Renormalon cancellation and Borel summability of the Gross-Neveu model mass gap

J.-L. Kneur and D. Reynaud

Physique Mathématique et Théorique, UMR-5825-CNRS, Université Montpellier II, F–34095 Montpellier Cedex 5, France.

Abstract

The exact mass gap of the $O(N)$ Gross-Neveu model is known, for arbitrary $N$, from non-perturbative methods. However, a “naive” perturbative expansion of the mass exhibits an infinite set of infrared renormalons at order $1/N$, formally similar to the QCD heavy quark pole mass renormalons, potentially leading to large $O(\Lambda)$ perturbative ambiguities. We examine the precise cancellation mechanism of such infrared renormalons, which avoids this (only apparent) contradiction, and operates without need of (Borel) summation contour prescription, usually preventing unambiguous separation of perturbative contributions. As a consequence we stress the direct Borel summability of the (genuine) perturbative expansion of the mass gap at order at least $1/N$. We briefly speculate on a possible similar behaviour of analogous non-perturbative QCD quantities.
1 Introduction

The $(1+1)$ dimensional $O(N)$ Gross-Neveu (GN) model[1, 2] often serves as a simpler toy model for more complicated theories like QCD, sharing with it the properties of asymptotic freedom and dynamical mass generation, while being an integrable model with many exact results available. The mass gap (associated with the breaking of the discrete chiral symmetry), for arbitrary $N$, has been established from non-perturbative (NP) methods, more precisely from exact S matrix results[3] associated with Thermodynamic Bethe Ansatz (TBA) methods[4]. Yet from a more general viewpoint, it can be an interesting issue (since still not fully clarified, in our opinion) to study the precise matching between those NP exact results on one side and the standard perturbative behaviour on the other side. This may give some more insight on the short/long distance physics interplay for more involved theories like QCD. In particular one apparent puzzle arises, once realizing, as we examine here, that the naive (standard) perturbative expansion of the GN mass suffers at next-to-leading $1/N$ order from potential ambiguities, due to the presence of severe infrared renormalons, which are indeed formally completely similar to the quark pole mass renormalons[5]. While the TBA GN mass gap[4], and a fortiori its next-to-leading $1/N$ expansion[6], are unambiguously determined. Actually, independently of TBA results, it is expected on general grounds that any truly NP calculation should be free of such ambiguities[7], i.e. that infrared renormalons are perturbative artifacts. But up to now only a few NP results have been explored from this perspective, even for integrable models, thus we find instructive to examine in some details how exactly the NP contributions, here fully controllable at least at $1/N$ order, organize themselves so that the necessary cancellation of such ambiguities (indeed an infinite series of ambiguities) occurs. Moreover, what is in fact generally expected (and illustrated in a few explicit calculations in the $(1+1)$ dimensional sigma model[8, 9]) is that the complete NP result is unambiguous, but the separation of its perturbative from its NP “operator product expansion” (OPE) contribution, is not. More precisely the two contributions are intrinsically related through the need of a definite prescription in choosing the integration path in the Borel plane, if using e.g. Borel resummation methods, necessary to avoid the renormalons in both parts and to resum the otherwise ill-defined factorially divergent perturbative series. In contrast, as we will show, no such prescription is needed for the infrared renormalon cancellation in the GN mass gap. More precisely, the cancellation is such that the “genuine” perturbative expansion (to be specified) at order $1/N$ is directly Borel summable and can be thus unambiguously separated from the NP part.
Perturbative renormalons in the $1/N$ mass

We start by briefly recalling the standard construction\cite{10, 4} of the mass gap of the $O(2N)$ $^1$ GN model at order $1/N$, with a slightly different approach. Here we consider in fact the model with the addition of a Lagrangian mass term, in order to define the pole mass in a somewhat more universal manner, making e.g. the link with analogous quantities in other models (QCD typically) more transparent. Obviously the true mass gap is to be considered in the chiral, massless Lagrangian limit. The Lagrangian thus reads

$$\mathcal{L}_{GN} = \bar{\Psi}i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi + \frac{g}{2}(\bar{\Psi}\Psi)^2$$ \hspace{1cm} (1)$$

and the graphs contributing to the two-point function at next-to-leading $1/N$ order are shown in Fig. 1. At this $1/N$ order, the renormalization procedure is relatively simple, since only the mass term is renormalized, which can be most simply performed by subtraction of the most divergent terms in a taylor expansion of the integrands\cite{10}. The expression corresponding to Fig.1 reads, after angular integration:

$$M_P = M \left[ 1 + r_1 \frac{1}{N} + O \left( \frac{1}{N^2} \right) \right]$$ \hspace{1cm} (2)$$

with $M = \Lambda \equiv \mu e^{-2\pi/(Ng(\mu))}$ the mass gap at leading $1/N$ order ($\mu$ is the arbitrary renormalization scale), and

$$r_1 = \frac{gN}{4\pi M} \left[ \int_0^\infty dq^2 \left( \frac{\hat{p}}{2p^2} \left[ 1 - \frac{q^2 + p^2 + M_P^2}{A} \right] - \frac{M_P}{A} \right) G[q^2] - 4G[0]M_P \int_0^\infty \frac{dq^2}{q^2} \zeta \ln \left[ \frac{\zeta + 1}{\zeta - 1} \right] G[q^2] \right]$$ \hspace{1cm} (3)$$

