ADELIC QUANTUM MECHANICS:
NONARCHIMEDEAN AND NONCOMMUTATIVE ASPECTS

GORAN DJORDJEVIĆ¹,², BRANKO DRAGOVICH³,⁴, LJUBIŠA NEŠIĆ¹ *
¹ Department of Physics, Faculty of Sciences, University of Niš, P.O. Box 91, 18001 Niš, Yugoslavia
² Sektion Physik, Universität München, Theresienstr. 37, D-80333 München, Germany
³ Institute of Physics, P.O.Box 57, 11001 Belgrade, Yugoslavia
⁴ Steklov Mathematical Institute, Gubkin St. 8, 117966, Moscow, Russia

Abstract.
We present a short review of adelic quantum mechanics pointing out its non-Archimedean and noncommutative aspects. In particular, $p$-adic path integral and adelic quantum cosmology are considered. Some similarities between $p$-adic analysis and $q$-analysis are noted. The $p$-adic Moyal product is introduced.

1. Introduction

There is now a common belief that the usual picture of spacetime as a smooth pseudo-Riemannian manifold should breakdown somehow at the Planck length $l_p \sim 10^{-33} cm$, due to the quantum gravity effects. We consider here two possibilities, which come from modern mathematics and mathematical physics: non-Archimedean geometry related to $p$-adic numbers, and noncommutative geometry with space coordinates given by non-commuting operators

$$[\hat{x}^i, \hat{x}^j] = i\hbar \delta^{ij} \quad (1)$$

*(gorandj@junis.ni.ac.yu)*
or by q-deformation $x^i x^j = qx^j x^i$. Some noncommutativity of configuration space should not be a surprise in physics since quantum phase space with the canonical commutation relation (9) is the well-known example of noncommutative geometry. We will mostly review our recent results concerning adelic quantum mechanics. We illustrate some features of adelic quantum mechanics by its application in quantum cosmology. A few remarkable similarities between non-Archimedean and noncommutative structures are noted. The usual Moyal product is extended to $p$-adic and adelic quantum mechanics.

Since 1987, there have been many interesting applications of $p$-adic numbers and non-Archimedean geometry in various parts of modern theoretical and mathematical physics (for a review, see [1, 2, 3]). However we restrict ourselves here to $p$-adic and adelic quantum mechanics as well as to some related topics. In particular, we review Feynman’s $p$-adic path integral method. A fundamental role of integral approach to $p$-adic and adelic quantum mechanics (and adelic quantum cosmology) is emphasized. The obtained $p$-adic probability amplitude for one-dimensional systems with quadratic Lagrangians has the form as that one in ordinary quantum mechanics.

It is well known that measurements give rational numbers $\mathbb{Q}$, whereas theoretical models traditionally use real $\mathbb{R}$ and complex $\mathbb{C}$ number fields. A completion of $\mathbb{Q}$ with respect to the $p$-adic norms gives the fields of $p$-adic numbers $\mathbb{Q}_p$ ($p$ is a prime number) in the same way as completion with absolute value yields $\mathbb{R}$. The paper of Volovich [4] initiated a series of articles on $p$-adic string theory and many other branches of theoretical and mathematical physics. The metric introduced by $p$-adic norm is the non-Archimedean (ultrametric) one. Possible existence of such space around the Planck length is the main motivation to study $p$-adic quantum models. However, $p$-adic analysis also plays a role in some areas of "macroscopic physics" as, for example: spin glasses, quasicrystals and some other complex systems.

In order to investigate possible $p$-adic quantum phenomena it is necessary to have the corresponding theoretical formalism. An important step in this direction is a formulation of $p$-adic quantum mechanics [5, 6]. Because of total disconnectedness of $p$-adic spaces and different valuations of variables and wave functions, the quantization is performed by the Weyl procedure. A unitary representation of the evolution operator $U_p(t)$ on the Hilbert space $L^2(\mathbb{Q}_p)$ of complex-valued functions of a $p$-adic argument is an appropriate way to describe quantum dynamics of $p$-adic systems. Recently formulated adelic quantum mechanics [7] successfully unifies ordinary and all $p$-adic quantum mechanics. The appearance of space-time discreteness in adelic formalism (see, e.g. [8]) is an encouragement for the
further investigations.

