THE VIRTUAL BLACK HOLE IN 2D QUANTUM GRAVITY
AND ITS RELEVANCE FOR THE S-MATRIX

DANIEL GRUMILLER∗

Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstr. 8-10
Vienna, A-1040, Austria

As shown recently 2d quantum gravity theories — including spherically reduced Einstein-gravity — after an exact path integral of its geometric part can be treated perturbatively in the loops of (scalar) matter. Obviously the classical mechanism of black hole formation should be contained in the tree approximation of the theory. This is shown to be the case for the scattering of two scalars through an intermediate state which by its effective black hole mass is identified as a “virtual black hole”. We discuss the lowest order tree vertex for minimally and non-minimally coupled scalars and find a non-trivial finite S-matrix for gravitational s-wave scattering in the latter case.

1. Introduction to first order gravity in two dimensions

Quantum gravity is beset with well-known conceptual problems. Probably the most challenging one is the dual rôle of geometric variables as fields which at the same time determine the local and global properties of the manifold on which they act. Due to this fact and because gravity is perturbatively non-renormalizable it is desirable to use non-perturbative methods (however, cf. the article of M. Reuter “Is Quantum Gravity Asymptotically Safe?” in this volume). Unfortunately, in \( d = 4 \) this is technically problematic.

Therefore, models in \( d = 2 \) are considered frequently in this context, most of which lack an important feature present in ordinary gravity: they contain no continuous physical degrees of freedom. One way to overcome this without leaving the comfortable realm of two dimensions is the inclusion of matter. We will restrict ourselves to the following class of actions:

\[
L_{dil} = \int d^2 x \sqrt{-g} \left[ R - U(X) (\nabla X)^2 + 2V(X) + F(X) (\nabla \phi)^2 \right].
\]

(1)

\( R \) denotes the 2d scalar curvature, \( X \) is the so-called dilaton field, \( \nabla \) the covariant derivative containing the Levi-Civita connection with respect to the metric \( g_{\mu\nu} \) and \( \phi \) is a massless scalar field. Adjusting the functions \( U(X) \), \( V(X) \) and \( F(X) \) properly, one obtains a variety of interesting models, the most prominent among them being the CGHS model\(^1\) and spherically reduced gravity\(^2\) (SRG). In this work we will focus on (originally four-dimensional) SRG.

∗e-mail: grumil@hep.itp.tuwien.ac.at
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The action (1) is locally and globally equivalent to a first order one\(^3,4\) depending on Cartan 1-forms \(e^\pm\) (we denote light-cone indices with \(\pm\)) and \(\omega\) (the abelian gauge structure of the two dimensional spin connection \(\omega^a{}_b = \epsilon_a{}^b \omega\) is used explicitly), the dilaton field \(X\), the auxiliary fields \(X^\pm\) and the Klein-Gordon field \(\phi\):

\[
L_{FO} = \int \left[ X^+(d - \omega) \wedge e^- + X^-(d + \omega) \wedge e^+ + Xd \wedge \omega - e^- \wedge e^+ V - F(X) d\phi \wedge \ast d\phi \right].
\] (2)

Actually, this equivalence holds for general dilaton theories. In the present spherically reduced case the “potential” in (2) becomes \(V = -1 - \frac{X^+ X^-}{2X}\). The coupling function \(F(X)\) will be chosen as a constant (minimally coupled case) or linear in \(X\) (non-minimally coupled case with proper \(s\)-wave factor). The latter case arises if one considers a minimally coupled massless scalar field in \(d = 4\), the factor \(X\) being a remnant of the \(4d\) measure \(\sqrt{-g^{(4)}} \propto |X| \sqrt{-g^{(2)}}\). Note that \(X\) is restricted to the half-line, which is why often alternatively the representation \(X = \exp(-2\Phi)\) is used with \(\Phi \in (-\infty, \infty)\).

2. Path integral quantization

The BRST analysis and path-integral quantization of models of the type (2) has been discussed for the minimally coupled case\(^5\) as well as for the more complicated non-minimally coupled case\(^6\). Using a convenient gauge for the Cartan variables (namely “temporal gauge”: \(\omega_0 = 0, e_0^+ = 0\) and \(e_0^- = 1\)) an exact integration of all geometric variables yields a non-local and non-polynomial action depending solely on the scalar field and on eventual external sources. We focus on the lowest order tree diagrams, i.e. on the (non-local) \(\phi^4\)-vertices. Thus, we use perturbation theory only in the matter sector, implicitly assuming that the typical energies of the scalar particles are small as compared to Planck energy. In addition to our local gauge fixing we had to impose certain boundary conditions. We chose for simplicity conditions yielding asymptotically Minkowski space-time.

