BRST-cohomology is given in terms of Q_0 (cf. (140)-(141) and [15]).

In the following proofs we will use Proposition 2 of [7], which is a formula for the (anti-)commutator of two interacting fields:[30]

$$[W^N_{g\mathcal{L}}(f), V^N_{g\mathcal{L}}(h)]_\mp = R^N_{g\mathcal{L}}(W, V)(f, h) \mp R^N_{g\mathcal{L}}(V, W)(h, f), \tag{235}$$

[30] This formula, the retarded support of the R-products (215) and some further, quite obvious requirements can be viewed as the defining properties of the retarded products. They determine a direct construction of the R_{n+1} ($n \in \mathbb{N}_0$) by induction on n. If wanted, the T-products can then be obtained by reversing (94), see [35], the second paper of [8] and [10].

where

$$R_{g\mathcal{L}}^N(W,V)(f,h) \stackrel{\mathrm{def}}{=} -\sum_{n=0}^{\infty} \frac{i^n}{n!} R_{n+2}^N((g\mathcal{L})^{\otimes n} \otimes Wf; Vh). \tag{236}$$

Due to the retarded support (215) of the R^N-products we call $R_{g\mathcal{L}}^N(W,V)(f,h)$ the retarded part (and $\mp R_{g\mathcal{L}}^N(V,W)(h,f)$ the advanced part) of $[W_{g\mathcal{L}}^N(f), V_{g\mathcal{L}}^N(h)]_\mp$.

Proof of $(\mathbf{Q}^{\mathbf{N}}_{\mathbf{g}\mathcal{L}})^{\mathbf{2}} = \mathbf{0}$: Generalized perturbative gauge invariance (209) implies the relation

$$[Q_0, R_n^N(\mathcal{L}f_1 \otimes ... \otimes \mathcal{L}f_{n-1}; \mathcal{L}_1^\mu f)]_+ = -i \sum_{l=1}^{n-1} R_n^N(\mathcal{L}f_1 \otimes ...$$
$$... \otimes \mathcal{L}_1^\rho \partial_\rho f_l \otimes ... \otimes \mathcal{L}f_n; \mathcal{L}_1^\mu f)) - iR_n^N(\mathcal{L}f_1 \otimes ... \otimes \mathcal{L}f_{n-1}; \mathcal{L}_2^{\mu\rho}\partial_\rho f). \tag{237}$$

Due to $\mathcal{L}_2^{\mu\rho} = -\mathcal{L}_2^{\rho\mu}$ we may require

$$T_n^N(\mathcal{L}, ..., \mathcal{L}, \mathcal{L}_2^{\mu\rho}) = -T_n^N(\mathcal{L}, ..., \mathcal{L}, \mathcal{L}_2^{\rho\mu}). \tag{238}$$

This is an additional normalization condition, which is compatible with the other normalization conditions. Similarly to **(N2)**, it can be satisfied by antisymmetrizing in $\mu \leftrightarrow \rho$ a $T_n^N(\mathcal{L}, ..., \mathcal{L}; \mathcal{L}_2^{\mu\rho})$ which satisfies the other normalization conditions. The corresponding $R_n^N(\mathcal{L}, ..., \mathcal{L}; \mathcal{L}_2^{\mu\rho})$ is then also antisymmetric in $\mu \leftrightarrow \rho$ and, hence, we have $\mathcal{L}_{2\,g\mathcal{L}}^{N\,\rho\mu} = -\mathcal{L}_{2\,g\mathcal{L}}^{N\,\mu\rho}$.

By means of (227) and $Q_0^2 = 0$ we obtain

$$2(Q_{g\mathcal{L}}^N)^2 = [Q_{g\mathcal{L}}^N, Q_{g\mathcal{L}}^N]_+ = 2[Q_0, \mathcal{L}_{1\,g\mathcal{L}}^{N\,\mu}(H\partial_\mu g)]_+ + [\mathcal{L}_{1\,g\mathcal{L}}^{N\,\nu}(H\partial_\nu g), \mathcal{L}_{1\,g\mathcal{L}}^{N\,\mu}(H\partial_\mu g)]_+. \tag{239}$$

Using (237) and (236) we obtain

$$2[Q_0, \mathcal{L}_{1\,g\mathcal{L}}^{N\,\mu}(H\partial_\mu g)]_+ = -2i\mathcal{L}_{2\,g\mathcal{L}}^{N\,\mu\rho}(\partial_\rho(H\partial_\mu g)) - 2R_{g\mathcal{L}}^N(\mathcal{L}_1^\nu, \mathcal{L}_1^\mu)(\partial_\nu g, H\partial_\mu g). \tag{240}$$

In the first term the $(\partial_\rho H)(\partial_\mu g)$-part vanishes by (232) and the $H\partial_\rho\partial_\mu g$-part is zero because of (238). In the remaining second term we first take (230) into account and then apply (235)

$$\begin{aligned} 2[Q_0, \mathcal{L}_{1\,g\mathcal{L}}^{N\,\mu}(H\partial_\mu g)]_+ &= -2R_{g\mathcal{L}}^N(\mathcal{L}_1^\nu, \mathcal{L}_1^\mu)(H\partial_\nu g, H\partial_\mu g) \\ &= -[\mathcal{L}_{1\,g\mathcal{L}}^{N\,\nu}(H\partial_\nu g), \mathcal{L}_{1\,g\mathcal{L}}^{N\,\mu}(H\partial_\mu g)]_+. \end{aligned} \tag{241}$$

Inserting this into (239) we see that $Q_{g\mathcal{L}}^N$ is in fact nilpotent. $\square$

Proof of the formula

$$[Q_{g\mathcal{L}}^N, W_{g\mathcal{L}}^N(f)]_\mp = -iW_{g\mathcal{L}}^{\prime\nu\,N}(\partial_\nu f) + i\big(G((\mathcal{L}, \mathcal{L}_1)g, Wf) + G((W, W')f, \mathcal{L}g)\big)_{g\mathcal{L}}^N \tag{242}$$

for the BRST-transformation of the interacting fields: the T-product identity (220) can be translated into an identity for the R-products

$$
\begin{aligned}
&[Q_0, R^N_{n+1}(\mathcal{L}f_1 \otimes ... \otimes \mathcal{L}f_n; Wf)]_\mp = \\
&-i\sum_{l=1}^n R^N_{n+1}(\mathcal{L}f_1 \otimes ... \otimes \mathcal{L}^\nu_1 \partial_\nu f_l \otimes ... \otimes \mathcal{L}f_n; Wf) \\
&-iR^N_{n+1}(\mathcal{L}f_1 \otimes ... \otimes \mathcal{L}f_n; W'_\nu \partial^\nu f) \\
&+\sum_{k=1}^n R^N_n\Big(\mathcal{L}f_1 \otimes ...\hat{k}... \otimes \mathcal{L}f_n; [G((\mathcal{L},\mathcal{L}_1)f_k, Wf) + G((W,W')f, \mathcal{L}f_k)]\Big),
\end{aligned}
\tag{243}
$$

similarly to (172). For $W \in [\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2\}]$ the G-term is absent, because (243) is then derived from (209). Keeping this in mind we treat the cases $W \notin [\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2\}]$ and $W = \mathcal{L}_j$ simultaneously.

Let $f \in \mathcal{D}(\mathcal{O})$. Using (227) we obtain

$$
\begin{aligned}
[Q^N_{g\mathcal{L}}, W^N_{g\mathcal{L}}(f)]_\mp = \sum_{n=0}^\infty \frac{i^n}{n!}[Q_0, R^N_{n+1}(\mathcal{L}g \otimes ... \\
... \otimes \mathcal{L}g; Wf)]_\mp + [\mathcal{L}^{N\,\nu}_{1\,g\mathcal{L}}(H\partial_\nu g), W^N_{g\mathcal{L}}(f)]_\mp.
\end{aligned}
\tag{244}
$$

By (243) and (236) the first term is equal to

$$
-iW'^{\nu\,N}_{g\mathcal{L}}(\partial_\nu f) + i\big(G((\mathcal{L},\mathcal{L}_1)g, Wf) + G((W,W')f, \mathcal{L}g)\big)^N_{g\mathcal{L}} - R^N_{g\mathcal{L}}(\mathcal{L}^\nu_1, W)(\partial_\nu g, f).
\tag{245}
$$