\footnote{From now on, all expressions correspond to the $O(2N)$ model, for easiest comparison with the exact\cite{4} and $1/N$\cite{6} mass gap results.}
with \( A \equiv \sqrt{(q^2 + p^2 + M_P^2)^2 - 4p^2 M_P^2} \) (\( q \) is Euclidean but \( p \) Minkowskian), \( \zeta \equiv (1 + 4M_P^2/q^2)^{1/2} \), and the dressed propagator (wavy line) is

\[
G[q^2] = \left[ 1 + \frac{gN}{2\pi} \ln \frac{M_P^2}{\mu^2} + \zeta \ln \left[ \frac{\zeta + 1}{\zeta - 1} \right] \right]^{-1} = \left[ \frac{gN}{2\pi} \zeta \ln \left[ \frac{\zeta + 1}{\zeta - 1} \right] \right]^{-1} + \mathcal{O}(1/N) .
\]  

(4)

To obtain the pole mass at the next-to-leading \( 1/N \) order, and in the massless limit, it is sufficient to replace \( \hat{\rho} = M_P = M[1 + \mathcal{O}(1/N)] \) in Eq.(3),(4) up to \( 1/N^2 \) terms. This reads

\[
r_1 = \frac{1}{4} \int_0^{\mu^2} \frac{d q^2}{M^2} (1 - \zeta) \left[ \zeta \ln \left[ \frac{\zeta + 1}{\zeta - 1} \right] \right]^{-1} - \frac{1}{2} \int_0^{\mu^2} \frac{d q^2}{q^2 \zeta^2} + \mathcal{O}(1/N) - \text{subtraction}
\]  

(5)

where a “factorization” scale \( \mu \) regularizing the integrals was introduced. The renormalization by subtraction will remove the divergent part of integrands, also regularized in term of \( \mu \). This sharp momentum cut-off procedure is rather similar to the one in ref. [6], giving results perfectly consistent with these authors, as we will see in next section, and is convenient for our purpose here, where we are primarily interested in the asymptotic behaviour of (5) (thus for \( \mu \gg \Lambda \), but kept finite).

For simplicity we let aside in this section the tadpole graph, as its exact expression in (5) does not give any factorially divergent perturbative coefficients when expanded\(^2\). [It gives however a finite contribution to the mass gap, as we shall see in next section when a precise evaluation of the mass gap and its asymptotic behaviour will be considered.] It is easily realized that a naive perturbative expansion of (5) leads to infrared renormalons and related \( \mathcal{O}(\Lambda/\mu) \) ambiguities. Starting from the purely perturbative information means replacing in (5) the dressed scalar propagator by the effective coupling\(^3\):

\[
r_1 = \frac{1}{4} \int_0^{\mu^2} \frac{d q^2}{M^2} (1 - \zeta) \left[ g(q^2) - 2g \frac{M^2}{q^2} + \mathcal{O}\left( \frac{M^2}{q^2} \right)^2 \right]  
\]  

(6)

where the \( \mathcal{O}(M^2/q^2) \) “NP” corrections to the effective coupling can be obtained explicitly here from a systematic expansion of \( \zeta \) for large \( q^2 \). Remark that in (5), the \( 1 - \zeta \) factor

\(^2\)Due to the second equality in (4), valid at this \( 1/N \) order, which makes the tadpole graph simplifying to purely polynomial integrals. Beyond the \( 1/N \) order, one could still in principle choose the arbitrary scale \( \mu \) such that (4) holds.

\(^3\)In the sequel we rescale \( b_0 g \equiv N/(2\pi) \) for \( g \), to define the \( 1/N \) expansion properly, and absorbing as well the \( 2\pi \) factor just for convenience.
corresponds to the skeletal “one-loop” integrand of the first graph in Fig.1, while the remaining \( \zeta \)-dependence is nothing but the dressed scalar propagator. Thus the perturbative form of (6) is quite generic, also applying to e.g. the QCD pole quark mass [but with obviously a different \( q^2 \)-dependence replacing the \( 1 - \zeta \) term]. The simplest standard procedure to exhibit the IR renormalons is by expanding the \( 1 - \zeta \) term for small \( q^2 \):