For minimally coupled massless scalars we obtained a non-trivial vertex, but only a trivial result for the scattering amplitude: It was either divergent or – if certain regularity conditions were imposed by hand – it vanished (the black hole was “plugged” by these boundary conditions). For massive scalars we obtained a non-trivial result for the \(S\)-matrix\(^7\).

The more interesting non-minimally coupled case yielded two vertices even for massless particles (cf. fig. 1). The explicit expressions for the quantities \(V^{(4)}(x, y)_a\) and \(V^{(4)}(x, y)_b\) contain simple polynomials multiplied by a signum function depending of one coordinate pair (e.g. \(x^0, y^0\)) and they are local in the other pair. These properties are also present in the line element discussed in the subsequent section\(^6,8\).
3. The virtual black hole

After obtaining a (perturbative or exact) solution for the matter degrees of freedom one can reconstruct the geometry. We did this for the lowest order tree-graph solution and found in both scenarios (minimal and non-minimal coupling) an effect which we called “virtual black hole” (VBH). We will briefly discuss this effect now.

In all theories of the type (2) there exists a conserved quantity\(^9\), even in the presence of matter\(^4\). Its geometric part is essentially equivalent to the so-called “mass aspect function”, a quantity widely used in numerical calculations to signal an apparent horizon\(^10\). It has a discontinuity (a feature which is also present in the minimally coupled case) which is inherited by the effective line element\(^a\)

\[
(ds)^2 = 2drdu + \left(1 - \frac{2m(r,u)}{r} - a(r,u)r\right)(du)^2, \tag{3}
\]

with \(m(r,u) = \delta(u-u_0)\theta(r_0-r)m_0(r_0)\) and \(a(r,u) = \delta(u-u_0)\theta(r_0-r)a_0(r_0)\), \(m_0\) and \(a_0\) being smooth functions of \(r_0\). The \(\theta\)-function is responsible for the discontinuity: For \(r \to \infty\) neither the Schwarzschild term proportional to \(m\) nor the Rindler term proportional to \(a\) is present.

This discontinuity effect in the mass-aspect function has been called “virtual black hole”. It is also reflected in the scalar curvature \(R\), i.e. it is not merely an artifact of an unsuitable coordinate system. However, this VBH geometry should not be taken at face value, since the only observable in our context is the \(S\)-matrix.

4. Scattering amplitude

Using asymptotic \(s\)-waves for the scalar field one can construct the Fock space and calculate the \(S\) matrix element with ingoing modes \(q, q'\) and outgoing ones \(k, k'\)

\[
T(q, q'; k, k') = -\frac{i\hbar}{2(4\pi)^{1/2}kk'qq'|3/2}E^3\tilde{T} \tag{4}
\]

\(^a\)The outgoing Bondi-Sachs form \((u = t - r)\) of the line element is a direct consequence of the chosen gauge.
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with the total energy $E = q + q'$, the coupling constant $\kappa = 8\pi G_N$, 

$$
\tilde{T}(q,q';k,k') := \frac{1}{E^3} \left[ \Pi \ln \frac{\Pi^2}{E^6} + \frac{1}{\Pi} \sum_{p \in \{k,k',q,q'\}} p^2 \ln \frac{p^2}{E^2} \right. 
\left. \cdot \left( 3kk'qq' - \frac{1}{2} \sum_{r \neq p} \sum_{s \neq r,p} (r^2 s^2) \right) \right],
$$

(5)

and the momentum transfer function $\Pi = (k + k')(k - q)(k' - q)$. The interesting part of the scattering amplitude is encoded in the scale independent (!) factor $\tilde{T}$. This remarkably simple result has been discussed in more detail elsewhere$^{6,8}$. The idea that black holes must be considered in the $S$-matrix together with elementary matter fields has been put forward some time ago$^{11}$. Our approach has allowed for the first time to derive (rather than suppose) the existence of the black hole states in the quantum scattering matrix. So far, we were able to perform actual computations in the first non-trivial order only. The next order calculations which should yield an insight into the information paradox are in progress.

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**References**