Cosmic Strings in a Braneworld Theory with Metastable Gravitons

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Abstract

If the graviton possesses an arbitrarily small (but nonvanishing) mass, perturbation theory implies that cosmic strings have a nonzero Newtonian potential. Nevertheless in Einstein gravity, where the graviton is strictly massless, the Newtonian potential of a cosmic string vanishes. This discrepancy is an example of the van Dam–Veltman–Zakharov (VDVZ) discontinuity. We present a solution for the metric around a cosmic string in a braneworld theory with a metastable four-dimensional graviton. This theory possesses those features that yield a VDVZ discontinuity in massive gravity, but nevertheless is generally covariant and classically self-consistent. Although the cosmic string in this theory supports a nontrivial Newtonian potential far from the cosmic string, one can recover the Einstein solution in a region near the cosmic string. That latter region grows proportionally to the evaporation length of the graviton (analogous to an inverse graviton mass), suggesting the lack of a VDVZ discontinuity in this theory. Moreover, the presence of scale dependent structure in the metric may have consequences for the search for cosmic strings through gravitational lensing techniques.

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General relativity is a theory of gravitation that supports a massless graviton with two degrees of freedom. However, if one were to describe gravity with a massive tensor field, general covariance is lost and the graviton would possess five degrees of freedom. In the limit of vanishing mass, these five degrees of freedom may be decomposed into a massless tensor (the graviton), a massless vector (a graviphoton which decouples from any conserved matter source) and a massless scalar. This massless scalar persists as an extra degree of freedom in all regimes of the theory. Thus, a massive gravity theory is distinct from Einstein gravity, even in the limit where the graviton mass vanishes. This discrepancy is a formulation of the van Dam–Veltman–Zakharov (VDVZ) discontinuity [1,2].

The most accessible physical consequence of the VDVZ discontinuity is the gravitational field of a star or other compact, spherically symmetric source. The ratio of the strength of the static (Newtonian) potential to that of the gravitomagnetic potential is different for Einstein gravity compared to massive gravity, even in the massless limit. Indeed the ratio is altered by a factor of order unity. Thus, such effects as light deflection by a star or perihelion precession of an orbiting body would be affected significantly if the graviton had even an infinitesimal mass.

This discrepancy appears for the gravitational field of any compact object. An even more dramatic example of the VDVZ discontinuity occurs for a cosmic string. A cosmic string has no static potential in Einstein gravity; however, the same does not hold for a cosmic string in massive tensor gravity. One can see why using the momentum space perturbative amplitudes for one-graviton exchange between two sources $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$:

\[
V_{\text{massless}}(q^2) \sim -\frac{1}{M_P^2} \frac{1}{q^2} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T^\alpha_\alpha \right) \tilde{T}^{\mu\nu},
\]

\[
V_{\text{massive}}(q^2) \sim -\frac{1}{M_P^2} \frac{1}{q^2 + m^2} \left( T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} T^\alpha_\alpha \right) \tilde{T}^{\mu\nu}.
\]

The potential between a cosmic string with $T_{\mu\nu} = \text{diag}(T^2, -T^2, 0, 0)$ and a test particle with $\tilde{T}_{\mu\nu} = \text{diag}(2\tilde{M}^2, 0, 0, 0)$ is

\[
V_{\text{massless}} = 0, \quad V_{\text{massive}} \sim -\frac{T^2 \tilde{M}}{M_P^2} \ln r,
\]

where the last expression is taken in the limit $m \to 0$. Thus in a massive gravity theory, we expect a cosmic string to attract a static test particle, whereas in general relativity, no such attraction occurs. The attraction in the massive case can be attributed to the exchange of the remnant light scalar mode that comes from the decomposition of the massive graviton modes in the massless limit.
Nevertheless, the presence of the VDVZ discontinuity is more subtle than just described. Vainshtein suggests that the discontinuity is derived from only the lowest order, tree-level approximation and that this discontinuity does not persist in the full classical theory [3]. However, doubts remain [4] since no self-consistent theory of massive tensor gravity exists. One can shed light on the issue of nonperturbative continuity versus perturbative discontinuity by studying a recent class of braneworld theories\(^1\) with a metastable four-dimensional graviton [8–10]. The metastable graviton has the same tensor structure as that for a massive graviton and perturbatively has the same VDVZ problem in the limit that the graviton tenuousness vanishes. In this model the momentum space perturbative amplitude for one-graviton exchange is

\[ V(q^2) \sim -\frac{1}{M_p^2 q^2 + qr_0^{-1}} \left( T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} T^\alpha_{\alpha} \right) \tilde{T}^{\mu\nu}, \tag{1.4} \]

where the scale \(r_0\) is scale over which the graviton evaporates off the brane. But unlike a massive gravity theory, this braneworld model provides a self-consistent, generally covariant environment in which to address the nonperturbative solutions in the limit as \(r_0 \to \infty\). Indeed, exact cosmological solutions [11] in this theory already suggest that there is no VDVZ discontinuity at the nonperturbative classical level [12].

