Non-Abelian Thermal Large Gauge Transformations in 2+1 Dimensions

F. T. Brandt\textsuperscript{a}, Ashok Das\textsuperscript{b}, Gerald V. Dunne\textsuperscript{c}, J. Frenkel\textsuperscript{a} and J. C. Taylor\textsuperscript{d}

\textsuperscript{a} Instituto de Física, Universidade de São Paulo, São Paulo, SP 05315-970, BRAZIL
\textsuperscript{b} Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171, USA
\textsuperscript{c} Department of Physics, University of Connecticut, Storrs, CT 06269-3046, USA
\textsuperscript{d} Department of Applied Mathematics, and Theoretical Physics, University of Cambridge, Cambridge, UK

We discuss several different constructions of non-Abelian large gauge transformations at finite temperature. Pisarski’s ansatz with even winding number is related to Hopf mappings, and we present a simple new ansatz that has any integer winding number at finite temperature.

I. INTRODUCTION

The question of large gauge invariance at finite temperature has led to a lot of activity in recent years. The basic problem is that, while finite temperature perturbation theory is invariant order by order under small gauge transformations, it is not invariant, order by order, under large gauge transformations. The nontrivial periodicity conditions satisfied by finite temperature large gauge transformations mix all orders of perturbation theory. This phenomenon leads to complications in computing finite temperature effective actions when there is an anomalous symmetry such as parity that can produce anomalous terms in the effective action. This problem has mostly been studied in the context of finite temperature Chern-Simons theories, both in 0 + 1 and 2 + 1 dimensions [1–4]. In Chern-Simons theories this is a serious problem, because a non-Abelian Chern-Simons term shifts by a constant under a large gauge transformation. At zero temperature a consistent quantum theory can still be defined if the coefficient of the Chern-Simons term takes discrete values [5], and, remarkably, these discrete coefficients are precisely what is found for Chern-Simons terms that are induced by radiative quantum loops [6,7]. At finite temperature the induced coefficient is temperature dependent and so does not take discrete values, suggesting the possibility of a violation of large gauge invariance [8]. The resolution of this problem has now been understood when the large gauge transformation is essentially Abelian in the sense that it is the nontrivial winding of an Abelian field around the Euclidean time $S^1$ that produces the shift in the Chern-Simons term [1–4]. Very briefly, at finite temperature, the effective action consists of an infinite number of parity-violating terms, of which the Chern-Simons term is only the first, in such a way that the complete effective action is invariant under large gauge transformations, although a truncation of the effective action at any order leads to violation of this symmetry. However, this mechanism has only been explicitly verified for a very restricted type of Abelian large gauge transformations. The situation for a genuinely non-Abelian large gauge transformation, whose non-vanishing winding number comes from its mapping from $S^2 \times S^1$ into the non-Abelian gauge group, has not yet been fully understood. However, explicit computations of the finite temperature multi-leg amplitudes in non-Abelian theories indicate that the structure of the non-Abelian parity-odd parts is much richer at finite temperature than at zero temperature or in finite temperature Abelian theories [9].

Clearly, an important part of this entire discussion is to understand the properties of the finite temperature large gauge transformations themselves. Here one strikes immediately a conceptual difference between zero and nonzero temperature. For simplicity, we consider the gauge group to be $SU(2)$. The group manifold $SU(2)$ can be identified with $S^3$. At zero temperature, the winding can be thought of as resulting from a map of space-time to the group space, $S^3 \to S^1$. These windings take integer values since the homotopy group is $\pi_3(S^3) = \mathbb{Z}$. On the other hand, at finite temperature the space-time manifold becomes $S^2 \times S^1$, in the imaginary time formalism of thermal field theories. Thus, at finite temperature, homotopy groups are not sufficient to characterize these maps, since both $\pi_1(S^3)$ and $\pi_2(S^3)$ are trivial. This is because there can only be trivial maps between $S^n \to S^n$, when $m < n$. Therefore, it might appear naively that there can be no large gauge transformations at finite temperature. This naive expectation is not correct: indeed an explicit ansatz for finite temperature large gauge transformations has been presented by Pisarski [8]. However, this ansatz is restricted to even winding numbers only. Furthermore, it does not have a smooth zero temperature limit to a zero temperature large gauge transformation.