We turn to the second term in (244) and use (235)

$$
[\mathcal{L}^{N\,\nu}_{1\,g\mathcal{L}}(H\partial_\nu g), W^N_{g\mathcal{L}}(f)]_\mp = R^N_{g\mathcal{L}}(\mathcal{L}^\nu_1, W)(H\partial_\nu g, f) \mp R^N_{g\mathcal{L}}(W, \mathcal{L}^\nu_1)(f, H\partial_\nu g).
\tag{246}
$$

The second term vanishes due to the support properties (215) and (231). Because of (230) we may omit the factor H in the first term. Hence, we find that $[\mathcal{L}^{N\,\nu}_{1\,g\mathcal{L}}(H\partial_\nu g), W^N_{g\mathcal{L}}(f)]_\mp$ cancels out with the last term in (245) and it remains the assertion (242). $\square$

5 Taking anomalies into account

5.1 General procedure and axial anomaly

We recall that we understand by the expression 'anomaly' any term that violates the master Ward identity **(N)** and cannot be removed by an admissible finite renormalization of the T-products. The aim of this section is to take anomalies into account in the formulation of the MWI. This has consequences for the normalization condition ($\tilde{\mathbf{N}}$): We want to normalize $T((\tilde{\partial}^\nu V) \otimes W_1 f_1 \otimes ...)$ such

that (31) holds true: in the special case $W=1$, $V,W_1,...,W_n\in\mathcal{P}_0$ the r.h.s. of ($\tilde{\mathbf{N}}$) should agree with the r.h.s. of **(N)**. Hence, we will take the anomalies into account also in ($\tilde{\mathbf{N}}$).

We proceed inductively with respect to the order n. Since we are not aware of a general proof that second order loop diagrams (i.e. $n=1$ in **(N)**) are anomaly-free[31], we start with that case. We set

$$-a^{(2)\nu}_{V,W}(g,f)\stackrel{\mathrm{def}}{=}T_2(V\partial^\nu g\otimes Wf)+T_2((\partial^\nu V)g\otimes Wf)$$
$$+i\sum_{\chi,\psi\in\mathcal{G}}(\pm)T_1\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}g,\frac{\partial W}{\partial\psi}f\big)\Big),\quad V,W\in\tilde{\mathcal{P}}_0. \tag{247}$$

For later purpose we let $V,W\in\tilde{\mathcal{P}}_0$ (not only $V,W\in\mathcal{P}_0$ as in the MWI **(N)**). Due to causal factorization of the T-products we know that $a^{(2)}_{V,W}(g,f)$ is local. Therefore, there exists a unique $b^{(2)}_{V,W}(g,f)\in\mathcal{D}(\mathbb{R}^4,\mathcal{P}_0)$ with[32]

$$a^{(2)\nu}_{V,W}(g,f)=T_1(b^{(2)\nu}_{V,W}(g,f)). \tag{248}$$

Let us now assume that we have already defined $a^{(m+1)}_{V,W_1,...,W_m}$ and $b^{(m+1)}_{V,W_1,...,W_m}\in\mathcal{D}(\mathbb{R}^4,\mathcal{P}_0)$ for all $m<n$. We then set

$$-a^{(n+1)\nu}_{V,W_1,...,W_n}(g,f_1,...,f_n)\stackrel{\mathrm{def}}{=}T_{n+1}(V\partial^\nu g\otimes W_1f_1\otimes...\otimes W_nf_n)$$
$$+T_{n+1}((\partial^\nu V)g\otimes W_1f_1\otimes...\otimes W_nf_n)$$
$$+i\sum_{m=1}^n\sum_{\chi,\psi\in\mathcal{G}}(\pm)T_n\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}g,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes...\hat{m}...\otimes W_nf_n\Big)$$
$$+\sum_{k=1}^{n-1}\sum_{1\le m_1<...<m_k\le n}(\pm)T_{n+1-k}\Big(b^{(k+1)\nu}_{V,W_{m_1},...,W_{m_k}}(g,f_{m_1},...,f_{m_k})$$
$$\otimes W_1f_1\otimes...\hat{m}_1...\hat{m}_k...\otimes W_nf_n\Big), \tag{249}$$

where $V,W_1,...,W_n\in\tilde{\mathcal{P}}_0$ and the possible signs $(\pm)$ come from the permutation of Fermi operators. By causal factorization and the definition of the $(b^{(k+1)})_{k<n}$ we conclude that $a^{(n+1)}_{V,W_1,...,W_n}(g,f_1,...,f_n)$ is local and, hence, that there exists a unique $b^{(n+1)}_{V,W_1,...,W_n}(g,f_1,...,f_n)\in\mathcal{D}(\mathbb{R}^4,\mathcal{P}_0)$ with

$$a^{(n+1)}_{V,W_1,...,W_n}(g,f_1,...,f_n)=T_1(b^{(n+1)}_{V,W_1,...,W_n}(g,f_1,...,f_n)). \tag{250}$$

Obviously

$$b^{(n+1)}:\mathcal{D}(\mathbb{R}^4,\tilde{\mathcal{P}}_0)^{\otimes n+1}\to\mathcal{D}(\mathbb{R}^4,\mathcal{P}_0),$$
$$Vg\otimes W_1f_1\otimes...\otimes W_nf_n\to b^{(n+1)}_{V,W_1,...,W_n}(g,f_1,...,f_n) \tag{251}$$

[31]In our proof of charge conservation **(N5) (charge)** (which is given in appendix B of [7]) vacuum polarization plays an exceptional role, even to second order: our general argumentation does not yield $\partial^\mu_x T_2(j_\mu(x)j_\nu(y))=0$, an explicit calculation was necessary.

[32]Remember $T_1(\mathcal{D}(\mathbb{R}^4,\mathcal{P}_0))=T_1(\mathcal{D}(\mathbb{R}^4,\tilde{\mathcal{P}}_0))$ and that T_1 is injective on $\mathcal{D}(\mathbb{R}^4,\mathcal{P}_0)$.

is linear and symmetrical in all factors except the first one. As a consequence of the normalization condition **(N3)** the $b^{(n+1)}_{V,W_1,...,W_n}$, $V,W_1,...,W_n \in \mathcal{P}_0$, n fixed, are not independent. For example let V be a sub-polynomial of V' and W_k a sub-polynomial of W'_k, $\forall k=1,...,n$ ($V=V'$ and $W_k=W'_k$ is admitted). Then $b^{(n+1)}_{V,W_1,...,W_n}\neq 0$ implies $b^{(n+1)}_{V',W'_1,...,W'_n}\neq 0$[33]. However, the $b^{(n+1)}_{V,W_1,...,W_n}$ are independent for *different* n, because the violations of the MWI coming from subdiagrams are taken into account in (249) by the terms $\sum_{k=1}^{n-1}\sum(\pm)T_{n+1-k}(b^{(k+1)}(...)\otimes ...)$.

Obviously the $b^{(k)}$ (249)-(250) depend on the normalization of the T-products. We assume that the latter fulfil **(N0)-(N3)** and ($\tilde{\mathbf{N}}$) in the following modified form:

($\tilde{\mathbf{N}}$)
$$\begin{aligned} &T_{n+1}((\tilde\partial^\nu V)Wg\otimes W_1f_1\otimes ...\otimes W_nf_n)=\\ &T_{n+1}((\partial^\nu V)Wg\otimes W_1f_1\otimes ...\otimes W_nf_n)\\ &+i\sum_{m=1}^n\sum_{\chi,\psi\in\tilde{\mathcal{G}}}(\pm)T_n\Big(\Delta^\nu_{\chi,\psi}\big(\frac{\partial V}{\partial\chi}Wg,\frac{\partial W_m}{\partial\psi}f_m\big)\otimes W_1f_1\otimes ...\hat m...\otimes W_nf_n\Big)\\ &+\sum_{k=1}^n\sum_{1\le m_1<...<m_k\le n}(\pm)T_{n+1-k}\Big(b^{(k+1)\nu}_{V,W_{m_1},...,W_{m_k}}(g,f_{m_1},...,f_{m_k})W\\ &\otimes W_1f_1\otimes ...\hat m_1...\hat m_k...\otimes W_nf_n\Big), \end{aligned} \tag{252}$$

where $V,W,W_1,...,W_n\in\tilde{\mathcal{P}}_0$. Note that the sum over k in the last term runs here up to n. Setting $W=1$ and using the definition (249)-(250), we in fact obtain (31) (even for $V,W_1,...,W_n\in\tilde{\mathcal{P}}_0$), which is the main reason for this modification of ($\tilde{\mathbf{N}}$). In sect. 2 this implication relied on the validity of the MWI **(N)**. This assumption is not needed here to get (31).