We would like to continue this program and investigate the gravitational field of compact objects in the same braneworld theory with a metastable four-dimensional graviton. In this regard, one would ideally like to identify the nonperturbative metric of a spherical, Schwarzschild-like source. That problem, however, possess considerable, though not insuperable, computational difficulties.

Instead, we investigate the metric of a cosmic string as a close alternative formulation of the VDVZ problem for a compact source. The advantage of this system is its relative simplicity, as well as the clarity with which the VDVZ discontinuity manifests itself. After laying out the framework in which the problem is phrased, we identify various regimes where one can linearize the cosmic string metric. We then argue that these regimes may be smoothly matched onto each other in the fully nonlinear solution, subject to potential global restrictions. The resulting cosmic string metric indicates there is no discontinuity in the fully nonperturbative theory. It also provides an understanding as to how different phases appear in different regions near and far away from the string source. We conclude with some comments regarding the consequences of this solution.

\(^1\)There has been a recent revival of interest in the VDVZ discontinuity in the context of braneworld theories. These studies have focused on variations of the Randall–Sundrum braneworld scenario where the brane tension is slightly detuned from the bulk cosmological constant. The localized four-dimensional graviton acquires a small mass, allowing one to study the VDVZ problem in an effective massive four-dimensional gravity theory. For some examples of such work, see [5–7].
II. THE SOLUTION

A. Preliminaries

We wish to address the issues raised in the previous section using a braneworld theory of gravity with an infinite volume bulk and a metastable four-dimensional graviton [8]. Consider a four-dimensional braneworld embedded in a five-dimensional spacetime. The bulk is empty; all energy-momentum is isolated on the brane. The action is

\[ S_{(5)} = -\frac{1}{2} M^3 \int d^5x \sqrt{|g|} \tilde{R} + \int d^4x \sqrt{-g^{(4)}} L_m . \]  

The quantity \( M \) is the fundamental five-dimensional Planck scale. The first term in Eq. (2.1) corresponds to the Einstein-Hilbert action in five dimensions for a five-dimensional metric \( g_{AB} \) (bulk metric) with Ricci scalar \( R \). In addition, we consider an intrinsic curvature term which is generally induced by radiative corrections by the matter density on the brane [8]:

\[ -\frac{1}{2} M_P^2 \int d^4x \sqrt{-g^{(4)}} R^{(4)} . \]  

Here, \( M_P \) is the observed four-dimensional Planck scale (see [8–10] for details). Similarly, Eq. (2.2) is the Einstein-Hilbert action for the induced metric \( g_{\mu\nu}^{(4)} \) on the brane, \( R^{(4)} \) being its scalar curvature. The induced metric is

\[ g_{\mu\nu}^{(4)} = \partial_\mu X^A \partial_\nu X^B g_{AB} , \]

where \( X^A(x^\mu) \) represents the coordinates of an event on the brane labeled by \( x^\mu \).

We wish to find the spacetime around a perfectly straight, infinitely thin cosmic string. In this context, the following coordinate choice is useful:

\[ ds^2 = N^2(r, z)(dt^2 - dx^2) - A^2(r, z)dr^2 - B^2(r, z)dz^2 - r^2 \sin^2 z \, d\phi^2 , \]

where the string is located at \( r = 0 \) for all \((t, x)\). These coordinates are depicted in Fig. 1. If spacetime were flat (i.e., \( N = A = 1, B = r \)), we would choose the brane to be located at \( z = \frac{\pi}{2} \). In general, one may choose a set of coordinates such that this condition holds true even when spacetime is not flat. However, we will find it useful to choose a set of coordinates with a slightly less stringent constraint. We choose the brane to be located at \( z = \frac{\pi \alpha}{2} \), where the parameter \( \alpha \) is to be specified by the brane boundary conditions.