\[
1 - \zeta = 1 - \frac{2M}{q} \left( 1 + \frac{q^2}{4M^2} \right)^{1/2} \simeq 1 - \frac{2M}{q} + \mathcal{O}\left( \frac{q}{2M} \right) \tag{7}
\]

and expanding the effective coupling \( g(q^2) \) at one loop order of renormalization group (RG) in powers of \( b_0g(\mu)\ln[q^2/\mu^2] \). From (7) it is seen that the leading singularity comes from the \( q^{-1} \) term which, combined with the \( (g(\mu)\ln[q^2/\mu^2])^n \) terms of the effective coupling expansion, produces factorially divergent perturbative coefficients:

\[
r_1^{\text{leading}} \sim -\frac{\mu}{M} \sum_{p=0}^{\infty} 2^p p! g^{p+1}(\mu) \tag{8}
\]

The non sign-alternation of those factorial coefficients implies[7] that the corresponding series is not Borel summable: the Borel integral corresponding to (8) reads

\[
BI[r_1^{\text{leading}}] \sim -\frac{\mu}{M} \int_0^{\infty} dt e^{-t/g} (1 - 2t)^{-1} \tag{9}
\]

so that the pole at \( t_0 = 1/2 \) on the integration path gives the leading ambiguity for the pole mass

\[
\delta M_{\text{leading}} \sim \lim_{\epsilon \to 0} \left[ \int_{\infty - i\epsilon}^{\infty + i\epsilon} - \int_{\infty - i\epsilon}^{\infty - i\epsilon} \right] (dt e^{-t/g} (1 - 2t)^{-1}) = \pm i\pi \mu e^{-t_0/g} \propto \Lambda \tag{10}
\]

Note that this leading renormalon ambiguity of \( \mathcal{O}(\Lambda) \) is completely similar to the one derived for the quark pole mass in QCD[5]. Although our above derivation (starting from purely perturbative information) is very standard, the fact that the GN model perturbative pole mass at \( 1/N \) order also has the specific structure (8)–(10) of infrared renormalons, was perhaps not clearly appreciated before, to our knowledge\(^4\). Similarly, we can easily check from (3) that the leading asymptotic behaviour of perturbative coefficients of the two-point function for \( p^2 \gg M^2 \) is \( \sim \sum_p p! g^p \), which accordingly gives a less severe ambiguity of \( \mathcal{O}(\Lambda^2/p^2) \), where again one can note the similarity with the QCD off-shell

\(^4\)In ref.[1] appeared already the (earliest) discussion on ultraviolet renormalons in a field theory framework, which were shown to be harmless (Borel summable) for the asymptotically free GN model.
\( p^2 \gg M^2 \) quark correlation function case [5, 7].

It is in fact possible to go a step further and to characterize at arbitrary next orders the renormalon properties of the \( 1/N \) GN mass gap. Consider the second order (exact) RG dependence of the effective coupling [11]:

\[
g/g(q^2) \equiv f = 1 + b_0 g \ln \left( \frac{q^2}{\mu^2} \right) + b_1 \frac{g}{b_0 g} \ln \left( \frac{f(1 + b_1/b_0 g f^{-1})}{1 + b_1/b_0 g} \right)
\]

(11)

defining \( f \) recursively, with \( g \equiv g(\mu) \) and the beta function \( \beta(g) = -2b_0 g^2 - 2b_1 g^3 - \cdots \) (where for clarity we reintroduce the original coupling and RG coefficients, i.e. before rescaling of \( g \)). We can put Eq. (5), with (11), directly into the form of a Borel integral, after a convenient change of variable [12], defining the (Borel) variable

\[
b_0 t = \frac{1 - f}{1 + b_1/b_0 g}.
\]

(12)

Taking expression (6), but using now Eq. (11), we find after some algebra

\[
r_{\text{RG}}^2 = -\frac{\mu}{2M} \int_0^\infty dt e^{-\frac{\mu}{2M} t} (1 - b_0 t)^{-1 - C} \left[ 1 - \frac{\mu}{2M} e^{-\frac{\mu}{2M} t} (1 - b_0 t)^{-C} \right]
\]

\[
+ \frac{\sqrt{\pi}}{2} \sum_{p=1}^{\infty} \frac{(\mu^2/4M^2)^p}{p! \Gamma[3/2 - p]} e^{-p\alpha t} (1 - b_0 t)^{-2pC}
\]