In this paper, we explore the geometric properties of these finite temperature large gauge transformations, and discuss several other explicit ansatzes. Some of these also have only even winding number, but we also present a very simple ansatz for a finite temperature large gauge transformation which has any integer winding number and which has a smooth zero T limit. In Section II, we briefly recall the zero temperature large gauge transformations. In Section III, we discuss Pisarski’s ansatz that leads to genuinely non-Abelian large gauge transformations with even
winding number. This construction is shown, in Section IV, to be related to Hopf maps. In Section V, we show how a coset construction, at finite temperature, gives only trivial winding. In Section VI, we show how one can enlarge the group space to have a nontrivial map from $S^2 \times S^1$. However, such a construction also leads to only even winding numbers. In Section VII, we present an alternate construction, borrowing from ideas in 0 + 1 dimensions, that leads to an arbitrary winding number. Section VIII contains some brief conclusions and we collect some useful formulae about winding numbers in an appendix.

II. ZERO TEMPERATURE

At zero temperature, a simple ansatz for an $SU(2)$ large gauge transformation is [10]

$$g(x) = \exp \left( \frac{m\pi i \vec{x} \cdot \vec{\sigma}}{\sqrt{x^2 + \lambda^2}} \right)$$

where $\vec{x} \in \mathbb{R}^3$, $\vec{\sigma}$ are the Pauli matrices, $m$ is an integer, and $\lambda$ is an arbitrary scale parameter. For later comparison, it is convenient to express $g$ as

$$g(x) = \cos \left( \frac{m\pi r}{\sqrt{r^2 + \lambda^2}} \right) 1 + i (\hat{x} \cdot \vec{\sigma}) \sin \left( \frac{m\pi r}{\sqrt{r^2 + \lambda^2}} \right)$$

where $r^2 = \vec{x}^2$, and $\hat{x} = \frac{\vec{x}}{r}$ is the three-dimensional unit position vector. Note that $g(r = 0) = 1$, $g(r = \infty) = \cos(m\pi) 1$

The winding number of an $SU(2)$ group element is

$$W[g] = \frac{1}{24\pi^2} \int d^3x \epsilon_{\mu\nu\lambda} \text{Tr} (g^{-1}_\mu gg^{-1}_\nu gg^{-1}_\lambda g)$$

It is a straightforward exercise to verify that, for the zero temperature ansatz (1), the winding number of $g$ is equal to the integer $m$:

$$W[g] = m$$

Geometrically, $W[g]$ is the number of times the map $g : S^3 \rightarrow S^3$ winds around the target manifold $SU(2) \sim S^3$ as the coordinates cover the base manifold $\mathbb{R}^3$ (which can be compactified to $S^3$ if $g$ has a limit that is independent of the angles at $r = \infty$). In other words, the integrand in (4) is just ($m$ times) the Jacobian involved in changing variables from Cartesian coordinates to the angular coordinates used to parameterize $S^3$ (see, e.g., [11]).

III. FINITE TEMPERATURE: PISARSKI’S ANSATZ

At finite temperature, in the imaginary time formulation, a gauge transformation $g(\vec{x}, t)$ must be strictly periodic [12] in the Euclidean time $t$, with period $\beta = \frac{1}{T}$ being the inverse temperature [13–15]:

$$g(t = 0) = g(t = \beta)$$

Since the coordinate manifold is now $S^2 \times S^1$ rather than $S^3$, it is natural to separate the $t$ dependence. Pisarski [8] proposed the following elegant ansatz for a finite temperature $SU(2)$ large gauge transformation:

$$g(\vec{x}, t) = \exp \left( \frac{2m\pi t}{\beta} \hat{n}(\vec{x}) \cdot \vec{\sigma} \right)$$

$$= \cos \left( \frac{2m\pi t}{\beta} \right) 1 + i (\hat{n} \cdot \vec{\sigma}) \sin \left( \frac{2m\pi t}{\beta} \right)$$

where $m$ is an integer, and $\hat{n}(\vec{x})$ is a three-component unit vector depending only on the two-component spatial coordinate vector $\vec{x} \in \mathbb{R}^2$. If we require that $\hat{n}(\vec{x})$ have an angle-independent limit at spatial infinity ($|\vec{x}| = \infty$), then
\( \mathbb{R}^3 \) is compactified to \( \mathbb{S}^2 \), and \( \hat{n} \) defines a map \( \hat{n} : \mathbb{S}^2 \rightarrow \mathbb{S}^2 \). A straightforward calculation using the ansatz (7) for \( g(\hat{x}, t) \) shows that the winding number (4) of \( g \) is related to the index of the map \( \hat{n} \) in the following simple manner:

\[
W[g] = 2m \, w[\hat{n}] \tag{8}
\]

where

\[
w[\hat{n}] = \frac{1}{8\pi} \int d^2 x \, e^{abc} \epsilon^{ij} \hat{n}^a \partial_i \hat{n}^b \partial_j \hat{n}^c \tag{9}
\]

This winding number of \( \hat{n} \) is itself an integer, which implies (as noted in [8]) that the winding number of \( g \) with the ansatz (7) is necessarily an even integer.