Also the modified ($\tilde{\mathbf{N}}$) fixes the normalization of the T-products of symbols with external derivatives in terms of T-products without external derivatives, namely by the following recursive procedure: For a monomial $W=\prod_s\tilde\partial^{a^{(s)}}\varphi_{r_s}\in\tilde{\mathcal{P}}_0$, $\varphi_r\in\mathcal{P}_0$, $a^{(s)}\in(\mathbb{N}_0)^4$, we define $|W|=\sum_s|a^{(s)}|$ where $|a^{(s)}|=a_0^{(s)}+...+a_3^{(s)}$. Let the normalization of $T_n(W_1,...,W_n)$ with $|W_1|+...+|W_n|=0$ (i.e. $W_1,...,W_n\in\mathcal{P}_0$) be given for all $n\in\mathbb{N}$. Then the determination of the $b^{(n)}_{W_1,...,W_n}$ and of the normalization of $T_n(W_1,...,W_n)$ with $|W_1|+...+|W_n|>0$ goes in a double inductive way: one makes a first induction with respect to the order n and for each fixed n a second induction with respect to $|W_1|+...+|W_n|$. More precisely let $T_l(W_1,...,W_l)$ and $b^{(l)}_{W_1,...,W_l}$ be given for all $l\le n$ and $W_1,...,W_l\in\tilde{\mathcal{P}}_0$, and also for $l=n+1$ if $|W_1|+...+|W_{n+1}|<d$ $(d\in\mathbb{N})$. Then we determine by ($\tilde{\mathbf{N}}$) (252) the normalization of the $T_{n+1}(W_1,...,W_{n+1})$ with $|W_1|+...+|W_{n+1}|=d$ (this

[33]To see this we consider the $(V',W'_1,...,W'_n)$-diagrams in which the additional factors of $V',W'_1,...,W'_n$ are external legs. By amputating these external legs we obtain all $(V,W_1,...,W_n)$-diagrams. **(N3)** requires that the non-amputated and amputated diagrams are equally normalized.

step does not take place for $d = 0$, because T_{n+1} is given in that case). More precisely we use ($\tilde{\mathbf{N}}$) for $|V| + |W| + |W_1| + ... + |W_n| = d - 1$. Note that all $b^{(k+1)}$ and all T-products which appear in this case on the r.h.s. of ($\tilde{\mathbf{N}}$) are inductively given. Finally, from (249)-(250) we obtain the $b^{(n+1)}_{V,W_1,...,W_n}$ with $|V| + |W_1| + ... + |W_n| = d$. Again we point out that thereby all terms which appear on the r.h.s. of (249) are inductively known.

To formulate the modified MWI we specialize to the case $V, W_1, ..., W_n \in \mathcal{P}_0$. Let us consider the set $\mathcal{T}$ of all sequences of T-products $(T_n)_{n\in\mathbb{N}}$ which satisfy the requirements of sect. 2.2 (in particular causality and the normalization conditions **(N0)-(N3)**) and the modified ($\tilde{\mathbf{N}}$). We now define

$$\mathcal{A}((T_n)_{n\in\mathbb{N}}) \stackrel{\mathrm{def}}{=} (\bar{b}^{(n+1)})_{n\in\mathbb{N}}, \qquad \forall (T_n)_{n\in\mathbb{N}} \in \mathcal{T}, \tag{253}$$

where $\bar{b}^{(n+1)}$ is the restriction of $b^{(n+1)}$ (251) to $\mathcal{D}(\mathbb{R}^4, \mathcal{P}_0)^{\otimes n+1}$. The image $\mathcal{A}(\mathcal{T})$ of this map is model dependent and it is usually hard work to get information about $\mathcal{A}(\mathcal{T})$. If the zero-sequence (i.e. $\mathbf{0} \stackrel{\mathrm{def}}{=} (0, 0, ...)$) is an element of $\mathcal{A}(\mathcal{T})$, which means that the model is anomaly-free, we are in the situation of sect. 2.4: the MWI is then the normalization condition which forbids all $(T_n)_{n\in\mathbb{N}}$ which are not an element of $\mathcal{A}^{-1}(\mathbf{0})$. If $\mathbf{0} \notin \mathcal{A}(\mathcal{T})$ we choose a suitable (usually as simple as possible) $\mathbf{b} \in \mathcal{A}(\mathcal{T})$ and *the master Ward identity is then the normalization condition that solely sequences* $(T_n)_{n\in\mathbb{N}} \in \mathcal{A}^{-1}(\mathbf{b})$ *are allowed.*

We illustrate this by the example of the axial anomaly. Let $\mathcal{P}_\psi$ be the linear space which is generated by $\mathcal{L} \stackrel{\mathrm{def}}{=} A_\mu j^\mu$, j_A, j_π (cf. (90)) and all sub-monomials thereof. According to Bardeen [1] the most simple $\mathbf{b} \in \mathcal{A}(\mathcal{T})$ reads

$$\begin{gathered}
b^{(n+1)}_{V,W_1,...,W_n} = 0, \quad \forall\, n+1 \neq 3, \quad V, W_1, ...W_n \in \mathcal{P}_\psi, \\
b^{(3)\nu}_{j_{A\nu}, j^{\mu_1}, j^{\mu_2}}(g, f_1, f_2) = C\epsilon^{\mu_1\mu_2\rho\tau} g(\partial_\rho f_1)(\partial_\tau f_2), \\
b^{(3)\nu}_{j_{A\nu}, \mathcal{L}, \mathcal{L}}(g, f_1, f_2) = C\epsilon^{\mu_1\mu_2\rho\tau} g\partial_\rho(f_1 A_{\mu_1})\partial_\tau(f_2 A_{\mu_2}), \\
b^{(3)\nu}_{j_{A\nu}, \mathcal{L}, j^\mu}(g, f_1, f_2) = b^{(3)\nu}_{j_{A\nu}, j^\mu, \mathcal{L}}(g, f_2, f_1) = C\epsilon^{\mu_1\mu_2\rho\tau} g\partial_\rho(f_1 A_{\mu_1})\partial_\tau f_2, \\
b^{(3)}_{j_A, j_A, j_A}(g, f_1, f_2) = \frac{1}{3} b^{(3)}_{j_A, j, j}(g, f_1, f_2)
\end{gathered} \tag{254}$$

and $b^{(3)}_{V,W_1,W_2} = 0$ for all other $(V, W_1, W_2) \in (\mathcal{P}_\psi)^{\times 3}$, where C is a well-known, fixed, complex number. Then particular cases of the MWI read

$$\begin{gathered}
T_{n+1}(j^\mu \partial_\mu f \otimes \mathcal{L}g_1 \otimes ... \otimes \mathcal{L}g_n) = 0, \\
T_{n+1}(j^\mu_A \partial_\mu f \otimes \mathcal{L}g_1 \otimes ... \otimes \mathcal{L}g_n) = 2m T_{n+1}(j_\pi f \otimes \mathcal{L}g_1 \otimes ... \otimes \mathcal{L}g_n) \\
+ \sum_{1 \leq m_1 < m_2 \leq n} C\epsilon^{\mu_1\mu_2\rho\tau} T_{n-1}(f\partial_\rho(g_{m_1} A_{\mu_1})\partial_\tau(g_{m_2} A_{\mu_2}) \otimes \mathcal{L}g_1 \otimes ...\hat{m}_1...\hat{m}_2... \otimes \mathcal{L}g_n),
\end{gathered} \tag{255}$$

which imply

$$j^\mu_{g\mathcal{L}}(\partial_\mu f) = 0,$$

$$-j^\mu_{A\,g\mathcal{L}}(\partial_\mu f) = 2m j_{\pi\,g\mathcal{L}}(f) - g_0^2 \frac{C}{8}\epsilon^{\mu_1\mu_2\rho\tau}(F_{\rho\mu_1}F_{\tau\mu_2})_{g\mathcal{L}}(f), \tag{256}$$

where we assume $g(x) = g_0 = \text{const.}, \forall x \in \text{supp}\, f$.

Non-vanishing anomalies $b^{(m+1)}_{V,W_1,\ldots,W_m}$ are not an obstacle to fulfil the normalization condition **(N4)** and hence the field equation (95) (see sect. 4.1), because (92) still solves **(N4)** (4.1). But the axial anomaly appears as an additional term in the charge conservation **(N5) (charge)** (97) and in the generalized perturbative gauge invariance **(N6)** (148) and hence also in the master BRST-identity, if axial fermions are present. However, for the non-Abelian gauge models studied in sects. 4.5.1-2, we expect that the master BRST-identity and ghost number conservation can be satisfied, and that therefore our local construction of observables works. But this remains to be proved.