This paper is organized as follows. We start with a short introduction to $p$-adic numbers, adeles and their functions. After that, $p$-adic and adelic quantum mechanics based on the Weyl quantization and Feynman’s path integral are presented. In Section 4 we review our previous results concerning one-dimensional $p$-adic propagator. In Section 5 we will see how adelic quantum mechanics can be useful in investigation of the very early universe, where in a natural way space-time discreteness emerges in minisuperspace models of adelic quantum cosmology. In the last Section we give some of interesting relations between non-Archimedean and non-commutative analyses. We also define and discuss the corresponding $p$-adic Moyal product.

2. $p$-Adic numbers and adeles

Any $x \in \mathbb{Q}_p$ can be presented in the form \( x = p^\nu(x_0 + x_1p + x_2p^2 + \cdots) \), $\nu \in \mathbb{Z}$, \( (2) \)

where $x_i = 0, 1, \cdots, p - 1$ are digits. $p$-Adic norm of any term $x_ip^{\nu+i}$ in the canonical expansion (2) is $| x_ip^{\nu+i} |_p = p^{-(\nu+i)}$ and the strong triangle inequality holds, i.e. $| a+b |_p \leq \max\{| a |_p, | b |_p \}$. It follows that $| x |_p = p^{-\nu}$ if $x_0 \neq 0$. There is no natural ordering on $\mathbb{Q}_p$. However one can introduce a linear order on $\mathbb{Q}_p$ by the following definition: $x < y$ if $| x |_p < | y |_p$ or when $| x |_p = | y |_p$ there exists such index $m \geq 0$ that digits satisfy $x_0 = y_0, x_1 = y_1, \cdots, x_{m-1} = y_{m-1}, x_m < y_m$.

Derivatives of $p$-adic valued functions $\varphi : \mathbb{Q}_p \to \mathbb{Q}_p$ are defined as in the real case, but with respect to the $p$-adic norm. There is no integral \( \int \varphi(x)dx \) in a sense of the Lebesgue measure [2], but one can introduce \( \int_a^b \varphi(x)dx = \Phi(b) - \Phi(a) \) as a functional of analytic functions $\varphi(x)$, where $\Phi(x)$ is an antiderivative of $\varphi(x)$. In the case of map $\varphi : \mathbb{Q}_p \to \mathbb{C}$ there is well-defined Haar measure. We use here the Gauss integral

\[
\int_{\mathbb{Q}_v} \chi_v(ax^2 + bx)dx = \lambda_v(a) \left| 2a \right|_v^{1/2} \chi_v(- \frac{b^2}{4a}) , \quad a \neq 0, \quad (3)
\]

where index $v$ denotes real ($v = \infty$) and $p$-adic cases, i.e. $v = \infty, 2, 3, 5, \cdots$. $\chi_v$ is an additive character: $\chi_\infty(x) = \exp(-2\pi ix), \chi_p(x) = \exp(2\pi i \{x\}_p)$, where $\{x\}_p$ is the fractional part of $x \in \mathbb{Q}_p$. $\lambda_v(a)$ is the complex-valued arithmetic function [2].

An adele [10] is an infinite sequence $a = (a_\infty, a_2, \ldots, a_p, \ldots)$, where $a_\infty \in \mathbb{R} \equiv \mathbb{Q}_\infty, a_p \in \mathbb{Q}_p$ with a restriction that $a_p \in \mathbb{Z}_p$ for all but a finite set $S$ of primes $p$. The set of all adeles $\mathbb{A}$ may be regarded as a subset of...
direct topological product $\mathbb{Q}_\infty \times \prod_p \mathbb{Q}_p$ whose elements satisfy the above restriction, \textit{i.e.}

$$\mathbb{A} = \cup_S \mathbb{A}(S), \quad \mathbb{A}(S) = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \not\in S} \mathbb{Z}_p.$$ (4)