(13)

where \( C = b_1/(2b_0^2) \) and \( \alpha = 1/g + b_1/b_0 \). To obtain (13) we expanded Eq. (7) in powers of \( q^2/M^2 \), and used Eqs. (11, 12). This gives the complete (leading and all subleading orders) series of infrared renormalons (initially corresponding to cuts at \( b_0 t_p = 1/2, 1, \ldots (2p + 1)/2, p \in \mathbb{N}^* \)). The change of variable (12) makes calculations more convenient since expression (13) has only a cut at \( t \geq 1/b_0 \). Now we can calculate the ambiguity to all (perturbative) orders (of course still limited to \( 1/N \) order), which we define by the difference of contour above and below the cut. We find, again after some algebra:

\[
(2\pi i)^{-1} \delta M^P = \pm \frac{\Lambda}{b_0} \left[ \frac{(2e)^{-C}}{\Gamma[1 + C]} - \frac{e^{-2C}}{2M \Gamma[1 + 2C]} \right]
\]

\[
+ \frac{\sqrt{\pi}}{2} (2e)^{-C} \sum_{p=1}^{\infty} \frac{(2p + 1)(2p+1)^C}{p! \Gamma[3/2 - p] \Gamma[1 + (2p + 1)C]} \frac{(2e)^{-C} \Lambda}{2M} 2 \gamma
\]

(14)

where we used essentially

\[
\lim_{\epsilon \to 0} \left[ \int_0^{\infty + i \epsilon} - \int_0^{\infty - i \epsilon} \right] dt e^{-\alpha t} (1 - \beta t)^\gamma = 2\pi i \frac{e^{-\alpha/\beta} \beta^\gamma \alpha^{-(1+\gamma)} \Gamma[-\gamma]}{-\gamma}
\]

(15)
and identified the \( \overline{\text{MS}} \) scale \( \bar{\Lambda} = \mu e^{-1/(2b_0 g)}(b_0 g)^{-\zeta} [1 + (b_1/b_0)g]^{\zeta} \) consistently at second RG order. Eq.(14) thus gives the full series of ambiguities due to infrared renormalons (for the first graph in Fig. 1), in the form of power corrections of order \( \Lambda^p \), with the first term in the bracket of (14) the leading order ambiguity of \( \mathcal{O}(\Lambda) \). If \( b_1 = 0 \) (e.g. at first RG order) expression inside the bracket of Eq. (10) simplifies to \( (1 + r^2/4)^{1/2} - r/2 \), \( r \equiv \Lambda/M \) (indeed sufficient at this \( 1/N \) order, since from Eq. (2) \( r_1 \) is already the \( 1/N \) term.)

The above derivation of (14) only uses information that is in fact purely perturbative: the effective coupling at second RG order, Eq. (11), and the specific GN mass \( q^2 \) kinematic dependence, Eq. (7) of the (one-loop) skeletal first graph in Fig. 1. Accordingly, a similar derivation is possible for the QCD quark pole mass renormalon properties (indeed the equivalent of the information in (14) in the QCD case is also known, though perhaps expressed in a slightly different form[7]).

Actually, by considering only the first graph of Fig. 1 we slightly oversimplified the complete renormalon picture for the GN model: clearly, the second tadpole graph also gives renormalons, if considered purely perturbatively. These are easily analyzed similarly to Eqs. (6)-(7), and the expanded integrand for \( q^2 \to 0 \) leads to renormalon poles at \( b_0 t_p = p \in \mathbb{N}^* \). Incidentally, the pole at \( b_0 t = 1 \) exactly cancels the one in (13), but the leading as well as all subleading poles for \( b_0 t_p \geq 3/2 \) in (13) remain uncancelled.

In summary, we thus observe from the structure of (14) that even resumming the full integrand \( 1 - \zeta \) (taking the full series of sub-leading renormalons) does not remove in any way the leading ambiguities (even if there are some ”accidental” cancellations among subleading poles at this purely perturbative level, bewteen the two graphs of Fig. 1 as above discussed). This is not at all surprising, since as emphasized (13),(14) are still perturbative calculations, from which one cannot hope a priori to cancel renormalons.

### 3 Borel summability of the exact \( \mathcal{O}(1/N) \) mass

Alternatively, since expression (5) is exact at \( 1/N \) order, we can try to calculate exactly the expression of the mass gap, i.e. without truncating (5) to its perturbative expansion. Indeed integral (5) can be evaluated analytically exactly: after a convenient change of variable \( \zeta^{-1} = \tanh(\frac{\theta}{2}) \) we obtain

\[
r_1 = \frac{1}{2} \left[ Ei[-\theta] - \ln \theta - \gamma_E + \ln(\ln \frac{\mu^2}{M^2}) - 2 \ln(\cosh[\theta/2]) + \ln \frac{\mu^2}{M^2} \right]
\]  (16)
with \( \chi = (1+4M^2/\mu^2)^{1/2} \equiv 1/\tanh(\theta/2) \) (i.e. \( \theta = \ln[(\chi+1)/(\chi-1)] \geq 0 \)), and \( Ei(-x) \equiv -\int_x^\infty dt e^{-t}/t \quad (x \geq 0) \) the Exponential Integral function. The term \(-2\ln(\cosh(\theta/2))\) in (16) corresponds to the (unsubtracted) tadpole graph of Fig.1, and the terms \( \ln \ln(\mu^2/M^2) \equiv -\ln g \) and \( \ln \mu^2/M^2 \equiv 1/g \) are the subtraction terms for the first and tadpole graphs, respectively. One can easily check the finiteness of (16), if letting the “cutoff” \( \mu \rightarrow \infty \).