5.2 Energy momentum tensor: conservation and trace anomaly

We follow the procedure in [27]. Classically the canonical energy momentum tensor is the Noether current belonging to translation invariance (in time and space). Turning to QFT we consider a real, free, scalar field ϕ of mass $m \geq 0$. (In the formalism of appendix A we set $\varphi \stackrel{\text{def}}{=} \chi \stackrel{\text{def}}{=} \phi$ and choose $\epsilon = 1$.) The free canonical energy momentum tensor reads

$$\Theta^{\mu\nu}_{0\,\text{can}} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial^\rho\phi\partial_\rho\phi + \frac{1}{2}g^{\mu\nu}m^2\phi^2, \tag{257}$$

and this tensor is conserved due to the Klein-Gordon equation: $\partial_\mu\Theta^{\mu\nu}_{0\,\text{can}} = 0$.

Now we add an interaction of the form

$$\mathcal{L} = \lambda\phi^4. \tag{258}$$

The interacting canonical energy momentum tensor is not simply the interacting field belonging to $\Theta^{\mu\nu}_{0\,\text{can}}$, it has an additional term

$$\Theta^{\mu\nu}_{\text{can}\,g\mathcal{L}}(f) = \Theta^{\mu\nu}_{0\,\text{can}\,g\mathcal{L}}(f) + g^{\mu\nu}\mathcal{L}_{g\mathcal{L}}(gf). \tag{259}$$

Let $W_1, ..., W_n$ be polynomials in ϕ (without any derivative). Applying twice the definition (249)-(250) we obtain the relation

$$\begin{aligned} &T(W_1g_1 \otimes ... \otimes W_ng_n \otimes \Theta^{\mu\nu}_{0\,\text{can}}\partial_\mu f) = \\ &-i\sum_{k=1}^n T(W_1g_1 \otimes ... \otimes (\partial^\nu W_k)g_kf \otimes ... \otimes W_ng_n) - \mathcal{A}_1(f, g_1, ..., g_n) = \\ &-\mathcal{A}_1(f, g_1, ..., g_n) + i\sum_{k=1}^n \Big(T(W_1g_1 \otimes ... \otimes W_k\partial^\nu(g_kf) \otimes ... \\ &... \otimes W_ng_n) + \mathcal{A}_{2,k}(f, g_1, ..., g_n)\Big), \end{aligned} \tag{260}$$

where

$$\mathcal{A}_1(f,g_1,...,g_n)\stackrel{\rm def}{=}\sum_{j=1}^n\sum_{m_1<...<m_j}T(b^{(j+1)}_{\Theta^{\mu\nu}_{0\,{\rm can}},W_{m_1},...,W_{m_j}\,\mu}(f,g_{m_1},...,g_{m_j})$$

$$\otimes W_1g_1\otimes...\hat{m}_1...\hat{m}_j...\otimes W_ng_n),$$

$$\mathcal{A}_{2,k}(f,g_1,...,g_n)\stackrel{\rm def}{=}\sum_{l=1}^{n-1}\sum_{m_1<...<m_l\,(m_j\neq k)}T(b^{(l+1)\nu}_{W_k,W_{m_1},...W_{m_l}}(g_kf,g_{m_1},...,g_{m_l})$$

$$\otimes W_1g_1...\hat{m}_1...\hat{m}_l...\otimes W_ng_n). \tag{261}$$

In [27] it is shown that there exists a normalization (which is compatible with **(N0) - (N3)**[34]) such that $-\mathcal{A}_1+i\sum_{k=1}^n\mathcal{A}_{2,k}=0$. In the following we use this normalization. Then the identity (260) and (259) imply

$$\Theta^{\mu\nu}_{{\rm can}\,g\mathcal{L}}(\partial_\mu f)=-\mathcal{L}_{g\mathcal{L}}((\partial^\nu g)f). \tag{262}$$

The energy momentum tensor is only conserved in space-time regions in which the coupling 'constant' g is constant, in agreement with the fact that translation invariance is broken by a non-constant g.

Unfortunately the trace of the canonical energy momentum tensor does not vanish, even for free fields. Following [27] and references cited therein, we assume $m=0$ (and still $\mathcal{L}=\lambda\phi^4$) and introduce the improved energy momentum tensor. In (interacting) classical field theory it is defined by

$$\Theta^{{\rm class}\,\mu\nu}_{\rm imp}\stackrel{\rm def}{=}\Theta^{{\rm class}\,\mu\nu}_{\rm can}-\frac{1}{3}[\partial^\mu(\phi^{\rm class}\partial^\nu\phi^{\rm class})-g^{\mu\nu}\partial^\rho(\phi^{\rm class}\partial_\rho\phi^{\rm class})], \tag{263}$$

where $\Theta^{{\rm class}\,\mu\nu}_{\rm can}$ is given by the same formulas (257)-(259) as in QFT. This improved tensor is conserved and traceless. The latter relies on the field equation.

Now we are going to construct the corresponding tensor in QFT. We apply the definition (249)-(250) to $T((\phi\partial^\nu\phi)\partial^\mu f\otimes...)$ and $(\tilde{\mathbf{N}})$ (252) to $T((\phi\tilde{\partial}^\mu\partial^\nu\phi)f\otimes...)$. So we obtain

$$-T((\phi\partial^\nu\phi)\partial^\mu f\otimes W_1g_1\otimes...\otimes W_ng_n)=T((\partial^\mu\phi\partial^\nu\phi)f\otimes W_1g_1\otimes...\otimes W_ng_n)$$

$$+T((\phi\tilde{\partial}^\mu\partial^\nu\phi)f\otimes W_1g_1\otimes...\otimes W_ng_n)+\mathcal{A}^{(n+1)\,\mu\nu}_{W_1,...,W_n}(f,g_1,...,g_n), \tag{264}$$

where

$$\mathcal{A}^{(n+1)\,\mu\nu}_{W_1,...,W_n}(f,g_1,...,g_n)\stackrel{\rm def}{=}\sum_{k=1}^n\sum_{m_1<...<m_k}T(b^{(k+1)\,\mu}_{\phi\partial^\nu\phi,W_{m_1},...,W_{m_k}}(f,g_{m_1},...,g_{m_k})$$

$$\otimes W_1g_1\otimes...\hat{m}_1...\hat{m}_k...\otimes W_ng_n). \tag{265}$$

Here we have normalized $T(\partial^\nu\phi,W_{m_1},...,W_{m_k})$ according to (92) **(N4)**, which implies

$$b^{(k+1)}_{\partial^\nu\phi,W_{m_1},...,W_{m_k}}=0 \tag{266}$$

[34] So far no external derivatives are present. Hence, $(\tilde{\mathbf{N}})$ plays no role.

(there are no anomalies for tree-like diagrams, cf. sect 2.4). Hence, there are no $T(b^{(k+1)}...)$-terms in the application of ($\tilde{\mathbf{N}}$) to $T((\phi\tilde{\partial}^\mu\partial^\nu\phi)f\otimes W_1g_1\otimes ...)$. In particular it follows

$$T((\phi\tilde{\partial}^\mu\partial^\nu\phi)f\otimes W_1g_1\otimes ...\otimes W_ng_n)=T((\phi\tilde{\partial}^\nu\partial^\mu\phi)f\otimes W_1g_1\otimes ...\otimes W_ng_n). \tag{267}$$

from ($\tilde{\mathbf{N}}$). For the interacting fields the identity (264) implies

$$-(\phi\partial^\nu\phi)_{g\mathcal{L}}(\partial^\mu f)=(\partial^\mu\phi\partial^\nu\phi)_{g\mathcal{L}}(f)+(\phi\tilde{\partial}^\mu\partial^\nu\phi)_{g\mathcal{L}}(f)+\mathcal{A}_g^{\mu\nu}(f), \tag{268}$$

where

$$\mathcal{A}_g^{\mu\nu}(f)\stackrel{\mathrm{def}}{=}\sum_{n=1}^\infty i^n\sum_{r=1}^n\frac{(-1)^{n-r}}{r!(n-r)!}\bar{T}((g\mathcal{L})^{\otimes(n-r)})\mathcal{A}^{(r+1)\,\mu\nu}_{\mathcal{L},...,\mathcal{L}}(f,g,...,g). \tag{269}$$