$\mathbb{A}$ is a topological space, and can be considered as a ring with respect to the componentwise addition and multiplication. An elementary function on adelic ring $\mathbb{A}$ is

$$\varphi(x) = \varphi_\infty(x_\infty) \prod_p \varphi_p(x_p) = \prod_v \varphi_v(x_v)$$ (5)

with the main restriction that $\varphi(x)$ must satisfy $\varphi_p(x_p) = \Omega(|x_p|_p)$ for all but a finite number of $p$, where

$$\Omega(|x|_p) = \begin{cases} 1, & 0 \leq |x|_p \leq 1, \\ 0, & |x|_p > 1 \end{cases}$$ (6)

is a characteristic function on the set of $p$-adic integers $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}$. It should be noted that the Fourier transform of the characteristic function (vacuum state) $\Omega(|x_p|)$ is $\Omega(|k_p|)$.

All finite linear combinations of elementary functions (5) make the set $\mathcal{D}(\mathbb{A})$ of the Schwartz-Bruhat functions. The Fourier transform of $\varphi(x) \in \mathcal{D}(\mathbb{A})$ (that maps $\mathcal{D}(\mathbb{A})$ onto $\mathcal{D}(\mathbb{A})$) is

$$\tilde{\varphi}(y) = \int_{\mathbb{A}} \varphi(x) \chi(xy) dx = \int_{\mathbb{R}} \varphi_\infty(x) \chi_\infty(xy) dx \prod_p \int_{\mathbb{Q}_p} \varphi_p(x) \chi_p(xy) dx,$$ (7)

where $dx = dx_\infty dx_2 \ldots dx_p \ldots$ is the Haar measure on $\mathbb{A}$. The Hilbert space $L^2(\mathbb{A})$ is a space of complex-valued functions $\psi_1(x), \psi_2(x), \ldots$, with the scalar product and norm

$$(\psi_1, \psi_2) = \int_{\mathbb{A}} \tilde{\psi}_1(x) \psi_2(x) dx, \quad ||\psi|| = (\psi, \psi)^{1/2} < \infty.$$ (8)

A basis of the above space may be given by the orthonormal eigenfunctions of an evolution operator [7].

3. Adelic quantum mechanics

In foundations of standard quantum mechanics (over $\mathbb{R}$) one usually starts with a representation of the canonical commutation relation

$$[\hat{q}, \hat{k}] = i\hbar,$$ (9)
where \( q \) is a coordinate and \( k \) is the corresponding momentum. It is well known that the procedure of quantization is not unique. In formulation of \( p \)-adic quantum mechanics [5, 6] the multiplication \( \hat{q} \psi \rightarrow x \psi \) has no meaning for \( x \in \mathbb{Q}_p \) and \( \psi(x) \in \mathbb{C} \). Also, there is no possibility to define \( p \)-adic "momentum" or "Hamiltonian" operator. In the real case they are infinitesimal generators of space and time translations, but, since \( \mathbb{Q}_p \) is disconnected field, these infinitesimal transformations become meaningless. However, finite transformations remain meaningful and the corresponding Weyl and evolution operators are \( p \)-adically well defined. For the one dimensional systems which classical evolution can be described by

\[
z_t = T_t z, \quad z_t = \begin{pmatrix} q(t) \\ k(t) \end{pmatrix}, \quad z = \begin{pmatrix} q(0) \\ k(0) \end{pmatrix},
\]

(10)

where \( q(0) \) and \( k(0) \) are initial position and momentum, respectively, and \( T_t \) is a matrix. Canonical commutation relation in \( p \)-adic case can be represented by the Weyl operators \((h = 1)\)

\[
\hat{Q}_p(\alpha) \hat{\psi}_p(x) = \chi_p(\alpha x) \psi_p(x)
\]

(11)

\[
\hat{K}_p(\beta) \psi(x) = \psi_p(x + \beta).
\]

(12)