Explicit examples of \( \hat{n}(\bar{x}) \) with nontrivial integer windings as maps from \( \mathbb{S}^2 \) to \( \mathbb{S}^2 \) are provided by the \( CP^1 \) instantons [11]. For example, the map (here \( z = \frac{e^{i+ia^2}}{\lambda} \))

\[
\hat{n} = \frac{1}{1 + |z|^2} \begin{pmatrix} 2 \text{Re}(z) \\ 2 \text{Im}(z) \\ 1 - |z|^2 \end{pmatrix} \tag{10}
\]

has \( w[\hat{n}] = 1 \). Geometrically, this can simply be viewed as the stereographic projection connecting the unit sphere \( \mathbb{S}^2 \) to the plane. If the \( \mathbb{S}^2 \) is embedded into \( \mathbb{R}^3 \), then we can write this as \( \hat{n} = \hat{x} \).

Clearly, the ansatz (7) is manifestly periodic in Euclidean time, with period \( \beta \). Further, it gives a nonzero winding number (8). Thus, it is indeed a genuine non-Abelian finite temperature large gauge transformation. However, this construction also raises several questions. First, why are only even winding numbers allowed? Second, what is the geometrical interpretation of this construction in terms of mappings from \( \mathbb{S}^2 \times \mathbb{S}^1 \) to \( SU(2) \)? Third, why is it that this ansatz does not survive the zero temperature (\( \beta \rightarrow \infty \)) limit and reduce to the zero temperature large gauge transformation in (1)?

A partial answer to the first question can be given as follows. Notice that if we consider the two-dimensional instanton in (10) for which we can identify \( \hat{n} \) with the three-dimensional unit vector \( \hat{x} \), then the finite T ansatz (7) is very similar to the zero temperature ansatz (2). The only difference is that in the zero T case, as \( r \) goes from 0 to \( \infty \), the trigonometric factors go through \( m \) half cycles, while in the finite T case, as \( t \) goes from 0 to \( \beta \), the trigonometric factors go through \( m \) full cycles. This difference is forced by the strict periodicity condition at finite temperature, and means that the finite T ansatz is wrapping twice as often around \( SU(2) \). A deeper geometrical interpretation of this phenomenon is presented in the next section. Let us also point out here that Eq. (7), considered as a map from space-time to the group manifold, has no inverse, because \( \hat{n}(\bar{x}) \) shows that the winding number (4) of \( g \) is related to the index of the map \( \hat{n} \). Let us also point out here that Eq. (7), considered as a map from space-time to the group manifold, has no inverse, because \( \hat{n}(\bar{x}) \) shows that the winding number (4) of \( g \) is related to the index of the map \( \hat{n} \) in the following simple manner:

\[
H[\hat{n}] = \frac{1}{8\pi^2} \int d^3 x \epsilon^{ijk} a_i f_{jk} \tag{12}
\]

where \( a_j \) is an associated Abelian gauge field (connection).

### IV. RELATION TO HOPF MAPS

With every \( SU(2) \) group element, \( g : \mathbb{R}^3 \rightarrow SU(2) \), it is possible to associate a three-component unit vector \( \hat{N} \in \mathbb{S}^2 \) in such a way that the winding number of \( g \) (viewed as a map from \( \mathbb{S}^3 \rightarrow \mathbb{S}^1 \)) is equal to the Hopf index of \( \hat{N} \) (viewed as a map from \( \mathbb{S}^3 \rightarrow \mathbb{S}^2 \)). It is important not to confuse \( \hat{N}(\bar{x}) \) with \( \hat{n}(\bar{x}) \) of the previous section. \( \hat{N} \) maps from a three-dimensional base manifold into \( \mathbb{S}^2 \), while \( \hat{n} \) mapped from a two-dimensional base manifold into \( \mathbb{S}^2 \). This Hopf construction has been intensely studied recently in the physics literature in a wide variety of contexts ranging from knot solitons [16–18], to magnetic helicity [19], to zero modes of Abelian Dirac operators [20], to Abelian projections [21,22]. The specific relation between \( g \) and \( \hat{N} \) is

\[
\hat{N}^a = \frac{1}{2} \text{tr} (\sigma_3 \, g^{-1} \sigma_3 g) \tag{11}
\]

It is simple to verify that \( \hat{N}^2 = 1 \), so that \( \hat{N} \in \mathbb{S}^2 \). Algebraically, \( g \) is the local unitary transformation that diagonalizes \( \hat{N} \cdot \hat{\sigma} \) to its asymptotic value (at spatial infinity) of \( \sigma_3 \). The Hopf index of \( \hat{N} \) is

\[
H[\hat{N}] = \frac{1}{8\pi^2} \int d^3 x \epsilon^{ijk} a_i f_{jk} \tag{12}
\]
\[ a_j = -\frac{i}{2} tr (\sigma_3 g^{-1} \partial_j g) \]  