Without further knowledge about $b^{(k+1)\,\mu}_{\phi\partial^\nu\phi,\mathcal{L},...,\mathcal{L}}$ we cannot interpret $\mathcal{A}_g^{\mu\nu}(f)$ as an interacting field and $\mathcal{A}_g^{\mu\nu}(f)$ needs not to be symmetrical in $\mu\leftrightarrow\nu$.[35] By means of (267) we find

$$(\phi\partial^\nu\phi)_{g\mathcal{L}}(\partial^\mu f)=(\phi\partial^\mu\phi)_{g\mathcal{L}}(\partial^\nu f)-\mathcal{A}_g^{\mu\nu}(f)+\mathcal{A}_g^{\nu\mu}(f). \tag{271}$$

Now we define the improvement tensor

$$I^{\mu\nu}_{g\mathcal{L}}(f)\stackrel{\mathrm{def}}{=}-(\phi\partial^\nu\phi)_{g\mathcal{L}}(\partial^\mu f)+g^{\mu\nu}(\phi\partial_\rho\phi)_{g\mathcal{L}}(\partial^\rho f). \tag{272}$$

By using (271) we find that it is conserved up to anomalous terms (i.e. terms which violate the MWI)

$$I^{\mu\nu}_{g\mathcal{L}}(\partial_\mu f)=\mathcal{A}_g^{\mu\nu}(\partial_\mu f)-\mathcal{A}_g^{\nu\mu}(\partial_\mu f). \tag{273}$$

[35] Due to **(N0)** the anomaly $a^{(k+1)\,\mu}_{\phi\partial^\nu\phi,\mathcal{L},...,\mathcal{L}}=T_1(b^{(k+1)\,\mu}_{\phi\partial^\nu\phi,\mathcal{L},...,\mathcal{L}})$ has the form

$$\begin{aligned} a^{(k+1)\,\mu}_{\phi\partial^\nu\phi,\mathcal{L},...,\mathcal{L}}(f,g,...,g)=&\int dx\,f(x)\int dy_1...dy_k\,g(y_1)...g(y_k)\\ &\{P_4^{\mu\nu}(\partial_1,...,\partial_k)\delta(y_1-x,...,y_k-x)\\ &\sum_{j\le l}:\phi(y_j)\phi(y_l):P^{\mu\nu}_{2,jl}(\partial_1,...,\partial_k)\delta(y_1-x,...,y_k-x)\\ &+\sum_{j\le l\le r\le s}:\phi(y_j)\phi(y_l)\phi(y_r)\phi(y_s):P^{\mu\nu}_{0,jlrs}\delta(y_1-x,...,y_k-x)\}, \end{aligned} \tag{270}$$

where $P^{\mu\nu}_{m,...}(\partial_1,...,\partial_k)$ is a polynomial of degree m in the partial derivatives $\partial_{y_1},...,\partial_{y_k}$, and the expression in the $\{...\}$-bracket is symmetrical under permutations of $y_1,...,y_k$. $P^{\mu\nu}_{0,jlrs}$ is $\sim g^{\mu\nu}$ and, hence, symmetrical in $\mu\leftrightarrow\nu$. But, e.g. for $k=2$ the terms $(\square_1\partial_1^\mu\partial_2^\nu+\square_2\partial_2^\mu\partial_1^\nu)\delta(y_1-x,y_2-x)$ and $(:\ \phi^2(y_1)\ :\ \partial_1^\mu\partial_2^\nu+\ :\ \phi^2(y_2)\ :\ \partial_2^\mu\partial_1^\nu)\delta(y_1-x,y_2-x)$ have not this ($\mu\leftrightarrow\nu$)-symmetry and their contributions to $a^{(k+1)\,\mu}_{\phi\partial^\nu\phi,\mathcal{L},...,\mathcal{L}}(\partial_\mu f,g,...,g)-a^{(k+1)\,\nu}_{\phi\partial^\mu\phi,\mathcal{L},...,\mathcal{L}}(\partial_\mu f,g,...,g)$ and hence to $I^{\mu\nu}_{g\mathcal{L}}(\partial_\mu f)=\mathcal{A}_g^{\mu\nu}(\partial_\mu f)-\mathcal{A}_g^{\nu\mu}(\partial_\mu f)$ (273) do not vanish.

To compute the trace, we first mention

$$T((\phi\tilde{\partial}^\mu\partial_\mu\phi)f\otimes W_1g_1\otimes ...\otimes W_ng_n)=$$

$$i\sum_{m=1}^n T(W_1g_1\otimes ...\otimes \phi\frac{\partial W_m}{\partial\phi}fg_m\otimes ...\otimes W_ng_n) \tag{274}$$

which is a consequence of ($\tilde{\mathbf{N}}$) and (266). Therefore,

$$(\phi\tilde{\partial}^\mu\partial_\mu\phi)_{g\mathcal{L}}(f)=-(\phi\frac{\partial\mathcal{L}}{\partial\phi})_{g\mathcal{L}}(fg)=-4\mathcal{L}_{g\mathcal{L}}(fg). \tag{275}$$

From (272), (268) and (275) we obtain

$$\frac{1}{3}I^\mu_{\mu\,g\mathcal{L}}(f)=-(\partial^\mu\phi\partial_\mu\phi)_{g\mathcal{L}}(f)+4\mathcal{L}_{g\mathcal{L}}(fg)-\mathcal{A}^\mu_{g\,\mu}(f)=\Theta^\mu_{\mathrm{can}\,\mu\,g\mathcal{L}}(f)-\mathcal{A}^\mu_{g\,\mu}(f). \tag{276}$$

The improved energy momentum tensor is defined analogously to (263), namely

$$\Theta^{\mu\nu}_{\mathrm{imp}\,g\mathcal{L}}(f)\stackrel{\mathrm{def}}{=}\Theta^{\mu\nu}_{\mathrm{can}\,g\mathcal{L}}(f)-\frac{1}{3}I^{\mu\nu}_{g\mathcal{L}}(f). \tag{277}$$

Our results (262), (273) and (276) yield that it is conserved and traceless up to anomalous terms:

$$\begin{aligned}\Theta^{\mu\nu}_{\mathrm{imp}\,g\mathcal{L}}(\partial_\mu f) &= \Theta^{\mu\nu}_{\mathrm{can}\,g\mathcal{L}}(\partial_\mu f)-\frac{1}{3}(\mathcal{A}^{\mu\nu}_g(\partial_\mu f)-\mathcal{A}^{\nu\mu}_g(\partial_\mu f))\\ &= -\mathcal{L}_{g\,\mathcal{L}}((\partial^\nu g)f)-\frac{1}{3}(\mathcal{A}^{\mu\nu}_g(\partial_\mu f)-\mathcal{A}^{\nu\mu}_g(\partial_\mu f)),\\ \Theta^\mu_{\mathrm{imp}\,\mu\,g\mathcal{L}}(f) &= \mathcal{A}^\mu_{g\,\mu}(f).\end{aligned} \tag{278}$$

In the literature ([27] and references cited therein) it is shown that the anomalous terms can be removed by suitable normalization in one of the two equations in (278), but not simultaneously in both. Usually one puts the priority on the conservation and allows for a trace anomaly. The latter breaks the dilatation invariance and gives rise for anomalous dimensions of the interacting fields.

Remarks: (1) In the case $m>0$ the scaling degree of certain terms is relatively lowered, e.g. in the term $\sim m^2$ of $\Theta^{\mu\nu}_{0\,\mathrm{can}}$ (257). Hence, it is possible that the central extensions violate the MWI, our result of sect. 3.3 does not contradict the appearance of the trace anomaly.