Now, to the relation (9) in the real case, corresponds

\[
\hat{Q}_p(\alpha) \hat{K}_p(\beta) = \chi_p(\alpha \beta) \hat{K}_p(\beta) \hat{Q}_p(\alpha)
\]

(13)

in the \( p \)-adic one. It is possible to introduce the family of unitary operators

\[
\hat{W}_p(z) = \chi_p(-\frac{1}{2}qk) \hat{K}_p(\beta) \hat{Q}_p(\alpha), \quad z \in \mathbb{Q}_p \times \mathbb{Q}_p,
\]

(14)

that is a unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements \((z, \alpha)\) with the group product

\[
(z, \alpha) \cdot (z', \alpha') = (z + z', \alpha + \alpha' + \frac{1}{2}B(z, z')),
\]

(15)

where \( B(z, z') = -kq' + qk' \) is a skew-symmetric bilinear form on the phase space.

Dynamics of a \( p \)-adic quantum model is described by a unitary operator of evolution \( U(t) \) without using the Hamiltonian. Instead of that, the evolution operator has been formulated in terms of its kernel \( K_t(x, y) \)

\[
U_p(t) \psi(x) = \int_{\mathbb{Q}_p} K_t(x, y) \psi(y)dy.
\]

(16)

The next section will be devoted to the path integral formulation and calculation of the quantum propagator \( K_t(x, y) \) on \( p \)-adic spaces.
In this way [5] \( p \)-adic quantum mechanics is given by a triple

\[
(L_2(\mathbb{Q}_p), W_p(z_p), U_p(t_p)).
\]

(17)

Keeping in mind that standard quantum mechanics can be also given as the corresponding triple, ordinary and \( p \)-adic quantum mechanics can be unified in the form of adelic quantum mechanics [7]

\[
(L_2(A), W(z), U(t)).
\]

(18)

\( L_2(A) \) is the Hilbert space on \( A \), \( W(z) \) is a unitary representation of the Heisenberg-Weyl group on \( L_2(A) \) and \( U(t) \) is a unitary representation of the evolution operator on \( L_2(A) \).

The evolution operator \( U(t) \) is defined by

\[
U(t)\psi(x) = \int_A K_t(x, y)\psi(y)dy = \prod_v \int_{Q_v} K_t^{(v)}(x_v, y_v)\psi^{(v)}(y_v)dy_v.
\]

(19)

The eigenvalue problem for \( U(t) \) reads

\[
U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x),
\]

(20)

where \( \psi_{\alpha\beta} \) are adelic eigenfunctions, \( E_\alpha = (E_\infty, E_2, ..., E_p, ...) \) is corresponding energy, indices \( \alpha \) and \( \beta \) denote energy levels and their degeneration. Note that any adelic eigenfunction has the form

\[
\Psi(x) = \Psi_\infty(x_\infty) \prod_{p \in S} \Psi_p(x_p) \prod_{p \not\in S} \Omega(\mid x_p \mid_p), \quad x \in A,
\]

(21)

where \( \Psi_\infty \in L_2(\mathbb{R}) \), \( \Psi_p \in L_2(\mathbb{Q}_p) \). Adelic quantum mechanics takes into account also \( p \)-adic quantum effects and may be regarded as a starting point for construction of a more complete superstring and M-theory. In the low-energy limit adelic quantum mechanics becomes ordinary one.

4. \( p \)-Adic path integrals

A suitable way to calculate propagator in \( p \)-adic quantum mechanics is by \( p \)-adic generalization of Feynman’s path integral. For the classical action \( \bar{S}(x'', t''; x', t') \) which is a polynomial quadratic in \( x'' \) and \( x' \) it is well known that in ordinary quantum mechanics the Feynman path integral is

\[
\mathcal{K}(x'', t''; x', t') = \left( \frac{i}{\hbar} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right)^{1/2} \exp \left( \frac{2\pi i}{\hbar} \bar{S}(x'', t''; x', t') \right).
\]