Assuming the cosmic string dominates the energy-momentum content of the spacetime, we ignore the matter effects of the brane itself, except through the intrinsic curvature term.
FIG. 1. A schematic representation of a spatial slice through a cosmic string located at $A$. The coordinate $x$ along the cosmic string is suppressed. The coordinate $r$ represents the 3-dimensional distance from the cosmic string $A$, while the coordinate $z$ denotes the polar angle from the vertical axis. In the weak gravity limit, the braneworld is the horizontal plane, $z = \frac{\pi}{2}$. The coordinate $\phi$ is the azimuthal coordinate. Note that everywhere except at the cosmic string, the unit vector in the direction of the $z$-coordinate extends perpendicularly from the brane into the bulk.

Eq. (2.2). Using the coordinate system specified by Eq. (2.4), the energy-momentum of the system is

$$T_{tt} = -T_{xx} = N^2(r, z) \frac{T^2 \delta(r)}{2\pi A(r, z) B(r, z) r \sin z},$$

(2.5)

where the parameter $T$ denotes the string tension and all other components of the energy-momentum tensor are zero. The Einstein equations dictated by the action Eqs. (2.1–2.2) is

$$\frac{1}{2r_0} G_{AB} + \frac{1}{B(r, z)} \delta \left( z - \frac{\pi\alpha}{2} \right) G^{(4)}_{AB} = \frac{1}{M_p^2} T_{AB},$$

(2.6)

where $G_{AB}$ is the five-dimensional Einstein tensor, $G^{(4)}_{AB}$ is the induced four-dimensional Einstein tensor induced on the brane, $T_{AB}$ is the energy-momentum on the brane Eq. (2.5), and we have defined a crossover scale

$$r_0 = \frac{M_p^2}{2 M^3}.$$  

(2.7)

This scale characterizes that distance over which metric fluctuations propagating on the brane dissipate into the bulk [8].

We assume a $\mathcal{Z}_2$–symmetric brane across $z = \frac{\pi\alpha}{2}$. Under this circumstance, one may solve Eqs. (2.5–2.6) by solving $G_{AB} = 0$ in the bulk, i.e., when $z < \frac{\pi\alpha}{2}$ and $r \neq 0$, such that the following brane boundary conditions apply at $z = \frac{\pi\alpha}{2}$:

Throughout this paper, we define the distributional $\delta(x)$ of the variable $x$, such that given any well-behaved function $f(x)$, $\int dx \, \delta(x) f(x) = f(0)$.
\[
\frac{1}{r_0} \frac{N_z}{N} = -\frac{B}{A^2} \left[ \frac{1}{2} \frac{N_r^2}{N^2} + \frac{1}{r} \frac{N_r}{N} \right] + \frac{\sqrt{1 - \beta^2}}{r_0 \beta} \\
\frac{1}{r_0} \frac{A_z}{A} = -\frac{B}{A^2} \left[ \frac{N_r}{N^2} - \frac{1}{2} \frac{N_r^2}{N^2} - \frac{N_r}{N} \frac{A_r}{A} - \frac{1}{r} \frac{A_r}{A} \right]
\]

(2.8)

where we have defined \( \beta = \sin \frac{\pi \alpha}{2} \) and where the subscript represents partial differentiation with respect to the corresponding coordinate. These boundary conditions are generated by the intrinsic curvature term induced by the action Eq. (2.2). We also impose boundary conditions to ensure continuity of the metric and its derivatives at \( z = 0 \).

We wish to find the full five-dimensional spacetime metric induced by a thin cosmic string situated within the braneworld. The problem defined by Eqs. (2.5–2.6) is dependent only on the scale \( r_0 \) and the dimensionless parameter \( T^2/M_p^2 \). We are interested in the problem when \( r_0 \to \infty \) with all other parameters held fixed. Since \( r_0 \) represents the only scale in the problem, this statement implies we are interested in the system when \( r \ll r_0 \) with \( T^2/M_p^2 \) fixed.

### B. The Einstein solution

Before we attempt to solve the full five-dimensional problem given by Eqs. (2.5–2.6) and Eq. (2.8), it is useful to review the cosmic string solution in simply four-dimensional Einstein gravity \([13,14]\). For a cosmic string with energy momentum Eq. (2.5), the exact metric for a cosmic string may be represented by the line element:

\[
d s^2 = d t^2 - d x^2 - \left( 1 - \frac{T^2}{2 \pi M_p^2} \right)^{-2} d r^2 - r^2 d \phi^2 .
\]

(2.9)

This represents a flat space with a deficit angle \( T^2/M_p^2 \). Thus, there is no Newtonian potential between a cosmic string and a static test particle. However, a test particle (massive or massless) suffers an azimuthal deflection of \( T^2/2\pi M_p^2 \) when scattered around the cosmic string. With a different coordinate choice, the line element can be rewritten as

\[
d s^2 = d t^2 - d x^2 - (y^2 + z^2)^{-T^2/2\pi M_p^2} [d y^2 + d z^2] .
\]