(13)

and \( f_{jk} = \partial_j a_k - \partial_k a_j \) is the corresponding Abelian field strength (curvature). Then it is straightforward to show that the Hopf index (12) of \( \hat{N} \) is equal to the winding number (4) of \( g \): 

\[ H[\hat{N}] = W[g] \]  

(14)

The geometrical interpretation of the Hopf index is as follows \([17,21]\). The pre-image of any fixed point \( \hat{N} \) on \( S^2 \) describes a closed loop in \( R^3 \). The Hopf index of \( \hat{N} \) is the linking number of the pre-image loops for any two points on \( S^2 \).

Now consider the zero and finite temperature large gauge transformations (1) and (7), respectively, in this Hopf map language. For the zero T group element in (1), the relation (11) leads to

\[
\hat{N}^a = \cos \left( \frac{2m\pi r}{\sqrt{r^2 + \lambda^2}} \right) \delta^a_3 + \left[ 1 - \cos \left( \frac{2m\pi r}{\sqrt{r^2 + \lambda^2}} \right) \right] \hat{x}^a \hat{x}^3 - \sin \left( \frac{2m\pi r}{\sqrt{r^2 + \lambda^2}} \right) \epsilon^{ab}_3 \hat{x}^b 
\]  

(15)

On the other hand, for the finite T group element in (7), the relation (11) leads to

\[
\hat{N}^a = \cos \left( \frac{4m\pi t}{\beta} \right) \delta^a_3 + \left[ 1 - \cos \left( \frac{4m\pi t}{\beta} \right) \right] \hat{n}^a \hat{n}^3 - \sin \left( \frac{4m\pi t}{\beta} \right) \epsilon^{ab}_3 \hat{n}^b 
\]  

(16)

If we consider the particular \( CP^1 \) instanton \( \hat{n} \) to be the embedding unit vector \( \hat{x} \) of \( S^2 \) into \( R^3 \), then we can compare the two maps \( \hat{N} \) in (15) and (16). The difference, once again, is that the trigonometric factors wrap twice as often in the finite T case. Thus, geometrically speaking, the corresponding pre-images link through one another twice as often, thereby explaining the fact that the winding number of \( g \) in (8) is an even integer.

V. COSET CONSTRUCTION

In this section, we consider an even simpler construction at finite temperature. Recall, from the studies of thermal Abelian transformations, that they have the general form

\[ U(\bar{x},t) = e^{i(2\pi m t \sigma_3 \Omega(t,\bar{x}))} = e^{2i\pi m t} e^{i\Omega(t,\bar{x})} \]  

(17)

where \( m \) is an integer and \( \Omega \) is periodic and we choose:

\[ \Omega(\bar{x}, t = 0) = \Omega(\bar{x}, t = \beta) = 0 \]  

(18)

With such a construction, the Abelian group element \( U(\bar{x},t) \) is manifestly periodic, and brings out, in a natural and intuitive manner, the windings of the gauge transformation around the circle along the time direction.

We can try to generalize this to \( SU(2) \) in the following manner. First, let us recall that \( SU(2) \) has an Abelian subgroup. Therefore, let us write an element of \( SU(2) \) as

\[ g(\bar{x}, t) = h(t) u(\bar{x}) = e^{2m\pi i \hat{\sigma} \cdot \hat{x}} u(\bar{x}) \]  

(19)

This is clearly periodic. Now choose the space part \( u(\bar{x}) \) of the group element to be

\[ u(\bar{x}) = \exp \left[ i \pi \hat{x} \cdot \hat{\sigma} f(\rho) \right] \]  

(20)

where \( \hat{x} \) is the two-dimensional spatial unit vector, \( \hat{\sigma} \) represents the two spatial Pauli matrices, and \( \rho \) is the two dimensional radial coordinate. Then, it is easily shown that the winding number, in this case, becomes

\[
W[g] = \frac{1}{24\pi^2} \int_0^\beta dt \int d^2 x \epsilon_{\mu\nu\lambda} Tr \left( g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\lambda g \right) 
= m \left[ \cos(2\pi f(0)) - \cos(2\pi f(\infty)) \right] 
\]  

(21)

Thus, if we choose
where $\lambda$ is an arbitrary length scale, then,

$$W[g] = m(1 - (-1)^n)$$

This shows that the winding number can be nonzero, if $n$ is odd. But then, once again, the winding number $W[g]$ is necessarily an even integer. The problem, however, lies in the fact that the group element has the explicit form

$$g(\bar{x}, t) = e^{2m\pi i \hat{\bar{x}} \cdot \hat{\bar{\sigma}}_3} (\cos \pi f(\rho) + i \hat{\bar{x}} \cdot \hat{\bar{\sigma}} \sin \pi f(\rho))$$