(22)
The p-adic generalization of the Feynman path integral was suggested in [5] and can be written on a p-adic line as

$$K_p(x''; t'') = \int \chi_p \left( -\frac{S[q]}{\hbar} \right) Dq = \int \chi_p \left( -\frac{1}{\hbar} \int_{t'}^{t''} L(q, \dot{q}, t) dt \right) \prod_t dq(t).$$  \hspace{1cm} (23)

In (23) we take $h \in \mathbb{Q}$ and $q, t \in \mathbb{Q}_p$. This path integral is elaborated, for the first time, for the harmonic oscillator [11]. It was shown that there exists the limit

$$K_p(x''; t'') = \lim_{n \to \infty} K^{(n)}_p(x'', t''; x', t') = \lim_{n \to \infty} N^{(n)}_p(t'', t')$$

$$\times \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} \chi_p \left( -\frac{1}{\hbar} \sum_{i=1}^{n} \dot{S}(q_i, t_i; q_{i-1}, t_{i-1}) \right) dq_1 \cdots dq_{n-1},$$ \hspace{1cm} (24)

where $N^{(n)}_p(t'', t')$ is the corresponding normalization factor for the harmonic oscillator. The subdivision of p-adic time segment $t_0 < t_1 < \cdots < t_{n-1} < t_n$ is made according to linear order on $\mathbb{Q}_p$ and $|t_i - t_{i-1}|_v \to 0$ for every $i = 1, 2, \cdots, n$, when $n \to \infty$. In the similar way we have calculated path integrals for: a particle in a constant external field [12], some minisuperspace cosmological models and a relativistic free particle [8], as well as for a harmonic oscillator with a time-dependent frequency [12].

p-Adic classical mechanics has the same analytic form as in the real case. If $q(t) = \bar{q}(t) + y(t)$ denotes a possible quantum path, with conditions $y(t'') = y(t''') = 0$, where $\bar{q}(t)$ is a p-adic classical path with $\delta S[\bar{q}] = 0$, we have the following action for quadratic Lagrangians:

$$S[q] = S[\bar{q}] + \frac{1}{2} \delta^2 S[\bar{q}] = S[\bar{q}] + \frac{1}{2} \int_{t'}^{t''} \left( y \frac{\partial}{\partial q} + \dot{y} \frac{\partial}{\partial \dot{q}} \right)^{(2)} L(q, \dot{q}, t) dt.$$ \hspace{1cm} (25)

Putting (25) into (23), and using condition

$$\int_{\mathbb{Q}_p} K^*_p(x''; t'') K_p(z, t''; x', t') dx' = \delta_p(x'' - z),$$ \hspace{1cm} (26)

with quadratic expansion of action as well as the general form of the normalization factor

$$N_p(t'', t') = |N_p(t'', t')|_{\infty} A_p(t'', t'),$$

we obtain general expression for the propagator (for some details, see [13])

$$K_p(x'', t''; x', t') = \lambda_p \left( -\frac{1}{2\hbar} \frac{\partial^2 \tilde{S}}{\partial x'' \partial x'} \right) \left| \frac{1}{\hbar} \frac{\partial^2 \tilde{S}}{\partial x'' \partial x'} \right| \chi_p \left( -\frac{1}{\hbar} \tilde{S}(x'', t''; x', t') \right).$$ \hspace{1cm} (27)
This result exhibits some very important properties. For instance, replacing an index $p$ with $v$ in (27) we can write quantum-mechanical amplitude $K$ in ordinary and all $p$-adic cases in the same compact form. It points out a generic behaviour of quantum propagation in Archimedean and non-Archimedean spaces and emphasizes the fundamental role of the Feynman path integral method in quantum theory. Also, considering the most general quadratic $p$-adic Lagrangian $L(x, \dot{x}, t) = a(t)x^2 + 2b(t)x + c(t)x^2 + 2d(t)x + e(t)$ with analytic coefficients, we found a connection [14] between these coefficients and the simplest $p$-adic quantum state $\Omega(|x|_p)$, that is necessary for existence of adelic quantum dynamics. For space-time discreteness in adelic models, see [8].