(2) We are going to show that the trace anomaly is of order $\mathcal{O}(g^2)$ for the interaction (258). We have to verify that (260), (264), (267) and (274) can be fulfilled without any anomalous terms $\mathcal{A}^{...}_{...}$ to first order in g. Due to **(N3)** we have

$$T_2(\partial^a\phi\tilde{\partial}^b\partial^c\phi,\phi^4)(x,y)=:\partial^a\phi\partial^{b+c}\phi(x)\phi^4(y): \tag{279}$$

$$+4\langle\Omega,T_2(\tilde{\partial}^b\partial^c\phi,\phi)(x,y)\Omega\rangle:\partial^a\phi(x)\phi^3(y): \tag{280}$$

$$+4\langle\Omega,T_2(\partial^a\phi,\phi)(x,y)\Omega\rangle:\partial^{b+c}\phi(x)\phi^3(y): \tag{281}$$

$$+6\langle\Omega,T_2(\partial^a\phi\tilde{\partial}^b\partial^c\phi,\phi^2)(x,y)\Omega\rangle:\phi^2(y):. \tag{282}$$

For the tree diagrams (279), (280) and (281) the MWI holds true. An anomaly must come from the loop diagram (282), which is the two-legs sector. We define the normalization of $\langle\Omega, T_2(\phi\tilde{\partial}^\mu\partial^\nu\phi, \phi^2)\Omega\rangle$ by (264) with $\mathcal{A}^{(2)\,\mu\nu}_{\phi^2} \equiv 0$. The $(\mu \leftrightarrow \nu)$-symmetry (267) holds, because all tensors of rank two are $\sim g^{\mu\nu}$ or $\sim p^\mu p^\nu$, where p is the momentum belonging to the relative coordinate $(x-y)$. The T-products on the r.h. sides of (260) and (274) have four legs for $n = 1$ and $W_1 = \mathcal{L} = \lambda\phi^4$. Hence, it remains to show that there exits a normalization such that

$$\partial^x_\mu\langle\Omega, T_2(\partial^\mu\phi\partial^\nu\phi, \phi^2)(x,y)\Omega\rangle = \frac{1}{2}\partial^\nu_x\langle\Omega, T_2(\partial^\rho\phi\partial_\rho\phi, \phi^2)(x,y)\Omega\rangle \tag{283}$$

(which is (260)) and

$$\partial^x_\mu\langle\Omega, T_2(\phi\partial^\mu\phi, \phi^2)(x,y)\Omega\rangle = \langle\Omega, T_2(\partial_\mu\phi\partial^\mu\phi, \phi^2)(x,y)\Omega\rangle \tag{284}$$

(which is (274)). An explicit calculation shows that this can in fact be done[36]. □

6 Conclusions

The justifications to require the master Ward identity (as a normalization condition for the time-ordered products) are the following facts:

- In the classical limit $\hbar \to 0$ the MWI becomes an identity which holds always true [9].
- The MWI has many, far-reaching and important consequences (see sect. 4) which we would like to hold true in QFT.[37]
- It seems that the MWI can nearly always be satisfied: it is compatible with the other normalization conditions (sect. 3), and many consequences of the MWI (e.g. the field equation, charge- and ghost-number conservation, conservation of the energy momentum tensor and perturbative gauge invariance ((148) with $j_1 = ... = j_n = 0$) for $SU(N)$-Yang-Mills theories) have already been proved in the literature (sect. 4). The only counter-examples we know are the usual anomalies of perturbative QFT.

7 Appendix A: Feynman propagators

Let $\varphi, \chi \in \mathcal{G}$ be the symbols corresponding to two massive or massless free fields (without derivatives) with the same mass and which satisfy the Klein-Gordon

[36]The C-number distributions in (282) for $b = 0$ and the relevant values of a and c have essentially been calculated in the second paper of [11] (sect. 2 and appendix C).

[37]We discovered (or invented) the MWI by searching for a local construction of observables in non-Abelian quantum gauge theories. (In [7] this construction is given for QED). We succeeded provided several normalization conditions are fulfilled, see [3]. In order to prove that the latter have a common solution we looked for a universal formulation of these normalization conditions - and found the MWI.

or wave equation

$$(\square + m^2)\varphi = 0, \qquad (\square + m^2)\chi = 0, \qquad m \geq 0, \tag{285}$$

and obey Bose or Fermi statistics. We assume that $T_1(\varphi g)$, $g \in \mathcal{D}(\mathbb{R}^4)$ (anti-)commutes with all free fields except $T_1(\chi h)$, $h \in \mathcal{D}(\mathbb{R}^4)$ and the same for φ and χ exchanged. The non-vanishing (anti-)commutator is given by

$$\Delta_{\varphi,\chi} = \epsilon D_m, \tag{286}$$

where D_m is the (massive or massless) Pauli-Jordan distribution to the mass m, ϵ is a sign which depends on (φ, χ) and we have extended the notation (22) to anti-commutators. For a bosonic real scalar field it is $\chi = \varphi$ and for a bosonic complex scalar field we have $\chi = \varphi^+$. In case of the fermionic ghost fields of non-Abelian gauge theories φ and χ must be different: $\varphi = \tilde{u}_a$, $\chi = u_a$, $\epsilon = 1$ where a is the colour index. Alternatively one may also set $\varphi = u_a$, $\chi = \tilde{u}_a$, $\epsilon = -1$. Spinor fields will be treated later.

According to our definition (33) of the Feynman propagators and the normalization condition **(N0)**, $\Delta^F_{\partial^a\varphi,\partial^b\chi}$ contains undetermined local terms if and only if

$$\omega \stackrel{\mathrm{def}}{=} \mathrm{sd}(\Delta^F_{\partial^a\varphi,\partial^b\chi}) - 4 \equiv -2 + |a| + |b| \geq 0, \tag{287}$$

namely

$$\Delta^F_{\partial^a\varphi,\partial^b\chi} = \epsilon(-1)^{|b|}[\partial^a\partial^b D^F_m + \sum_{|c|=0}^{\omega} C_c^{(a,b)}\partial^c\delta], \tag{288}$$

where D^F_m is the massive or massless Feynman propagator and the $C_c^{(a,b)} \in \mathbb{C}$ are constants. We give an explicit list of the undetermined terms for the lowest values of $|a| + |b|$:

$$\Delta^F_{\partial^\mu\varphi,\partial^\nu\chi} = -\epsilon[(\partial^\mu\partial^\nu D^F_m + Cg^{\mu\nu}\delta)], \tag{289}$$

$$\Delta^F_{\partial^\mu\partial^\nu\varphi,\chi} = \Delta^F_{\varphi,\partial^\mu\partial^\nu\chi} = \epsilon[\partial^\mu\partial^\nu D^F_m - \frac{1}{4}g^{\mu\nu}\delta] \tag{290}$$

$$\begin{aligned} -\Delta^F_{\partial^\mu\partial^\nu\varphi,\partial^\lambda\chi} &= \Delta^F_{\partial^\lambda\varphi,\partial^\mu\partial^\nu\chi} = \epsilon[\partial^\mu\partial^\nu\partial^\lambda D^F_m \\ &+ C_1 g^{\mu\nu}\partial^\lambda\delta - (\frac{1}{2} + 2C_1)(g^{\mu\lambda}\partial^\nu\delta + g^{\nu\lambda}\partial^\mu\delta)] \end{aligned} \tag{291}$$

$$\begin{aligned} \Delta^F_{\partial^\mu\partial^\nu\partial^\lambda\varphi,\chi} &= -\Delta^F_{\varphi,\partial^\mu\partial^\nu\partial^\lambda\chi} = \epsilon[\partial^\mu\partial^\nu\partial^\lambda D^F_m \\ &- \frac{1}{6}(g^{\mu\nu}\partial^\lambda\delta + g^{\mu\lambda}\partial^\nu\delta + g^{\nu\lambda}\partial^\mu\delta)], \end{aligned} \tag{292}$$

where we have taken account of Poincare covariance, symmetry with respect to exchange of Lorentz indices and

$$\Delta^F_{\partial^a\square\varphi,\partial^b\chi} = -m^2\Delta^F_{\partial^a\varphi,\partial^b\chi} = \Delta^F_{\partial^a\varphi,\partial^b\square\chi}. \tag{293}$$

With these formulas and $(\square + m^2)D_m^F = \delta$ we compute $\delta^\mu_{\chi,\psi} \stackrel{\mathrm{def}}{=} \partial^\mu \Delta^F_{\chi,\psi} - \Delta^F_{\partial^\mu\chi,\psi}$ (34):

$$\delta^\mu_{\varphi,\chi} = 0, \tag{294}$$

$$\delta^\mu_{\varphi,\partial^\nu\chi} = \epsilon C g^{\mu\nu}\delta, \tag{295}$$

$$\delta^\mu_{\partial^\nu\varphi,\chi} = \epsilon\frac{1}{4}g^{\mu\nu}\delta, \qquad \delta^\mu_{\partial_\mu\varphi,\chi} = \epsilon\delta, \tag{296}$$

$$\delta^\mu_{\partial^\tau\varphi,\partial^\nu\chi} = \epsilon[-(C+\frac{1}{2}+2C_1)g^{\tau\nu}\partial^\mu\delta + C_1 g^{\mu\tau}\partial^\nu\delta - (\frac{1}{2}+2C_1)g^{\mu\nu}\partial^\tau\delta], \tag{297}$$

$$\delta^\mu_{\partial_\mu\varphi,\partial^\nu\chi} = -\epsilon(1+C)\partial^\nu\delta, \tag{298}$$

$$\delta^\mu_{\partial_\mu\partial_\tau\varphi,\chi} = \epsilon\frac{3}{4}\partial_\tau\delta, \tag{299}$$

$$\delta^\mu_{\partial_\mu\partial^\tau\varphi,\partial^\nu\chi} = \epsilon[(-\frac{1}{2}+C_1)\partial^\nu\partial^\tau\delta + (\frac{1}{2}+2C_1)g^{\tau\nu}\square\delta - Cm^2 g^{\tau\nu}\delta]. \tag{300}$$