(2.10)

Again, there is no Newtonian potential between a cosmic string and a static test particle. However, in this coordinate choice, the deflection of a moving test particle can be interpreted as resulting from a gravitomagnetic force generated by the cosmic string. We can ask whether this Einstein solution is recovered on the brane in the limit of the theory where the graviton tenuousness vanishes. In this limit, gravity fluctuations originating on the brane are pinned on that surface indefinitely, implying that gravity should resemble a four-dimensional theory. However, the question remains whether the four-dimensional theory that results is Einstein gravity or some massless scalar-tensor theory instead.
C. The weak field limit

Let us first identify the solution to Eqs. (2.5–2.6) and Eq. (2.8) in the weak field limit. Here, we presume that the metric deviates from a flat metric where the perturbations are proportional to the strength of the source, \( \frac{T^2}{M_P^2} \), assuming this parameter is small. One can arrive at the metric around a cosmic string in this limit by taking the graviton propagator [8,12] and approximating the gravitational potential through one-particle graviton exchange between the cosmic string source and a test particle in a Minkowski spacetime with a flat braneworld. Rewriting the results using the form Eq. (2.4), we find that the metric on the brane (located now at \( \sin z = \beta = 1 \)) is specified by the line element

\[
d s^2 = N^2(r)|_{\sin z = 1} (d t^2 - d x^2) - A^2(r)|_{\sin z = 1} \, d r^2 - r^2 d \phi^2 \tag{2.11}
\]

with

\[
N(r)|_{\sin z = 1} = 1 + \frac{c}{3} \left( \frac{T^2}{2 \pi M_P^2} \ln \frac{r}{r_0} \right) + \mathcal{O} \left( \frac{T^4}{M_P^4} \right) \tag{2.12}
\]

\[
A(r)|_{\sin z = 1} = 1 + \frac{2c}{3} \frac{T^2}{2 \pi M_P^2} + \mathcal{O} \left( \frac{T^4}{M_P^4} \right), \tag{2.13}
\]

which represents a conical space with deficit angle \( \frac{2c T^2}{3 M_P^2} \) where \( c \) is an \( \mathcal{O}(1) \) quantity that parametrizes the energy-momentum contribution of both the matter source and the effective source associated with the intrinsic curvature on the brane (see Sec. III for details). Recall that for pure four-dimensional Einstein gravity, this metric is \( N(r) = 1 \) and \( A(r) = (1 - \frac{T^2}{2 \pi M_P^2})^{-1} \), which again represents a flat conical space with deficit angle \( \frac{T^2}{2 \pi M_P^2} \). Thus in the weak field limit, we expect not only an extra light scalar field generating the Newtonian potential found in \( N(r) \), but also a discrepancy in the deficit angle with respect to the Einstein solution.

We can ask the domain of validity of the solution Eqs. (2.12–2.13). Examining the boundary conditions Eqs. (2.8) and the bulk Einstein equations \( G_{AB} = 0 \) and comparing the size of terms neglected with respect to those included, we see that on the brane this solution is only valid when

\[
r_0 \sqrt{\frac{T^2}{M_P^2}} \ll r \ll r_0. \tag{2.14}
\]

The left-hand inequality of Eq. (2.14) is the one of interest. For values of \( r \) violating this condition, nonlinear contributions to the Einstein tensor become important and the weak field approximation breaks down. But this is precisely the regime we are interested in, since we wish to understand what happens when \( r \) and \( \frac{T^2}{M_P^2} \) are fixed and \( r_0 \to \infty \). We need to find a solution in this regime.

\[3\] Along the axis of symmetry (\( \sin z = 0 \)), the bulk Einstein equations imply that the weak field approximation is valid when \( r \gg r_0 \exp \left[ -\text{const} \times \frac{M_P^2}{T^2} \right] \).
D. The \( r/r_0 \to 0 \) limit

The weak field approximation breaks down when the condition Eq. (2.14) does not apply. Outside this domain of validity, nonlinear contributions to the Einstein equations become important. However, by relaxing the condition that the braneworld is extrinsically flat in the bulk space (i.e., by allowing the brane to be located at \( \sin z = \beta \neq 1 \)), a perturbative solution to the Einstein equations Eqs. (2.5–2.6) with the boundary conditions Eq. (2.8) can be found in the limit of interest when \( r \ll r_0 \) with \( T^2/M_p^2 \) held fixed. In the ansatz Eq. (2.4), we organize a perturbation around flat space,

\[
N(r, z) = 1 + \sum_{m=1}^{\infty} n_m(z) \left( \frac{r}{r_0} \right)^m \\
A(r, z) = 1 + \sum_{m=1}^{\infty} a_m(z) \left( \frac{r}{r_0} \right)^m \\
B(r, z) = r \left[ 1 + \sum_{m=1}^{\infty} b_m(z) \left( \frac{r}{r_0} \right)^m \right].
\]