It is worth mentioning that this approach can be extended to systems with the two, three and more dimensions, and results will be presented elsewhere. The above results are also a starting point for a further elaboration of adelic quantum mechanics and for a semiclassical computation of the $p$-adic path integrals with non-quadratic Lagrangians.

5. Adelic quantum cosmology

Adelic quantum cosmology [15] is an application of adelic quantum mechanics to the universe as a whole. It unifies ordinary and $p$-adic quantum cosmology. Here, path integral formalism occurs to be quite appropriate tool to take integration over both Archimedean and non-Archimedean geometries on the equal footing. In this approach we introduce $v$-adic complex-valued cosmological amplitudes by a functional integral

$$
\langle h''_{ij}, \phi'' \mid h'_{ij}, \phi' \rangle_v = \int D(g_{\mu\nu})_v D(\Phi)_v \chi_v (-S_v[g_{\mu\nu}, \Phi]).
$$

In practice, it is not possible to deal with full superspace (the space of all 3-metrics and matter field configurations). Instead, one exploits minisuperspace (a finite number of coordinates $(h_{ij}, \phi)$). After this simplification, $v$-factors of adelic minisuperspace propagator are given by the relation

$$
\langle q^{\alpha''} | q^{\alpha'} \rangle_v = \int dN K_v(q^{\alpha''}, N | q^{\alpha'}, 0),
$$

where $K_v$ is an ordinary quantum-mechanical propagator with fixed minisuperspace coordinates $q^{\alpha}$ and the lapse function $N$.

We illustrate adelic quantum cosmology by Bianchi I model ($k = 0$). Using Lorentz metric [16]

$$
ds^2 = \sigma^2 \left[ -\frac{N^2(t)}{a^2(t)} dt^2 + a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2 \right]
$$

(30)
and replacements:

\[ x = \frac{bc + a^2}{2}, \quad y = \frac{bc - a^2}{2}, \quad \dot{z}^2 = a^2 \dot{b} \dot{c}, \]

we obtain the corresponding action

\[ S_p[x, y, z] = \frac{1}{2} \int_0^1 dt \left[ -\frac{1}{N} \left( \frac{\dot{x}^2 - \dot{y}^2}{2} + \dot{z}^2 \right) - \lambda N (x + y) \right], \]

and equations of motion

\[ \ddot{x} + \lambda N^2 = 0, \quad \ddot{y} - \lambda N^2 = 0, \quad \ddot{z} = 0. \]

Taking into account conditions \( x(0) = x', \quad y(0) = y', \quad z(0) = z', \quad x(1) = x'', \quad y(1) = y'', \quad z(1) = z'' \), the quantum transition amplitude can be written as

\[ \mathcal{K}_p(x'', y'', z'', N|x', y', z', 0) = \frac{\lambda_p(-2N)}{4^N N^p} \chi_p \left( -\bar{S}(x'', y'', z'', N|x', y', z', 0) \right). \]

Conditions for the existence of the vacuum state \( \Omega(|x|_p)\Omega(|y|_p)\Omega(|z|_p) \) can be calculated from the equality

\[ \int_{|x'|_p \leq 1} \int_{|y'|_p \leq 1} \int_{|z'|_p \leq 1} \mathcal{K}_p(x'', y'', z'', N|x', y', z', 0) dx' dy' dz' = \Omega(|x''|_p)\Omega(|y''|_p)\Omega(|z''|_p), \]

and the simplest vacuum state is

\[ \Psi_p(x, y, z, N) = \begin{cases} \Omega(|x|_p)\Omega(|y|_p)\Omega(|z|_p), & |N|_p \leq 1, \quad |\lambda|_p \leq 1, \quad p \neq 2, \\ \Omega(|x|_2)\Omega(|y|_2)\Omega(|z|_2), & |N|_2 \leq \frac{1}{2}, \quad |\lambda|_2 \leq 2, \quad p = 2. \end{cases} \]