For *spinor fields* with mass $m \geq 0$ obeying the Dirac equation we have

$$\Delta_{\psi,\overline{\psi}} = -(i\gamma^\mu\partial_\mu + m)D_m \tag{301}$$

and find

$$\delta^\mu_{\gamma_\mu\psi,\overline{\psi}} = -i\delta, \tag{302}$$

$$\delta^\mu_{\overline{\psi}\gamma_\mu,\psi} = -i\delta. \tag{303}$$

8 Appendix B: Explicit results for Δ^μ used in the application of the MWI to the BRST-current

Let j_μ be the free BRST-current (119). We assume that each symbol in $W \in \mathcal{P}_0$ carries at most a first (internal) derivative (no higher derivatives). Then the following

$$\Delta^\mu_{\chi,\psi}\big(\frac{\partial j_\mu}{\partial\chi}g, \frac{\partial W}{\partial\psi}f\big), \qquad \chi,\psi \in \mathcal{G}, \tag{304}$$

do not vanish:

$$\chi = \partial_\tau A_a^\tau : \qquad \Delta^\mu_{\partial_\tau A_a^\tau, A_b^\nu}\big((\partial^\mu u_a)g, \frac{\partial W}{\partial A_b^\nu} f\big) = \frac{1}{4}(\partial^\mu u_a)\frac{\partial W}{\partial A_a^\mu} gf, \tag{305}$$

$$\begin{aligned}
&\Delta^\mu_{\partial_\tau A_a^\tau, \partial^\sigma A_b^\nu}\big((\partial^\mu u_a)g, \frac{\partial W}{\partial(\partial^\sigma A_b^\nu)} f\big) = \\
&(C_{A_a} + \frac{1}{2} + 2C_{1A_a})[(\tilde{\partial}^\mu \partial_\mu u_a)\frac{\partial W}{\partial(\partial_\nu A_a^\nu)} gf \\
&+(\partial_\mu u_a)\frac{\partial W}{\partial(\partial_\nu A_a^\nu)}(\partial^\mu g)f] \\
&-C_{1A_a}[(\tilde{\partial}^\sigma \partial^\nu u_a)\frac{\partial W}{\partial(\partial^\sigma A_a^\nu)} gf + (\partial^\mu u_a)\frac{\partial W}{\partial(\partial^\nu A_a^\mu)}(\partial^\nu g)f] \\
&+(\frac{1}{2} + 2C_{1A_a})[(\tilde{\partial}^\nu \partial^\sigma u_a)\frac{\partial W}{\partial(\partial^\sigma A_a^\nu)} gf \\
&+(\partial^\mu u_a)\frac{\partial W}{\partial(\partial^\mu A_a^\nu)}(\partial^\nu g)f],
\end{aligned} \tag{306}$$

$$\chi = \partial_\mu u_a : \qquad \begin{aligned}
&\Delta^\mu_{\partial_\mu u_a, \tilde{u}_b}\big((\partial_\tau A_a^\tau + m_a\phi_a)g, \frac{\partial W}{\partial \tilde{u}_b} f\big) = \\
&-(\partial_\tau A_a^\tau + m_a\phi_a)\frac{\partial W}{\partial \tilde{u}_a} gf,
\end{aligned} \tag{307}$$

$$\begin{aligned}
&\Delta^\mu_{\partial_\mu u_a, \partial_\nu \tilde{u}_b}\big((\partial_\tau A_a^\tau + m_a\phi_a)g, \frac{\partial W}{\partial(\partial_\nu \tilde{u}_b)} f\big) = \\
&-(1 + C_{u_a})[(\tilde{\partial}_\nu(\partial_\tau A_a^\tau + m_a\phi_a))\frac{\partial W}{\partial(\partial_\nu \tilde{u}_a)} gf \\
&+(\partial_\tau A_a^\tau + m_a\phi_a)\frac{\partial W}{\partial(\partial_\nu \tilde{u}_a)}(\partial_\nu g)f,
\end{aligned} \tag{308}$$

$$\chi = \partial_\mu \partial_\tau A_a^\tau : \qquad \begin{aligned}
&\Delta^\mu_{\partial_\mu \partial_\tau A_a^\tau, A_b^\nu}\big(-u_a g, \frac{\partial W}{\partial A_b^\nu} f\big) = \\
&\frac{3}{4}[(\tilde{\partial}^\nu u_a)\frac{\partial W}{\partial A_a^\nu} gf + u_a \frac{\partial W}{\partial A_a^\nu}(\partial^\nu g)f],
\end{aligned} \tag{309}$$

$$\begin{aligned}
&\Delta^\mu_{\partial_\mu \partial_\tau A_a^\tau, \partial^\sigma A_b^\nu}\big(-u_a g, \frac{\partial W}{\partial(\partial^\sigma A_b^\nu)} f\big) = (\frac{1}{2} - C_{1A_a}) \cdot \\
&[(\tilde{\partial}^\nu \tilde{\partial}^\sigma u_a)\frac{\partial W}{\partial(\partial^\sigma A_a^\nu)} gf + (\tilde{\partial}^\sigma u_a)\Big(\frac{\partial W}{\partial(\partial^\nu A_a^\sigma)} + \frac{\partial W}{\partial(\partial^\sigma A_a^\nu)}\Big)(\partial^\nu g)f \\
&+u_a \frac{\partial W}{\partial(\partial^\sigma A_a^\nu)}(\partial^\nu \partial^\sigma g)f] - (\frac{1}{2} + 2C_{1A_a})[(\tilde{\Box} u_a)\frac{\partial W}{\partial(\partial_\nu A_a^\nu)} gf \\
&+2(\tilde{\partial}_\nu u_a)\frac{\partial W}{\partial(\partial_\tau A_a^\tau)}(\partial^\nu g)f + u_a \frac{\partial W}{\partial(\partial_\tau A_a^\tau)}(\Box g)f] \\
&+C_{A_a} m_a^2 u_a \frac{\partial W}{\partial(\partial_\tau A_a^\tau)} gf,
\end{aligned} \tag{310}$$

$$\chi = u_a : \qquad \Delta^\mu_{u_a,\tilde{u}_b}\big(-(\partial_\mu(\partial_\tau A^\tau_a + m_a\phi_a))g, \frac{\partial W}{\partial \tilde{u}_b} f\big) = 0, \tag{311}$$

$$\Delta^\mu_{u_a,\partial_\nu \tilde{u}_b}\big(-(\partial_\mu(\partial_\tau A^\tau_a + m_a\phi_a))g, \frac{\partial W}{\partial(\partial_\nu \tilde{u}_b)} f\big) = C_{u_a}(\partial_\nu(\partial_\tau A^\tau_a + m_a\phi_a))\frac{\partial W}{\partial(\partial_\nu \tilde{u}_a)} gf, \tag{312}$$

$$\chi = \phi_a : \qquad \Delta^\mu_{\phi_a,\phi_b}\big((\partial_\mu u_a)g, \frac{\partial W}{\partial \phi_b} f\big) = 0, \tag{313}$$

$$\Delta^\mu_{\phi_a,\partial_\nu\phi_b}\big((\partial_\mu u_a)g, \frac{\partial W}{\partial(\partial_\nu \phi_b)} f\big) = -C_{\phi_a}(\partial_\mu u_a)\frac{\partial W}{\partial(\partial_\mu \phi_a)} gf, \tag{314}$$

$$\chi = \partial_\mu\phi_a : \qquad \Delta^\mu_{\partial_\mu\phi_a,\phi_b}\big(-u_a g, \frac{\partial W}{\partial \phi_b} f\big) = u_a\frac{\partial W}{\partial \phi_a} gf, \tag{315}$$

$$\Delta^\mu_{\partial_\mu\phi_a,\partial_\nu\phi_b}\big(-u_a g, \frac{\partial W}{\partial(\partial_\nu \phi_b)} f\big) = (1 + C_{\phi_a})[(\tilde{\partial}_\nu u_a)\frac{\partial W}{\partial(\partial_\nu \phi_a)} gf + u_a\frac{\partial W}{\partial(\partial_\nu \phi_a)}(\partial_\nu g)f], \tag{316}$$

where we have used the explicit expressions (294)-(300) for the δ^μ and the definition (36) of Δ^μ.