Order by order, the bulk Einstein equations form a series of ordinary differential equations for \((n_m(z), a_m(z), b_m(z))\). The boundary conditions for the coefficients may be set by Eqs. (2.8) as well as by continuity at \( z = 0 \). Evaluating the leading terms of this series, we find the resulting metric elements:

\[
N(r, z) = 1 + \left( \frac{1}{2\beta} \cos z \right) \frac{r}{r_0} + \mathcal{O}\left( q^2/r_0^2, r^2/r_0^4 \right) \\
A(r, z) = 1 - \left( \frac{1 - \beta^2}{2\beta} \frac{1}{\cos z} \right) \frac{r}{r_0} + \mathcal{O}\left( q^2/r_0^2, r^2/r_0^4 \right) \\
B(r, z) = r \left[ 1 + \left( \frac{1 - \beta^2}{2\beta} \frac{\sin^2 z}{\cos^3 z} \right) \frac{r}{r_0} + \mathcal{O}\left( q^2/r_0^2, r^2/r_0^4 \right) \right],
\]

where the parameter \( q \) is greater than unity for a nonvanishing string tension. We discuss this parameter in more detail in Sec. III. The coordinate \( z \) into the bulk again acts as a polar angle, but where the space it parametrizes has a deficit polar angle. The bulk is characterized by that part of space where \( \sin z < \beta \) (i.e., \( 0 \leq z < \frac{\pi\alpha}{2} \) and \( \pi(1 - \frac{\alpha}{2}) < z \leq \pi \)) and the two surfaces where \( \sin z = \beta \) are identified and together represent the braneworld. The identification of these two surfaces induces an extrinsic curvature contribution on the brane which is compensated by the braneworld’s intrinsic curvature. Note that the bulk is \( \mathcal{Z}_2 \)-symmetric across the brane. Figure 2 depicts a spatial slice through the cosmic string.

The matching conditions of the matter \( \delta \)-function cannot be satisfied by the five-dimensional Einstein tensor \( G_{AB} \) contribution to Eq. (2.6). The matching conditions can only be satisfied by the intrinsic curvature term on the brane \( B^{-1}\delta(z - \frac{\pi\alpha}{2})G_{AB}^{(4)} \). This is
FIG. 2. A spatial slice through the cosmic string located at $A$. As in Fig. 1 the coordinate $x$ along the cosmic string is suppressed. The solid angle wedge exterior to the cone is removed from the space, and the upper and lower branches of the cone are identified. This conical surface is the braneworld ($z = \frac{\pi \alpha}{2}$ or $\sin z = \beta$). The bulk space now exhibits a deficit polar angle (cf. Fig. 1). Note that this deficit in polar angle translates into a conical deficit in the braneworld space.
achieved by balancing the source delta-function in the second equation of Eqs. (2.8) with
the step in \( A(r, z) \) at \( r = 0 \) necessary to maintain elementary flatness at the location of the
string. This condition constrains the value of \( \beta \):

\[
\beta = 1 - \frac{T^2}{2\pi M_P^2},
\]  

(2.17)

which is equivalent to the Einstein result. Thus, the deficit polar angle in the bulk is \( \pi(1-\alpha) \)
where again \( \sin \frac{\pi\alpha}{2} = \beta \), while the deficit azimuthal angle in the brane itself is \( 2\pi(1-\beta) \).

The metric on the brane is determined by the line element

\[
ds^2 = N^2(r)|_{\sin z = \beta} \left( dt^2 - dx^2 \right) - A^2(r)|_{\sin z = \beta} dr^2 - \beta^2 r^2 d\phi^2
\]  

(2.18)

with

\[
N(r)|_{\sin z = \beta} = 1 + \frac{\sqrt{1 - \beta^2}}{2\beta} \frac{r}{r_0} + \frac{1}{8} \left[ 1 - \frac{3}{2} \left( \frac{1 - \beta^2}{\beta^2} \right) \right] \frac{r^2}{r_0^2} + \mathcal{O}(r^{q+1}/r_0^{q+1}, r^3/r_0^3)
\]  