According to (21) adelic wave function \( \Psi(x, t) \) offers more information on a physical system than only its standard part \( \Psi_\infty(x, t) \). In quantum-mechanical experiments, as well as in all measurements, numerical results belong to the field of rational numbers \( \mathbb{Q} \). For the Bianchi I model, as well as for any adelic quantum model, according to the usual interpretation of the wave function we have to consider \( |\Psi(x, t)|^2_\infty \) at rational space-time points. In the above adelic case we get

\[ |\Psi(x, y, z, N)|^2_\infty = |\Psi_\infty(x, y, z, N)|^2_\infty \prod_p \Omega(|x|_p)\Omega(|y|_p)\Omega(|z|_p) \]

\[ = \begin{cases} |\Psi_\infty(x, y, z, N)|^2_\infty, & x, y, z \in \mathbb{Z}, \\ 0, & x, y, z \in \mathbb{Q} \setminus \mathbb{Z}. \end{cases} \]
Here we used the following properties of the \( \Omega \)-function: \( \Omega^2(|x|_p) = \Omega(|x|_p) \), \( \prod_p \Omega(|x|_p) = 1 \) if \( x \in \mathbb{Z} \), and \( \prod_p \Omega(|x|_p) = 0 \) if \( x \in \mathbb{Q} \setminus \mathbb{Z} \). Thus, it means that positions \( x, y, z \) may have only discrete values: \( x = 0, \pm 1, \pm 2, \ldots \). Since the \( \Omega \)-function is invariant under the Fourier transformation, there is also discrete momentum space. When system is in some excited state, the sharpness of the discrete structure disappears and space demonstrates usual continuous properties. It is worth mentioning that a space-time discreteness is also noted in the framework of q-deformed quantum mechanics [17].

6. \( p \)-Adic analysis and q-analysis. The Moyal product

Some connections between \( p \)-adic analysis and quantum deformations has been noticed [18] in a variety of cases during the last ten years or so. It was shown [19] that the two parameter Sklyanin quantum algebra and its generalizations provide a promising connection between the \( p \)-adics and quantum deformation. A similar connection has been indicated by Macdonald’s paper [20] on orthogonal polynomials associated with the root systems. In [19] it was also pointed out that elliptic quantum group and its generalizations unify the \( p \)-adic and real versions of a Lie group (e.g. \( SL(2) \)). This result is connected with adelic approach and the possibility of establishing q-deformed Euler products.

In some other contexts it has been observed that the Haar measure on \( SU_q(2) \) coincides with the Haar measure on the field of \( p \)-adic numbers \( \mathbb{Q}_p \) if \( q = \frac{1}{p} \) [21]. Namely, Tomea-Jackson integral in q-analysis

\[
\int_0^1 f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n, \quad (37)
\]

and the integral in \( p \)-adic analysis

\[
\int_{|x|_p \leq 1} f(|x|_p) dx = (1 - \frac{1}{p}) \sum_{n=0}^{\infty} f(p^{-n}) p^{-n}, \quad (38)
\]

are equal if \( q = \frac{1}{p} \), i.e.

\[
\int_0^1 f(x) d_{1/p} x = \int_{|x|_p \leq 1} f(|x|_p) dx. \quad (39)
\]

In q-analysis there is the following differential operator (related to the q-deformed momentum in the coordinate representation [21])

\[
\partial_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}. \quad (40)
\]
In p-adic analysis, when one considers a complex-valued function \( f(x) \) depending on a p-adic variable \( x \) we are not able to use standard definition of differentiation. Instead of that it is possible to use Vladimirov’s operator

\[
D^\alpha \psi(x) = \frac{p - 1}{1 - p^{-1-\alpha}} \int \frac{f(x) - f(y)}{|x - y|_{p}^{\alpha+1}} dy
\]

which in a sense resembles (40). Moreover, there is a potential such that the spectrum of the p-adic Schrödinger-like (diffusion) equation [22]

\[
D\psi(x) + V(|x|_{p})\psi(x) = E\psi(x)
\]

is the same one as in the case of q-deformed oscillator found by Biedenharn [23] and Macfarlane [24] for \( q = 1/p \). For more details, see [21].