Now, as already stressed, here we are interested in the complete asymptotic behaviour, thus letting \( \mu \gg \Lambda \) but kept finite, retaining eventually all the power correction terms in \( \Lambda/\mu \). The function \( Ei(-x) \) for \( x > 0 \) has an asymptotic expansion with factorial but sign alternating coefficients, therefore explicitly Borel summable and perturbatively unambiguous. More precisely when re-expanding (16) in perturbation, we obtain

\[
M_P = M \left[ 1 + \frac{1}{2N} \left( 2\ln 2 - \gamma_E - \frac{M^2}{\mu^2} \sum_{n=0}^{\infty} (-1)^n n! \frac{g^{n+1}}{g^n} + O\left( \frac{M^2}{\mu^2} \right) \right) \right] \tag{17}
\]

where the higher order power correction terms, that we do not give explicitly here, can be obtained by a systematic expansion in \( M^2/\mu^2 \) of \( \theta \). The tadpole graph in Fig.1 contributes a constant term \( 2\ln 2 \), relevant to the mass gap determination, but does not contribute the factorial asymptotic behaviour of the perturbative series, as already mentioned. We stress that the specific \( 1 - \zeta \) form of the integrand in (5) plays an essential role for the Borel summability of expression (17), which is accordingly peculiar to the pole mass. In contrast, the off-shell two-point function expression (3) for arbitrary \( p^2 \) may be evaluated similarly non-perturbatively (at \( 1/N \) order), and does not lead to a Borel summable perturbative series, in consistency with the results obtained and discussed previously in ref. [10]. We shall come back to this specificity of the pole mass in more details in next section 4. We also stress that the use of the \( O(1/N) \) mass gap relation Eq.(2) in the Lagrangian mass \( m \rightarrow 0 \) limit, though it greatly simplifies the evaluation of Eq.(5), plays no particular role in the good asymptotic properties of Eq.(17): more precisely, starting from the exact \( 1/N \) mass expression (5), and introducing the arbitrary mass dependence \( m \), Borel summability is maintained with an asymptotic expansion similar to Eq.(17) for any value of the pole mass \( M_P \gg \Lambda \) (see Appendix).

Finally to obtain the correct mass gap \( M_P/\Lambda \) at next-to-leading \( 1/N \) order from expression (16), one should yet introduce the \( \overline{MS} \) basic scale above defined after Eq. (15). Dropping terms of higher \( O(1/N^2) \) order, we obtain

\[
M_P/\Lambda = 1 + \frac{\gamma_1}{N} + O\left( \frac{1}{N^2} \right) = 1 + \frac{1 + 2\ln 2 - \gamma_E}{2N} \tag{18}
\]

5 Technically, the simplest such terms can be expressed in terms of \( Ei(\theta) \), which accordingly (since \( \theta > 0 \)) has asymptotic expansion with same sign factorial coefficients and imaginary part \( \pm i\pi \).
in agreement with ref.[6]. Note however that our expression (16) differs in fact from ref.[6], more precisely by the $Ei[-\theta]$ term. This is simply because in ref.[6] all terms vanishing as inverse powers of $\mu$ were dropped, which is sufficient to identify the mass gap Eq. (18). From our result, this is consistent because at the (non-perturbative) level of Eq.(16), all those “power correction” contributions from $Ei(-\theta)$ can be unambiguously separated, thus dropped from Eq.(17), to let only the part relevant to determine the mass gap. The explicit Borel summability of the genuine perturbative expansion of the pole mass, Eq. (17), confirms the consistency of the whole procedure. An equivalent signature of Borel summability here is the absence of positive powers of $\mu$ in (17). Applying the same procedure to other quantities than the pole mass, one eventually ends up with asymptotic expansions with non sign-alternated factorial coefficients and $\mu^P$ power terms with $P > 0$ (as is clear from the above expression of $\theta$, cf. footnote 5). The latter $\mu^P$ terms in fact cancel when combining the OPE and perturbative parts, as illustrated in explicit calculations for some vacuum expectation value[8, 13] and off-shell correlation functions[9] of the $O(N)$ sigma model at $1/N$ order, but it means one cannot separate unambiguously the NP and perturbative contributions.

Coming back to our result Eq. (17), it appears however immediately in apparent contradiction with those obtained starting from purely pertubative expansions, (8–14) above. We shall examine in next section how to reconcile these two different pictures.