(2.19)

\[
A(r)|_{\sin z = \beta} = 1 - \frac{\sqrt{1 - \beta^2}}{\beta} \frac{r}{r_0} + \frac{7}{16} \left[ 1 + \frac{11}{14} \left( \frac{1 - \beta^2}{\beta^2} \right) \right] \frac{r^2}{r_0^2} + \mathcal{O}(r^{q+1}/r_0^{q+1}, r^3/r_0^2)
\]  

(2.20)

where the boundary conditions Eq. (2.8) are used to evaluate the \( \mathcal{O}(r^2/r_0^2) \) correction to the
metric elements on the brane. Note that this solution is in a coordinate system distinct from
that in the solution Eqs. (2.12–2.13). In particular, the extrinsic curvature of the brane in
this case is nonzero, whereas extrinsic curvature of the brane in the weak field approximation
not attributed to the fields themselves vanishes. In the limit when \( r_0 \to \infty \), the graviton
tenuousness vanishes and we recover a flat conical space with a deficit angle \( 2\pi(1-\beta) = \frac{T^2}{M_P^2} \),
the solution for a cosmic string in four-dimensional Einstein gravity.

The solution Eqs. (2.19–2.20) breaks down when the subleading terms in the expansion
becomes larger than the leading terms. When \( \beta \to 1 \) or alternatively when \( \frac{T^2}{M_P^2} \to 0 \), the
solution is only valid when

\[
r \ll \frac{r_0 \sqrt{1 - \beta^2}}{\beta} \sim r_0 \sqrt{\frac{T^2}{M_P^2}}.
\]  

(2.21)

This condition is also where the weak field approximation breaks down.\(^4\) Thus, the approx-
imation is valid in the limit of interest where \( r_0 \to \infty \) while all other parameters are kept
fixed. Consequently, the cosmic string solution Eqs. (2.16) does not suffer from a VDVZ
discontinuity, supporting the results found for cosmological solutions [12] in this braneworld
theory with a metastable four-dimensional graviton.

\(^4\)Along the axis of symmetry (\( \sin z = 0 \)), the bulk Einstein equations imply that this approximation
is valid when \( r \ll r_0 \frac{T^2}{M_P^2} \), again complementing the regime where the weak field approximation is
valid when one considers the global shape of the bulk space.
E. The $r/r_0 \to \infty$ limit

For completeness, we include the metric on the brane in the regime where $r \gg r_0$ while $\frac{T^2}{M_p^2}$ is fixed. The right-hand inequality of Eq. (2.14) represents the crossover from four-dimensional to five-dimensional behavior expected at the scale $r_0$. Graviton modes localized on the brane evaporate into the bulk over distances comparable to $r_0$. The presence of the brane becomes increasingly irrelevant as $r/r_0 \to \infty$ and a cosmic string on the brane acts as a codimension-three object in the full bulk. One can again use the propagator found in [8,12] to recover a Schwarzschild-like metric. Then, with the brane at $\sin z = 1$ and using the line element Eq. (2.11), the metric elements are

$$N(r)|_{\sin z = 1} = 1 - \frac{c}{3} \left( \frac{T^2}{2\pi M_p^2} \right) \frac{r_0}{r} + \mathcal{O}(r_0^2/r^2)$$ (2.22)

$$A(r)|_{\sin z = 1} = 1 + \frac{2c}{3} \left( \frac{T^2}{2\pi M_p^2} \right) \frac{r_0}{r} + \mathcal{O}(r_0^2/r^2),$$ (2.23)

recovering the Schwarzschild-like solution for codimension-three object in five-dimensional spacetime with

$$r_G = \frac{r_0 T^2}{2\pi M_p^2} = \frac{T^2}{4\pi M^3}$$ (2.24)

acting as the effective Schwarzschild radius.

The polar deficit angle solution for a strong cosmic string source Eqs. (2.16) breaks down when $\beta \to 0$, i.e., when the deficit angle of the solution approaches $2\pi$. But this limit corresponds to when the Schwarzschild $r_G > r_0$, or when the characteristic scale of source becomes greater than the evaporation length of a four-dimensional graviton off the brane. Nevertheless, the solution Eqs. (2.22–2.23) remains valid in the region far outside the Schwarzschild radius. Thus, we see that in the regime where $\frac{T^2}{M_p^2} > 2\pi$, the cosmic string ceases to act as a codimension-two object and its behavior corresponds to that of a codimension-three object in the full bulk.