Recently [25], it has been proposed a new pseudodifferential operator with rational part of p-adic numbers \( \{x\}_p \). In such case, energy levels for p-adic free particle exhibit discrete dependence on the corresponding momentum: \( \{E\}_p = \{k\}_p^2 \). Note also a proposal for q-deformation of Vladimirov’s operator [26].

We see that there are some interesting relations between p-adic and q-analysis, and in a sense between adelic quantum mechanics and noncommutative one. It would be fruitful to find some deeper reasons for these connections, between theories which pretend to give us more insights on the space-time structure at the Planck scale. By now it is not enough understood. It seems to be reasonable to formulate a noncommutative adelic quantum mechanics that may connect non-Archimedean and noncommutative effects and structures. As the first step in this direction one has to consider a p-adic and adelic generalization of the Moyal product.

Let us consider D-dimensional classical space with coordinates \( x^1, x^2, \ldots, x^D \). Let \( f(x) \) be a classical function \( f(x) = f(x^1, x^2, \ldots, x^D) \). Then, with the respect to the Fourier transformations, we have

\[
\hat{f}(k) = \int_{\mathbb{Q}_p^D} dx \chi_v(kx)f(x),
\]

\[
f(x) = \int_{\mathbb{Q}_v^D} dk \chi_v(-k\hat{x})\hat{f}(k).
\]

According to the usual Weyl quantization

\[
\hat{f}(x) = \int_{\mathbb{Q}_v^D} dk \chi_\infty(-k\hat{x})\hat{f}(k) \equiv f(\hat{x}).
\]

Let us now have two classical functions \( f(x) \) and \( g(x) \) with

\[
\hat{f}(x) = \int_{\mathbb{Q}_\infty^D} dk \chi_\infty(-k\hat{x})\hat{f}(k),
\]
\[ \hat{g}(x) = \int_{Q^D_{\infty}} dk \chi_\infty(-k\hat{x})\tilde{g}(k). \quad (47) \]

In the coordinate representation we can write the same above expressions replacing \( \hat{x} \) by \( x \) and extend it to all \( p \)-adic cases.

Now we are interested in product \( \hat{f}(x)\hat{g}(x) \). In the real case this operator product is of the form

\[ (\hat{f} \cdot \hat{g})(x) = \int \int dk dk' \chi_\infty(-k\hat{x})\chi_\infty(-k'\hat{x})\tilde{f}(k)\tilde{g}(k'). \quad (48) \]

Using the Baker-Campbell-Hausdorff formula, the relation (1) and then the coordinate representation one finds the Moyal product in the form

\[ (f \ast g)(x) = \int \int dk dk' \chi_p \left( -(k+k')x + \frac{1}{2} k_i k'_j \theta^{ij} \right) \tilde{f}(k)\tilde{g}(k'), \quad (49) \]

where we already used our generalization from \( \mathbb{Q}_\infty \) to \( \mathbb{Q}_p \). Note that in the real case we use \( k_i \rightarrow -(i/2\pi)(\partial/\partial x^i) \) and obtain the well known form

\[ (f \ast g)(x) = \chi_\infty \left( -\frac{\theta^{ij}}{2(2\pi)^2} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) f(y)g(z)|_{y=z=x}. \quad (50) \]

Thus, as the \( p \)-adic Moyal product we take

\[ (\hat{f} \ast \hat{g})(x) = \int_{Q^D_{p}} \int_{Q^D_{p}} dk dk' \chi_p (-x^i k_i + x^j k'_j + \frac{1}{2} k_i k'_j \theta^{ij})\tilde{f}(k)\tilde{g}(k'). \quad (51) \]

As the first step in adelization one can consider the Moyal product on \( \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p \) space. Various adelic aspects of the Moyal product will be presented elsewhere.

**Acknowledgments** Authors G.Dj. and B.D. wish to thank the co-directors of ARW "Noncommutative Structures in Mathematics and Physics" Profs. J. Wess and S. Duplij for their invitation to participate and give a talk. G.Dj. is partially supported by DFG Project “Noncommutative spacetime structure - Cooperation with Balkan Countries”. The work of B.D. was supported in part by RFFI grant 990100866.
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