4 Explicit cancellation of IR renormalons

How exactly the “bad” factorial coefficients with no sign alternation in Eq. (8) disappear, or more precisely transmute into “good” sign-alternated factorials, in the exact expression (17)? Clearly, it can only be that this necessary cancellation of the wrong sign factorials should occur with the NP power expansion contributions: the weak point of the standard perturbative renormalon picture is that we have expanded the $1 - \zeta$ term in the infrared low $q^2$ regime, while keeping the short distance, perturbative effective coupling form of the propagator. This is of course usually motivated from the fact that the latter information is a priori the only accessible one in more involved theories such as QCD. In the present case, as we know exactly the mass at $1/N$ order one may at first hope to infer such cancellations by examining e.g. the systematic short distance $q^2 \to \infty$ and long distance $q^2 \to 0$ power expansion of the integrand in (6), which are perfectly well-defined\(^6\).

\(^6\)E.g. the infrared $q^2 \to 0$ expansion of the dressed propagator is purely polynomial, with no log. dependence: $\left[ \zeta \ln[\frac{\zeta+1}{\zeta-1}] \right]^{-1} \simeq \frac{1}{2} - \frac{q^2}{24M^2} + \cdots$. 

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In fact, to see the cancellation operating needs a little more sophisticated analysis. Following e.g. refs.[8, 9], we introduce the Mellin-Barnes (MB) transform for a part of the integral Eq. (5), which then takes the form (again omitting here the tadpole graph for simpler illustration):

\[
r_1 = \frac{1}{4} \int_0^{\mu^2} \frac{dq^2}{M^2} \int_0^\infty dt \frac{1}{2\pi i} \oint ds K(s, t) \left( \frac{M^2}{q^2} \right)^{-s}
\]

where the Kernel (inverse MB transform) is defined in our case by

\[
K(s, t) = \int_0^\infty dx x^{s-1} (1 - \zeta) \zeta^{-1} \left[ \frac{\zeta - 1}{\zeta + 1} \right]^t.
\]

The MB transform method main purpose is that it will exhibit precisely the singularities of the integrand, in the Borel plane of interest. The sequel is just algebraic manipulation. Changing again variable \( \zeta^{-1} = \tanh(\phi/2) \), except for the \( 1 - \zeta \) term kept on purpose, using Eq. (7), as an expansion in \( q/M \), (20) can be evaluated exactly, and

\[
\int_0^{\mu^2} \frac{dq^2}{M^2} \left( \frac{M^2}{q^2} \right)^{-s} K(s, t) = \frac{\mu^2}{2M^2} \sum_{a \geq -1} 2^{-a} c_a \frac{\Gamma[1 + a - 2s]\Gamma[-a/2 + s + t]}{\Gamma[1 + a/2 - s + t] (1 + s)} \left( \frac{M^2}{\mu^2} \right)^{-s}
\]

defined for \( \text{Re}[s + t] > a/2 \) and \( \text{Re}[2s] < 1 + a \). The latter conditions are such that integral (20) converges, and play essential role in determining the singularities. The variable \( a \) in (21) is simply the power of \( q/M \sim \sinh \phi \) in expansion (7), with coefficient \( c_a \) respectively. Thus \( a = -1 \) with \( c_{-1} = -2 \) corresponds to the leading renormalon, and \( a = 0, 2p + 1, p \in \mathbb{N} \) to subleading ones. To evaluate the \( s \) integral one can simply close the contour on the left, and sum over residues of the poles included in this domain (since \( x^{-s} \equiv (M^2/q^2)^{-s} \) decays exponentially fast for the asymptotic regime \( q^2 \gg M^2 \) we are interested in). Expression (21) has (simple) poles at \( s = -1, s = a/2 - t - k \), and \( 2s = a + 1 + k, k \in \mathbb{N} \), where the latter poles do not contribute for the relevant contour. The final result is

\[
r_1 \sim \frac{\mu^2}{4M^2} \int_0^\infty dt \sum_{a \geq -1} \frac{c_a}{2^a} \frac{\Gamma[3 + a\Gamma[t - 1 - a/2] M^2]}{\Gamma[2 + t + a/2]} \frac{\mu^2}{\mu^2}
\]

\[
+ e^{-t/g} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[1 + 2t + 2k]}{k! \Gamma[1 + 2t + k] (1 + a/2 - t - k)} \left( \frac{M^2}{\mu^2} \right)^{k-a/2}
\]

where the first term in bracket corresponds to the residue of the pole at \( s = -1 \), while other terms correspond to the sum over residues of the poles at \( s = a/2 - t - k \), and we
used \((M^2/\mu^2)^t \equiv e^{-t/\gamma_E}\). In (22) one sees that the first term in the bracket and the summed terms both have poles at \(t(a, k) = 1 + a/2 - k\), which can occur at \(t > 0\) depending on \(a, k\) values. On more physical grounds, the contributions from the first term, the initially \(s = -1\) pole, originate from power terms \((M^2/q^2)^{-s}\) in Eq.(19), and correspond intuitively to non-perturbative "OPE" contributions, while the perturbative contributions are those multiplied by \(e^{-t/\gamma_E}\) in (22). For example the poles at \(t = 1/2\) for \(a = -1\) correspond to the leading order renormalon, with \(k = 0\). Indeed, keeping only the leading renormalon perturbative terms, \(\propto e^{-t/\gamma_E}\), for \(k = 0\), one recovers exactly Eq. (9).