III. INTERPOLATION BETWEEN PHASES

We wish to address the matching of the Einstein phase, Eqs. (2.19–2.20) to the weak field phase, Eqs. (2.12–2.13). Because these metrics are solutions of linearized Einstein equations, one may add to either any homogeneous solution and tune its coefficient such that it smoothly matches onto the other solution as one integrates the full equations outside the linear regime. Let us focus on the Einstein phase, Eqs. (2.19–2.20). A homogeneous solution in this phase may be identified by including nonanalytic contributions to the expansion.
Eqs. (2.15). If one includes contributions of the form \( f(z)r^q \), then the metric elements take the form
\[
N(r, z) = 1 + \sum_{m=1}^{\infty} n_m(z) \left( \frac{r}{r_0} \right)^m + n(z) \left( \frac{r}{r_0} \right)^q + \cdots
\]
\[
A(r, z) = 1 + \sum_{m=1}^{\infty} a_m(z) \left( \frac{r}{r_0} \right)^m + a(z) \left( \frac{r}{r_0} \right)^q + \cdots
\]
\[
B(r, z) = r \left[ 1 + \sum_{m=1}^{\infty} b_m(z) \left( \frac{r}{r_0} \right)^m + b(z) \left( \frac{r}{r_0} \right)^q + \cdots \right]
\]
with \( q \) to be determined. Then the bulk Einstein equation require the following relations:
\[
\frac{2}{dz} \frac{dn}{dz} = \frac{b(z)}{\tan z} - [2qn(z) - a(z)] \tan z
\]
\[
\frac{da}{dz} = -\frac{b(z)}{\tan z} - (q - 1)[2qn(z) - a(z)] \tan z
\]
\[
\frac{db}{dz} = [(q + 2)b(z) - 2a(z) - 2q(q - 1)n(z)] \tan z.
\]
The brane boundary conditions Eqs. (2.8) require \( n(z = \beta) = a(z = \beta) = 0 \) and continuity on the axis of symmetry requires \( b(z = 0) = 0 \). Because this solution is homogeneous, the overall scale is undetermined. The boundary conditions can only be satisfied by fixing \( q \) for any given \( \beta \). Integrating the system Eqs. (3.2), one can find the constraint for \( q \). One finds an infinite sequence of homogeneous solutions, all with \( q > 1 \) for nonvanishing string tensions such that in the limit \( \frac{R^2}{M_p^2} = 0 \), \( q(\beta = 1) = 2m + 1 \) with \( m = 0, 1, 2, \ldots \). The \( m \)-th solution in the sequence is characterized by \( m + 1 \) nodes in \( n(z) \). For the first homogeneous solution \((m = 0)\), when the string tension is small,
\[
q \to 1 + \sqrt{\frac{T^2}{\pi M_p^2}}.
\]
For the same solution, when the string tension is large and \( \beta \to 0 \),
\[
q \to \frac{2}{\beta} = \frac{2}{1 - \frac{R^2}{2\pi M_p^2}}.
\]
The dependence of \( q(\beta) \) on \( \frac{R^2}{M_p^2} \) is monotonic and smoothly transitions between these two regimes. Since, \( q(\beta) > 1 \) for all \( \beta \neq 1 \) and for all values of \( m \), the full solution Eqs. (2.16) is unaffected to leading order. The overall normalization of each of the homogeneous contributions resulting from \((n(z), a(z), b(z))\) can now be adjusted such that as one integrates out the full nonlinear bulk Einstein equations toward spatial infinity, the solution Eqs. (2.16) for all \( z \) matches smoothly with the bulk weak-field solution. One needs to evaluate the full nonlinear Einstein equations to verify this claim; however, barring any subtlety of the global structure of the equations, such a matching prescription is allowed in general.
Let us now address the coefficient $c$ in Eqs. (2.12–2.13) and Eqs. (2.22–2.23). In order to ascertain the coefficient that appears as the Schwarzschild mass, one needs to identify the analog of Gauss’ law. Following Landau and Lifshitz [15], Sec. 105, one can use the Einstein equations to evaluate the integral of the component $R^0_0$ of the Ricci tensor over all space (but only along a unit length of the cosmic string):

$$\int dV R^0_0 = \frac{1}{3} \frac{r_0 T^2}{M_P} + \frac{r_0}{3} \int_{\text{brane}} dv \left[ (G^{(4)})^0_0 - (G^{(4)})^r_r - (G^{(4)})^0_\phi \right].$$

(3.5)