This shows that, with the above choice of $f(\rho)$, the group element is independent of angles at spatial infinity, $\rho \to \infty$, for any fixed $t$, only if $n$ is an even integer. Therefore, while for even $n$ we can construct an acceptable group element with the required isotropy, the corresponding winding number is trivial. On the other hand, for odd $n$, the group element does give a nontrivial winding number (albeit even), but the group element does not possess the required isotropy properties. It is worth noting here that, for odd $n$, even though the group element is not isotropic at spatial infinity, $g^{-1} \partial_\xi g$ is, where $\xi$ is the polar angle. An alternate, simple way to see the vanishing of the winding number is to use formula (A5).

VI. ENLARGING THE GROUP MANIFOLD

The failure of the simplistic construction of the previous section is due to a mismatch between the base and the target manifolds at finite temperature. To see this more explicitly, note that an element of $SU(2)$ can be parameterized as

$$g(\bar{x}, t) = \exp \left( i \theta \hat{N} \cdot \hat{\bar{\sigma}} \right)$$

where $\theta = \theta(\bar{x}, t)$, and $\hat{N} = \hat{N}(\bar{x}, t)$ is a three-dimensional unit vector. (For example, the ansatz of Eq. (7) is a special case where the $t$ and $\bar{x}$ dependence is separated: $\theta(\bar{x}, t) = \theta(t)$, and $\hat{N}(\bar{x}, t) = \hat{n}(\bar{x})$.) Furthermore, $0 \leq \theta \leq \pi$ and the three components of the unit vector $\hat{N}$ can be traded for two angular variables, say $\psi, \phi$ with $0 \leq \psi \leq \pi$ and $0 \leq \phi \leq 2\pi$. Thus, an arbitrary element of $SU(2)$ can be parameterized in terms of three angles

$$g(\bar{x}, t) = g(\theta, \psi, \phi) , \quad 0 \leq \theta, \psi \leq \pi; 0 \leq \phi \leq 2\pi$$

where the coordinate dependence is contained in the angular parameters. This brings out explicitly the identification of $SU(2)$ with $S^3$ (of course, we do not need to worry about the center of the group).

In contrast, the space-time manifold, at finite temperature, is labeled by $(\bar{x}, t)$, where $0 \leq \frac{2\pi t}{\beta} \leq 2\pi$ defines the $S^1$. For the spatial coordinates we can use polar coordinates $(\rho, \xi)$ where $0 \leq \xi \leq 2\pi$. The radial coordinate $\rho$ can further be identified with an angular variable through a stereographic projection as

$$\frac{\rho}{\lambda} = 2 \tan \frac{\xi}{2}$$

so that we can describe the two dimensional space by $(\xi, \xi)$ with $0 \leq \xi \leq \pi$ and $0 \leq \xi \leq 2\pi ((\xi, \xi)$, for example, can be thought of as the polar angles of (10)). This is the $S^2$ associated with the spatial manifold. Thus, we can also represent the group element as

$$g = g(t, \xi)$$

and the requirement of asymptotic isotropy can now be written as

$$g(t, \xi = \pi, \xi) = g_0(t)$$

The difficulty for the maps from the space-time manifold to the group manifold is now clear. Namely, the parameters of the space-time manifold, $(\tau, \zeta, \xi)$, have ranges that are different from those in the group manifold, $(\theta, \psi, \phi)$; specifically, the coordinate parameter $\tau$ ranges from 0 to $2\pi$, while the group parameter $\theta$ ranges from 0 to $\pi$. Therefore, for a map with nontrivial winding to exist, we must somehow enlarge the group space. For example, if we enlarge the parameter space of the group by defining

$$f(\rho) = \frac{n\rho}{2\sqrt{\rho^2 + \lambda^2}}$$

(VI.4)
\[
\tilde{g}(\theta, \psi, \phi) = \begin{cases} 
  g(\theta, \psi, \phi) & \text{for } 0 \leq \theta \leq \pi \\
  g(2\pi - \theta, \pi - \psi, \pi + \phi) & \text{for } \pi \leq \theta \leq 2\pi
\end{cases}
\] (30)

then, the ranges of the parameters in the group space as well as the space-time manifold will match and a meaningful map may exist.