Now it can be easily checked that this \(t = 1/2\) pole in fact cancels exactly against the first NP term \(t = 1/2\) pole, and similarly for all subleading poles at \(t = 1, 3/2, \ldots (2p + 1)/2\). This is the announced cancellation. Moreover all cancellations happen, for a given pole at \(t = 1 + a/2 - k\), between NP and perturbative terms of the same \(k\) values. Since (22) is in the form of a Borel integral, and after cancellations all remaining poles occur at \(t < 0\), it defines a Borel summable series, whose leading terms just correspond to the asymptotic series defined in (17). To see that, it is simpler to alternatively proceed directly with (20) using the change of variable \(\zeta \rightarrow \phi\) for any terms in the integrand, i.e. without going through power expansion (7): after MB transform one ends up directly with a Borel integral where no poles at all appear at \(t > 0\), and which exactly gives the asymptotic series in Eq.(17), including correct finite terms \(-\gamma_E + \ldots\) [NB there are also poles at \(t = 0\) in (22), which as usual[7] simply corresponds to the UV divergences, and are removed consistently by the appropriate subtraction terms, that we do not display explicitly.] For instance, illustrating only the terms from the first graph in Fig.1, after cancellations of the \(t > 0\) poles the MB transformation gives

\[
\begin{align*}
    r_1 & \sim -\frac{\mu^2}{2M^2} \int_0^\infty dt \left[ \frac{M^2}{\mu^2} \frac{1}{t(1+t)} + e^{-t/\gamma_E} \left[ -\frac{1}{t^2} \frac{M^2}{\mu^2} + \frac{3+2t}{1+t} \left( \frac{M^2}{\mu^2} \right)^2 + \mathcal{O}\left( \frac{M^2}{\mu^2} \right)^3 \right] \right] \\
    & \sim -\frac{1}{2} \gamma_E + \frac{M^2}{\mu^2} g \sum_{n=0}^\infty (-1)^n n! g^n 
\end{align*}
\]

(23)

to compare with Eq.(17) (where we used e.g. \(\int_0^\infty dt \frac{1}{t} \left[ 1/(t+1) - e^{-t} \right] \equiv \gamma_E\)). Of course, proceeding in this "direct" way is nothing but a consistency check that the MB transform gives a correct alternative calculation of the asymptotic expansion of the exact integral Eq.(5), which we started from anyway. But our explicit separation of the expanded perturbative renormalon part (7) in connection with the MB method allows to visualize explicitly the renormalon cancellations order by order in (22). For completeness note that a very similar MB transform analysis can be performed for the tadpole graph renormalons, with similar cancellations which we do not display here.
One may alternatively understand perhaps more qualitatively the pole mass specific cancellations, by examining the asymptotic behaviour of the off-shell mass expression (3), first expanded in powers of $M^2/p^2$, and proceeding with the MB transform similarly to the above described procedure. [This is then a completely similar calculation than the one performed in details in ref.[9] for the $O(N)$ sigma model.] Skipping many details, we simply indicate sketchily that for the GN model, it would give coefficients of the $(M^2/p^2)^n$ terms with a form similar to Eq.(21), but with essentially the replacements $\mu^2 \rightarrow p^2; (1 + s)^{-1} \rightarrow \sum_i (1 + s + i)^{-1}, 0 \leq i \leq n - 1$. Those poles at $s = -1 - i$, of residues $\sim \Gamma[t - 1 - i]$, produce in turn poles at (integer) $t = 1, 2, ..n$. Again, the integral over $t$ may be cast into a Borel transform formally similar to (22), with a NP part and a perturbative part. However, in this off-shell case, the specific poles at $t_0 = 1, 2, ..$ within perturbative terms of order $(M^2/p^2)^n$, are cancelled by NP term of different order $n + t_0$. Since individual terms are singular, one cannot truncate the power expansion unless a definite integration contour prescription to avoid the poles is defined[9]. In other words, the separation between the NP and perturbative part in this $p^2 \neq M^2$ case is ambiguous, which means the non Borel summability of the perturbative series of the general two-point function (3) for arbitrary $p^2$, as obtained from a direct calculation, cf. remarks in section 3 (footnote 5). In contrast, for the pole mass any $(M^2/p^2)^n$ terms are replaced by 1, which “flatten” all orders of the $M^2/p^2$ expansion, so that the different cancellations of $t > 0$ poles now occur all at once.