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References

[1] Bardeen, W.A., “Anomalous Ward Identities in Spinor Field Theories”, Phys. Rev. **184** (1969) 1848

[2] Becchi, C., Rouet, A., and Stora, R., ”Renormalization of the Abelian Higgs-Kibble model”, *Commun. Math. Phys.* **42**, 127 (1975)

Becchi, C., Rouet, A., and Stora, R., ”Renormalization of gauge theories”, *Annals of Physics (N.Y.)* **98**, 287 (1976)

[3] Boas, F.M.,“Gauge Theories in Local Causal Perturbation Theory”, hep-th/0001014

[4] Bogoliubov, N.N., and Shirkov, D.V., *"Introduction to the Theory of Quantized Fields"*, New York (1959)

[5] Brunetti, R., and Fredenhagen, K., "Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds", *Commun. Math. Phys.* **208**, 623 (2000)

[6] Bresser, K., Pinter, G., Prange, D.,"The Lorentz invariant extension of scalar theories", hep-th/9903266

Prange, D.,"Lorentz Covariance in Epstein-Glaser Renormalization", hep-th/9904136

[7] Dütsch, M., and Fredenhagen, K., "A local (perturbative) construction of observables in gauge theories: the example of QED", *Commun. Math. Phys.* **203**, 71 (1999)

Dütsch, M., and Fredenhagen, K., "Deformation stability of BRST-quantization", hep-th/9807215, Proceedings of the conference 'Particles, Fields and Gravitation', Lodz, Poland (1998)

[8] Dütsch, M., and Fredenhagen, K.,"Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion", *Commun. Math. Phys.* **219**, 5 (2001)

Dütsch, M., and Fredenhagen, K., "Perturbative Algebraic Field Theory, and Deformation Quantization", hep-th/0101079, Proceedings of the Conference on Mathematical Physics in Mathematics and Physics, Siena, June 20-25 2000

[9] Dütsch, M., and Fredenhagen, K.,"The Master Ward Identity and Generalized Schwinger-Dyson Equations in Classical Field Theory", work in progress

[10] Dütsch, M., and Fredenhagen, K.,"Causal perturbation theory in terms of retarded products and perturbative algebraic field theory", work in progress

[11] Dütsch, M., Hurth, T., Krahe, K., and Scharf, G.: "Causal construction of Yang-Mills theories. I." *N. Cimento A* **106** 1029 (1993)

Dütsch, M., Hurth, T., Krahe, K. and Scharf, G., "Causal construction of Yang-Mills theories. II." *N. Cimento A* **107** 375 (1994)

Dütsch, M., Hurth, T., and Scharf, G., "Causal construction of Yang-Mills theories. III.", *N. Cimento A* **108** 679 (1995)

Dütsch, M., Hurth, T., and Scharf, G., "Causal construction of Yang-Mills theories. IV. Unitarity", *N. Cimento A* **108** 737 (1995)

Dütsch, M., "On gauge invariance of Yang-Mills theories with matter fields", *N. Cimento A* **109** 1145 (1996)

[12] Dütsch, M., Krahe, K., and Scharf, G.: "Interacting Fields in Finite QED", *N. Cimento A* **103** 871 (1990)

[13] Dütsch, M., Krahe, K., and Scharf, G.: "Gauge Invariance in Finite QED", *N. Cimento A* **103** 903 (1990)

[14] Dütsch, M., and Scharf, G.: "Perturbative Gauge Invariance: the Electroweak Theory", *Ann. Phys. (Leipzig)*, **8** (1999) 359

Aste, A., Dütsch, M., and Scharf, G.: "Perturbative Gauge Invariance: the Electroweak Theory II", *Ann. Phys. (Leipzig)*, **8** (1999) 389

[15] Dütsch, M., and Schroer, B.:"Massive Vector Mesons and Gauge Theory", *J. Phys. A* **33** (2000) 4317

[16] Dütsch, M., "Slavnov-Taylor identities from the causal point of view", *Int. J. Mod. Phys. A* **12** 3205 (1997)

[17] Dütsch, M., "Non-uniqueness of quantized Yang-Mills theories", *J. Phys. A* **29** 7597 (1996)

[18] Epstein, H., and Glaser, V., "The role of locality in perturbation theory", *Ann. Inst. H. Poincaré A* **19**, 211 (1973)

[19] Epstein, H., and Glaser, V.,"Adiabatic limit in perturbation theory" in G.Velo and A.S.Wightman (eds.): Renormalization theory (1976) 193-254

[20] Grigore, D. R., "On the uniqueness of the non-abelian gauge theories in Epstein-Glaser approach to renormalisation theory", hep-th/9806244, *Romanian J. Phys.* **44** 853 (1999)

Grigore, D. R., "The standard model and its generalisations in Epstein-Glaser approach to renormalisation theory', *J.Phys. A* **33** 8443 (2000)

Grigore, D. R., "The standard model and its generalisations in Epstein-Glaser approach to renormalisation theory II: the fermion sector and the axial anomaly", *J.Phys. A* **34** 5429 (2001)

Grigore, D. R., "The Renormalization of the Non-Abelian Gauge Theories in the Causal Approach", hep-th/0010226

[21] Itzykson, C., Zuber, J.-B.,"Quantum Field Theory", McGraw-Hill (1985)

[22] Kugo, T., and Ojima, I., "Local covariant operator formalism of non-abelian gauge theories and quark confinement problem", *Suppl. Progr. Theor. Phys.* **66**, 1 (1979)

[23] Krahe, F., "A causal approach to massive Yang-Mills theories", *Acta Phys. Polonica B* **27**, 2453 (1996)

[24] Lam, Y.-M.P., “Perturbation Lagrangian Theory for Scalar Fields - Ward-Takahashi Identity and Current Algebra”, *Phys. Rev.* **D6** 2145 (1972);

Lam, Y.-M.P.,“Equivalence Theorem on Bogoliubov-Parasiuk-Hepp-Zimmermann - Renormalized Lagrangian Field Theories”, *Phys. Rev.* **D7** 2943 (1973)

[25] Lowenstein, J.H., “Differential vertex operations in Lagrangian field theory”, *Commun. Math. Phys.* **24**, 1 (1971)

[26] Lowenstein, J.H., “Normal-Product Quantization of Currents in Lagrangian Field Theory”, *Phys. Rev. D* **4**, 2281 (1971)

[27] Prange, D.,”Energy Momentum Tensor and Operator Product Expansion in Local Causal Perturbation Theory”, hep-th/0009124

Prange, D.,”Energy Momentum Tensor in Local Causal Perturbation Theory”, *Ann. Phys. (Leipzig)* **10**, 497 (2001)

[28] Prange, D.,”Epstein-Glaser renormalization and differential renormalization”, *J. Phys. A* **32**, 2225 (1999)

[29] Piguet, O., and Sorella, S.P., *”Algebraic Renormalization”*, Springer-Verlag (1995)

[30] Pinter, G.,“Finite Renormalizations in the Epstein-Glaser Framework and Renormalization of the S-Matrix of ϕ^4-Theory”, *Ann. Phys. (Leipzig)* **10**, 333 (2001)

[31] Requardt, M.,”Symmetry conservation and integrals over local charge desities in quantum field theory”, *Commun. Math. Phys.* **50** 259 (1976)

[32] Scharf, G., *”Finite Quantum Electrodynamics. The causal approach”*, 2nd. ed., Springer-Verlag (1995)

[33] Scharf, G., *“Quantum Gauge Theories, a true ghost story”*, John Wiley and Sons, Inc. (2001)

[34] Scharf, G. “General massive gauge theory”, *N. Cimento A* **112** (1999) 619

[35] Steinmann, O., “Perturbation expansions in axiomatic field theory”, Lecture Notes in Physics **11**, Berlin-Heidelberg-New York: Springer-Verlag (1971)

[36] Stora, R., ”Local gauge groups in quantum field theory: perturbative gauge theories”, talk given at the workshop ’Local Quantum Physics’ at the Erwin-Schroedinger-institute, Vienna (1997)

[37] Stora, R., ”Differential algebras in Lagrangean field theory”, ETH-Zürich Lectures, January-February 1993;

Popineau, G., and Stora, R., ”A pedagogical remark on the main theorem of perturbative renormalization theory”, unpublished preprint (1982)

[38] Stora, R.:"Lagrangian field theory", summer school of theoretical physics about 'particle physics', Les Houches (1971)

[39] Zimmermann, W., in "Lectures on Elementary Particles and Quantum Field Theory", Brandeis Summer Institute in Theoretical Physics (1970), edited by S. Deser