In fact, with this enlargement, the winding number can be easily calculated to be

\[
W[\tilde{g}] = \frac{1}{24\pi^2} \int_0^\beta dt \int d^2x \epsilon_{\mu\nu\lambda} \text{Tr} \left( \tilde{g}^{-1} \partial_\mu \tilde{g}^{-1} \partial_\nu \tilde{g}^{-1} \partial_\lambda \tilde{g} \right)
\]

\[
= \frac{1}{2\pi^2} \int_0^\beta dt \int_0^\pi d\xi \int_0^{2\pi} d\xi \sin^2 \theta \sin \psi \left| \frac{\partial(\theta, \psi, \phi)}{\partial(t, \xi, \xi)} \right|
\]

\[
= \frac{1}{2\pi^2} \int_0^{2N\pi} d\theta \sin^2 \theta \int_0^\pi d\psi \sin \psi \int_0^{2N\pi} d\phi
\] (31)

Here, we have used the periodicity relations of the form

\[
\theta(2\pi, \zeta, \xi) = \theta(0, \zeta, \xi) + 2N\pi
\]

\[
\psi(t, 0, \xi) = 0, \quad \psi(t, \pi, \xi) = \pi
\]

\[
\xi(t, \zeta, 2\pi) = \xi(t, \zeta, 0) + 2N\pi
\] (32)

Therefore, we see that the enlargement of the parameters in the group manifold does allow us to construct a group element that leads to a nontrivial winding. In fact, it leads to an even winding number much like the one in Eq.(7).

VII. CONSTRUCTION FOR ARBITRARY WINDING

While the previous sections explain why the ansätze (7) and (25), (30) lead to even winding numbers, these ansätze still do not accommodate any integer winding number, nor do they have a smooth zero temperature limit. In this section, we present a construction that solves both these problems. This new ansatz is extremely simple, and is motivated by insights from the 0 + 1 dimensional models.

In 0 + 1 dimension, at zero temperature, an Abelian large gauge transformation can be written as

\[
U_{(0+1)}(t) = e^{i\Omega(t)}
\] (33)

where

\[
\Omega(t) = 2m \arctan(t) = -im \log \left( \frac{1 + it}{1 - it} \right)
\] (34)

The coefficient \(m\) is required to be an integer so that the Abelian group element

\[
U_{(0+1)}(t) = \left( \frac{1 + it}{1 - it} \right)^m
\] (35)

6
is analytic in $t$. This is why the winding number

$$W[U] = \frac{1}{2\pi} \int dt \frac{d\Omega}{dt} = m$$

is an integer. Now, to extend this to finite temperature, we can compactify the time direction with the redefinition of variables,

$$t \rightarrow \tan \frac{\pi t}{\beta}$$

so that the transformed time spans $-\beta/2 \leq t \leq \beta/2$. This changes the gauge transformation parameter $\Omega$ in (34) to

$$\Omega = -im \log \left( \frac{1 + i \tan \frac{\pi t}{\beta}}{1 - i \tan \frac{\pi t}{\beta}} \right) = \frac{2m\pi t}{\beta}$$

Then $U = \exp(2m\pi it/\beta)$ is precisely the familiar thermal Abelian large gauge transformation. In this compactified language, $m$ must be an integer in order to preserve the correct periodicity properties of matter fields to which the Abelian gauge field is coupled.

It is straightforward to generalize this idea to a non-Abelian gauge group in $2 + 1$ dimensions. For example, for $SU(2)$ we take the zero temperature ansatz (1) and map $x^3 = t \rightarrow \tan(\pi t/\beta)$ as in (37). This suggests the following ansatz:

$$g(\bar{x}, t) = \exp \left( m\pi i \left[ \frac{\frac{2}{\beta} \tan(\frac{\pi t}{\beta}) \sigma_3 + \bar{x} \cdot \bar{\sigma}}{\sqrt{\frac{2}{\pi t} \tan^2 \frac{\pi t}{\beta} + \bar{x}^2 + \lambda^2}} \right] \right)$$

where $m$ is an integer and $\lambda$ is an arbitrary length scale.

The Euclidean time axis has been compactified to $-\beta/2 \leq t \leq \beta/2$, and this group element is clearly periodic on this interval:

$$g(\bar{x}, t = -\frac{\beta}{2}) = g(\bar{x}, t = \frac{\beta}{2})$$

Furthermore, in the zero temperature limit, $\beta \rightarrow \infty$, the time coordinate covers the full range from $-\infty$ to $+\infty$, and the group element (39) reduces smoothly to the zero temperature large gauge transformation (1). It is a straightforward exercise to check that the winding number of the gauge transformation (39) is

$$W[g] = \frac{1}{24\pi^2} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt \int d^2x \epsilon_{\mu\nu\lambda} \text{Tr} \left( g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\lambda g \right) = m$$

which can be any arbitrary integer. So the winding number is nontrivial, and is not restricted to just even integers. This result can be seen in two different ways. First, it is worth pointing out that the nontrivial contribution, in an explicit calculation of the integrals, comes from the radial surface rather than the temporal direction. Alternatively, a very simple way to see that the winding number is $m$ is to note that the finite temperature ansatz is obtained from the zero temperature ansatz through a coordinate redefinition and, since the winding number is the integral of a three form, $W[g] = \frac{1}{24\pi^2} \int \text{Tr}((g^{-1} dg)^3)$, it is invariant under this coordinate redefinition. We, therefore, conclude that the simple ansatz (39) leads to a general non-Abelian thermal large gauge transformation.