The source now includes both the matter source and the source from the intrinsic curvature term Eq. (2.2). For static metrics, $R^0_0$ is a pure covariant divergence, implying that one can evaluate the left-hand side of this expression over an arbitrary boundary. Choose this surface to be at spatial infinity. Then, the metric is characterized by Eqs. (2.22–2.23), and

$$c \left( \frac{1}{3} \frac{r_0 T^2}{M_P} \right) = \frac{1}{3} \frac{r_0 T^2}{M_P} + \frac{r_0}{3} \int_{\text{brane}} dv \left[ (G^{(4)})^0_0 - (G^{(4)})^r_r - (G^{(4)})^0_\phi \right].$$

(3.6)

The second term has only contributions of order $\frac{r_0 T^2}{M_P}$ coming from the region $r \gg r_0$ and that contribution is proportional to $c$ as well. All contributions for $r \ll r_0$ are subleading in the string tension. Then, Eq. (3.6) gives the expression for the effective string tension in the weak field phase where the final expression for $c$ is $O(1)$.

IV. DISCUSSION

In different parametric regimes, we find different qualitative behaviors for the brane metric around a cosmic string. For an observer at a distance $r \gg r_0$ from the cosmic string, where $r_0$ characterizes the evaporation length of the graviton, the cosmic string appears as a codimension-three object in the full bulk. The metric is Schwarzschild-like in this regime. When $r \ll r_0$, brane effects become important, and the cosmic string appears as a codimension-two object on the brane. When the source is strong (i.e., when the tension, $T$, of the string is on the order of the four-dimensional Planck scale, $M_P$), the Einstein solution of a conical space with deficit angle $\frac{T^2}{M_P}$ holds with $O(r/r_0)$ corrections. However, if the source is weak (i.e., $\frac{T^2}{M_P} \rightarrow 0$), the Einstein solution with a deficit angle of $\frac{T^2}{M_P}$ holds on the brane only when $r \ll r_0 \sqrt{\frac{T^2}{M_P}}$. In the region on the brane when $r \gg r_0 \sqrt{\frac{T^2}{M_P}}$ (but still where $r \ll r_0$), the weak field approximation prevails, the cosmic string exhibits a nonvanishing Newtonian potential and space suffers a deficit angle different from $\frac{T^2}{M_P}$.

We identified a matching prescription that, subject to global restrictions, implies that the weak field phase, Eqs. (2.12–2.13), and the Einstein phase, Eqs. (2.19–2.20), can be chosen to be different linear regimes of a unique solution. Each linear solution becomes strongly
nonlinear outside of its domain of validity simply because the corresponding coordinate system in which each solution is linear differs from the other. The full nonlinear solution in this light is reminiscent of the ansatz introduced by Vainshtein [3] for the Schwarzschild solution in a massive gravity theory. Moreover, the presence of this weak field phase at large distances from the cosmic string may have non-negligible consequences for the observational search for cosmic strings through gravitational lensing techniques [16]. For a GUT scale \(10^{16} \text{ GeV}\) cosmic string, the Einstein deficit angle for the string is \(\sim 10^{-5}\). This implies that the light deflection by the string differs significantly from the predictions of general relativity at distances

\[
r \sim r_0 \sqrt{\frac{M_{\text{GUT}}^2}{M_P^2}} \sim 3 \text{ Mpc},
\]

where \(r_0\) is taken to be today’s Hubble radius (see Ref. [10,11,17] for phenomenological motivations for this choice).

The solution presented here supports the Einstein solution near the cosmic string in the limit that \(r_0 \to \infty\). This observation suggests that the braneworld theory under consideration does not suffer from a van Dam–Veltman–Zakharov (VDVZ) discontinuity, corroborating the findings for cosmological solutions in the same theory [12]. Far from the source, the gravitational field is weak, and the geometry of the brane within the bulk is not substantially altered by the presence of the cosmic string. Propagation of the light scalar mode is permitted. However near the source, the gravitational fields induce a nonperturbative extrinsic curvature in the brane. That extrinsic curvature suppresses the coupling of the scalar mode to matter and only the tensor mode remains, thus Einstein gravity is recovered. As one takes \(r_0 \to \infty\), the region where the source induces a large brane extrinsic curvature grows with \(r_0\), implying Einstein gravity is strictly recovered in this limit.

In this paper, we investigate the spacetime around a cosmic string on a brane in a five-dimensional braneworld theory that supports a metastable four-dimensional graviton. This system has the advantage of offering a semianalytic solution to the metric around a compact object, while still providing a clear example in which the VDVZ discontinuity manifests itself. The result may help shed light on the more difficult, more immediately relevant problem of a Schwarzschild-like spherical source in this braneworld theory. At the same time, the cosmic string solution is itself interesting and, should these objects exist in nature, would have testable phenomenological consequences.

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