5 Conclusion and perspectives for QCD?

In this paper we have exhibited in details the non-trivial cancellation of the perturbative IR renormalons of the GN mass gap at order $1/N$, implying the direct Borel summability of the genuine perturbative expansion, as defined by Eq.(17). Given the detailed cancellation mechanism, we are also confident that it should work similarly beyond $1/N$ order, though an explicit check has not been attempted. We should perhaps stress that the result Eq.(17) may be not a surprising one, as it could have been easily extracted from previous analysis of e.g. refs. [10, 6], if not explicitly displayed there. As motivated in introduction, our main purpose here was to illustrate in a calculable model how the NP contributions to the pole mass organize to eliminate completely the renormalon artifacts, even though the latter are unavoidable in the naive perturbation theory, similarly to the perturbative QCD quark pole masses.

It is thus tempting to speculate briefly on the possibility of NP renormalon cancellation in QCD similar to some extent to the mechanism discussed here. QCD in the massless
quark limit also has a mass gap, since the approximate chiral symmetry of the light quark sector is dynamically broken. Now if we assume that the dominant contribution to the NP pole mass is given by the first graph in Fig 1, with a gluon propagator (wavy line) dressed with massive, constituent quarks of mass $M_Q \sim \Lambda_{QCD} = \bar{\Lambda}$, one may expect this propagator to behave in the infrared in a way similar to Eq.(4). (Assuming also that such quark loops are complemented with appropriate QCD gauge sector contributions, so to match the correct beta function in a gauge invariant way, as indeed usually assumed in most QCD infrared renormalon issues[7]). We see no reason why such assumption on the NP behaviour would not be consistent with the usual perturbative behaviour, and in particular with the standard heavy quark pole mass renormalon picture[5], for which $M_Q \gg \bar{\Lambda}$. This is also irrespective of the fact that confinement in QCD ultimately makes the pole quark mass relevance somewhat elusive. What can be still theoretically relevant would be to have in this way a procedure to evaluate the NP contributions to the light constituent quark masses $M_Q \sim \bar{\Lambda}$ from first principles (and perhaps more interestingly for the NP order parameters related to chiral symmetry breaking). A detailed QCD analysis is obviously beyond the present scope and left for future work.

Let us finally mention that, indeed, there exist “direct” ways of modifying the asymptotic properties of the perturbative expansion of e.g. the mass gap, generically in asymptotically free models. It relies only on the purely perturbative information, but is based on a modification of the usual perturbative series. Such a method[14] appears to bypass the explicit cancellations of renormalons here exhibited, by directly removing the perturbative renormalon divergences, at least for adequate range of an (arbitrary) mass parameter values compatible with the chiral limit.

**Appendix: Asymptotic behaviour for $M_P \gg \Lambda$**

As mentioned in section 3 the use of the mass gap relation Eq.(2), strictly valid only at $\mathcal{O}(1/N)$ and in the chiral limit (Lagrangian mass $m \to 0$), plays in fact no particular role in the asymptotic properties Eq.(17), of the exact 1/N pole mass. Consider Eq.(5), but now for $M_P \gg \Lambda$ (which illustrates e.g. the massive theory case with $M_P \sim m \gg \Lambda$):

$$r_1 = \int d\phi [e^{-\frac{\phi}{2}} \cosh \frac{\phi}{2} - \frac{\phi}{2} (1 + \ln(M_P^2/\Lambda^2))^{-1}] (\ln(M_P^2/\Lambda^2) + \frac{\phi}{\tanh \frac{\phi}{2}})^{-1}$$

(24)

to be expanded in powers of $\ln^{-1}(M_P^2/\Lambda^2)$, where we used again $\phi = \ln[(\zeta + 1)/(\zeta - 1)]$. Each coefficient of such an expansion contains a logarithmic divergence, renormalizable by
subtraction (removing essentially the divergent \((1 + \phi)/2\) piece in the bracket of Eq. (24)).

We find finally that the renormalized series has the leading asymptotic behaviour

\[
\frac{r_1^{(as)}(M_P \gg \Lambda)}{\Lambda} \sim \sum_{p=0}^{\infty} (-1)^{p+1} \frac{p!}{\ln^{p+1} \left( \frac{M_P^2}{\Lambda^2} \right)}
\]

which agrees asymptotically with Eq. (17), provided \(M_P \sim \mu \gg \Lambda\), i.e. \(\ln(M_P^2/\Lambda^2) \sim 1/g\). The sign alternation in (25), leading to Borel summability, again makes the main difference with the behaviour obtained starting from the perturbative analysis of section 3.

References