Our ansatz in (39) is also partly motivated by the well-known form of the Harrington-Shepard caloron solutions [23,24] which are instantons in 4 dimensional Yang-Mills theory that are periodic in Euclidean time. These also have a trigonometrically compactified time coordinate, but the calorons are much more complicated than the finite temperature large gauge transformations constructed in (39) because the calorons are required to satisfy the non-trivial self-dual Yang-Mills equations after compactification. The large gauge transformation group element $g$ need not satisfy any particular differential equations; it simply is required to have a nontrivial winding number.

It is worth comparing the nature of the explicit ansatzes in Eqs. (7) and (39), at least for even winding numbers. Let us note the following properties of the ansatz in Eq. (7). For any fixed $t$, this group element defines a two sphere,
S^2, while for any fixed coordinate, it defines a circle, S^1. On the other hand, for even m, say m = 2, we can write the ansatz of Eq. (39) also in a similar form, namely,
\[ g(\bar{x}, t) = \exp \left( 2\pi i \theta \hat{\theta} \cdot \vec{\sigma} \right) \]
where \( \theta \) is the magnitude and \( \hat{\theta} \) the unit vector for the three-vector:
\[ \hat{\theta} = \left( \bar{x}, \frac{\beta \tan \pi t}{\beta} \right) \]
\[ \theta = \left( 1 - \frac{\bar{x}^2 \tan^2 \pi t}{\beta^2} + \bar{x}^2 + \lambda^2 \right)^{1/2} \]
Clearly, \( 0 \leq \theta \leq 1 \) and this looks qualitatively similar to the ansatz in Eq. (7). However, in the present case, \( \hat{\theta} = \hat{\theta}_a(\bar{x}, t) \) depends on both the spatial and temporal coordinates, while the unit vectors \( \hat{n}(\bar{x}) \), in Eq. (7) depend only on the spatial coordinates \( \bar{x} \). Also, although for any fixed \( t \), the present ansatz describes a two sphere, it does not describe a circle for any fixed spatial coordinates. In fact, note that \( \theta \) attains its maximum value 1 only at the points \( t = \pm \beta^2 \) or \( r = \infty \). Finally, we note that the construction in (39) is not completely analytic in the sense that the second derivative, with respect to \( t \), of the group element has a discontinuity at \( t = \beta/2 \). This non-analyticity may be the price one has to pay in order to get odd values for the winding number.

VIII. CONCLUSION

We conclude briefly by commenting that we have presented a very simple ansatz (39) for finite temperature large gauge transformations in SU(2). The generalization to other compact gauge groups is straightforward, just like at zero temperature. Our ansatz has a smooth zero temperature limit, and can accommodate any integer winding number. We have also discussed the properties of other ansätze for finite temperature large gauge transformations, and given several complementary geometric interpretations of why the ansatz (7) only produces even winding number. Given our ansatz, it will be interesting to study how the 2 + 1 dimensional parity-odd effective action responds to these genuinely non-Abelian finite temperature large gauge transformations.

Acknowledgment: AD would like to thank Prof. S. Okubo for many helpful discussions, and GD thanks R. MacKenzie for helpful correspondence. This work was supported in part by US DOE grant numbers DE-FG02-92ER40716, DE-FG-02-91ER40685 and by CNPq (Brasil).

APPENDIX A: SOME PROPERTIES OF LARGE GAUGE TRANSFORMATIONS:

In this appendix, we collect some formulae involving the winding numbers that simplify some of the calculations. Given a gauge transformation, \( g \), the winding number is defined as
\[ W[g] = \frac{1}{24\pi^2} \int d^3x \text{Tr} \epsilon_{\mu\nu\lambda} g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\lambda g \]
(A1)
A small gauge transformation is one for which the winding number vanishes (namely, it is contractible to identity), while the ones with a nontrivial winding number are known as large gauge transformations.

Winding numbers are additive, namely, if \( f = gh \), then (with appropriate asymptotic fall off),
(A2)
This can be seen in a simple manner as follows. Let us define
\[ X_\mu = g^{-1} \partial_\mu g, \quad Y_\mu = \partial_\mu hh^{-1}, \quad Z_\mu = f^{-1} \partial_\mu f = h^{-1}(X_\mu + Y_\mu)h \]
(A3)
It can now be easily seen that
\[ \epsilon_{\mu\nu\lambda} \text{Tr} (Z_\mu Z_\nu Z_\lambda) = \epsilon_{\mu\nu\lambda} \text{Tr} [X_\mu X_\nu X_\lambda + Y_\mu Y_\nu Y_\lambda - 3\partial_\lambda (X_\mu Y_\nu)] \]
(A4)
It follows now that, if $X_\mu, Y_\mu$ vanish sufficiently rapidly for asymptotic distances, the third term would vanish upon integration and we have

$$W[gh] = W[g] + W[h]$$

(A5)

This also implies that

$$W[g^{-1}] = -W[g]$$

(A6)

Let us next define gauge transformations belonging to a class as a set of gauge transformations, which can be continuously deformed to one another. Thus, for example, two transformations, $g_1, g_2$ belong to the same class, if there exists a set of gauge transformations (suppressing the dependence on coordinates)

$$g(a; \bar{x}, t) \equiv g(a)$$

(A7)

depending smoothly on a parameter ‘$a$’ (which may in reality stand for a set of parameters), such that

$$g(a_1) = g_1, \quad g(a_2) = g_2$$

(A8)

The class of small gauge transformations, then, corresponds to a special set of transformations, with the property that there exists a parameter $a_0$ for which

$$g(a_0) = 1$$

(A9)

Given a set of gauge transformations, $g(a)$, let us construct from these a one parameter family of gauge transformations as

$$f(a_1, a) = g^{-1}(a_1)g(a)$$

(A10)

where we assume that $a_1$ has a fixed value and $a$ is variable. Then, clearly,

$$f(a_1, a_1) = 1$$

(A11)

Consequently, it follows that $f(a_1, a)$ defines a class of small gauge transformations (for which the winding number vanishes). It follows now that

$$W[g(a)] = W[g(a_1)]$$

(A12)

Namely, every member of the set $g(a)$ belonging to a class of large gauge transformations have the same winding number and they differ from one another only by small gauge transformations.

Let us define (in connection with a $2 + 1$ dimensional thermal theory)

$$\rho^2 = x^2 + y^2, \quad r^2 = \rho^2 + t^2$$

(A13)

Then, as we have seen, for a fixed time, asymptotic isotropy implies

$$g(\bar{x}, t) \to g_0(t) \quad \text{as} \quad \rho \to \infty$$

(A14)

Since $g_0(t)$ is periodic, it describes a map from a circle to $SU(2)$ ($S^3$). Such a map is trivial since a circle on $S^3$ is contractible to a point. Therefore, $g_0(t)$ represents a small gauge transformation (An alternate way of seeing this is to note that it depends only on the time coordinate and, consequently, the winding number must vanish.). It follows that we can always define a new transformation

$$\tilde{g}(\bar{x}, t) = g_0^{-1}(t)g(\bar{x}, t)$$

(A15)

which belongs to the same class and has the simpler asymptotic form

$$\tilde{g}(\bar{x}, t) \to 1, \quad \text{as} \quad \rho \to \infty$$

(A16)

Let us next look at this transformation at a fixed time, say $t_0$, namely, $\tilde{g}(\bar{x}, t_0)$. This defines a map from $S^2$ to the group $SU(2)$ ($S^3$). This is also a trivial map (since it does not depend on the time coordinate, the winding number is zero). Therefore, we can define a gauge transformation
which will be in the same class as \( g, \tilde{g} \). Furthermore, it will have the property that

\[
\tilde{g}(\bar{x}, t) = \tilde{g}^{-1}(\bar{x}, t_0)\tilde{g}(\bar{x}, t)
\]

(A17)

This can be thought of as the generalization of the boundary condition at zero temperature to finite temperature. Namely, at zero temperature, the boundary condition corresponds to choosing \( \tilde{g} \rightarrow 1 \) as \( r \rightarrow \infty \). At finite temperature, on the other hand, we can think of the boundary of space-time to be at \( t = t_0 \) and \( \rho \rightarrow \infty \).

\[
\tilde{g}(\bar{x}, t_0) = 1, \quad \tilde{g}(\bar{x}, t) \rightarrow 1 \quad \text{as} \quad \rho \rightarrow \infty
\]

(A18)

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[12] Actually, in the presence of just adjoint fields, \( g \) can be periodic up to multiplication by an element of the center of the gauge group, which for \( SU(2) \) would mean \( g \) being either periodic or anti-periodic. This has interesting implications for monopoles in 2 + 1 dimensions: R. D. Pisarski, “Magnetic monopoles in topologically massive gauge theory”, Phys. Rev. D 34 (1986) 385. But with general fields such as fundamental matter fields, \( g \) must be strictly periodic.