INTEGRABLE BOUNDARY CONDITIONS
FOR THE O(N) NONLINEAR $\sigma$ MODEL

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Abstract We discuss the new integrable boundary conditions for the $O(N)$ non-linear $\sigma$ model and related solutions of the boundary Yang-Baxter equation, which were presented in our previous paper hep-th/0108039.

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1. Introduction

Two-dimensional nonlinear $\sigma$ (nl$\sigma$) models have been the subject of intense study during the past few years, since they may be used as toy models for the study of higher dimensional non-abelian gauge theories (Yang-Mills), they arise in several condensed matter and statistical mechanics problems, and there are powerful mathematical methods in 2d that allow one to have a deeper understanding of their structure. On top of that, they display a host of theoretical phenomena, such as asymptotic-freedom, dynamical mass generation, and $1/N$-expansions.

A natural generalization, both from the theoretical and experimental point of view, of a given 2d integrable model, is to consider it on the half-line [1]. This type of reduction arises in several problems, for example, when considering the radial part of the Schrödinger equation for a radial potential, in the study of quantum impurities, such as the Kondo effect, and in open string theory.

In this note we summarize the results obtained in [2], where we have found new integrable boundary conditions and related solutions of the

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boundary Yang-Baxter equation (bYBe) for the $O(N)$ nlσ model. The following discussion is informal and intended to a general audience, we refer to [2] for a more complete discussion.

2. The $O(N)$ Nonlinear Sigma Model

The lagrangian of the $O(N)$ nlσ model [3] is

$$L_{nlσ} = \frac{1}{2g_0^2} (\partial \vec{n})^2$$

where the field $\vec{n} = (n_1, n_2, \ldots, n_N)$ satisfies $\vec{n}^2 = 1$, and $g_0$ is a coupling constant. This constraint may be introduced in the lagrangian through a lagrangian multiplier $\lambda(x)$,

$$L_{nlσ} = \frac{1}{2g_0^2} (\partial \vec{n})^2 + \lambda(x)(\vec{n}^2 - 1)$$

The exact $S$-matrix for the $O(N)$ nlσ model was found by Zamolodchikov and Zamolodchikov in [4]. For a review on exact $S$-matrices see [5]. Since the $O(N)$ nlσ model is classically conformally invariant (no dimensionful quantities appear in the classical lagrangian (2)) the trace of the energy-momentum tensor, $T_{++} (= T_{--})$, vanishes. Upon quantization, conformal invariance is broken due to the introduction of an energy scale, such as an energy cut-off, for example. In any case we will see later that the classical conformal invariance is crucial in establishing the quantum integrability of this model. In light-cone coordinates the only non-vanishing components of the energy momentum tensor are $T_{++} = \partial_+ \vec{n} \cdot \partial_+ \vec{n}$ and $T_{--} = \partial_- \vec{n} \cdot \partial_- \vec{n}$. This means that energy-momentum conservation reads

$$\partial_- T_{++} = 0$$

and an analogue equation with $+ \leftrightarrow -$. Classically it is easy to see that (4) implies $\partial_- (T_{++})^n = 0$ for any integer $n$. Upon quantization this is no longer true since we have now a composite operator and one must be careful in defining the product of operators at the same space-time point. This means that the right-hand side of (4) will no longer be zero, or in other words, we have a quantum anomaly. Even though it is a hard problem to compute the exact form with all numerical coefficients of the
anomaly in (4), we can greatly fix its form, by using dimensional analysis and group theory. Let us look at what happens in the case \( n = 2 \). In this case the rhs of (4) has dimension 5, Lorentz weight 3 \( \frac{1}{2} \), and it is a scalar under the global \( O(N) \). This means that, whatever the anomaly is, it is a local operator that satisfies these three requirements. One can then proceed and make a list of the possible operators that contribute to the anomaly. The rhs of 4 will be, finally, a linear combination of these operators, the hard work been to have to compute the coefficients of this linear combination. In the case of \( \partial_-(T_{++})^2 = 0 \) a wonderful thing happens: all the operators that can contribute to the anomaly can be rewritten as a total derivative, with the help of the equation of motion and the constraint \( \vec{u} \cdot \vec{n} = 1 \). This is the so-called Goldschmidt-Witten argument [6] (see also [3], for an earlier version). For the complete list of these operators, we refer to [6]. By using this quantum conservation law the integrability of the \( O(N) \) nonlinear sigma model is established.

When considering a field theory on the half-line, bulk conservation laws may be broken, as it is clear in the case of linear momentum conservation (the boundary breaks translation symmetry). Therefore, one needs to impose suitable boundary conditions that will ensure that a given bulk conservation law will still hold after the introduction of the boundary. In equations, if we have a bulk conservation law of spin (Lorentz weight) \( s \)

\[
\partial_+ J^{(s+1)}_+ - \partial_- R^{(s-1)}_- \quad \text{and} \quad \partial_- J^{(s+1)}_- = \partial_+ R^{(s-1)}_+ \tag{5}
\]

it follows that

\[
Q_\pm = \int_{-\infty}^{+\infty} dx_1 \left( J^{(s+1)}_\pm - R^{(s-1)}_\mp \right) \tag{6}
\]

are conserved. After the introduction of a boundary, only (possibly) a combination of these charges will be conserved. The charge is

\[
\tilde{Q} = \int_{-\infty}^{0} dx_1 \left( J^{(s+1)}_+ - R^{(s-1)}_- + J^{(s+1)}_- - R^{(s-1)}_+ \right) + \Sigma(t) \tag{7}
\]

where \( \Sigma(t) \) is a local operator satisfying

\[
J^{(s+1)}_+ - J^{(s+1)}_- + R^{(s-1)}_- - R^{(s-1)}_+ \bigg|_{x=0} = \frac{d}{dt} \Sigma(t) , \tag{8}
\]

This is precisely where we have to impose the boundary conditions in order to have a non-trivial conserved charge in the presence of a boundary.

In [7] we have shown that if we impose Neumann boundary conditions, \( \partial_1 n_i |_{x=0} = 0 \), to \( k \) field components, and Dirichlet boundary conditions,
\( \partial_0 n_i |_{x=0} = 0 \), to the remaining \( N - k \) components, the condition (8) is satisfied and we have integrable boundary conditions. Note that there is no free parameter (coupling constant) in this case. These are diagonal boundary conditions, in the sense that scattering off the boundary does not change the \( O(N) \) index of the incoming particle. These boundary conditions break the bulk symmetry at the boundary to \( O(k) \times O(N - k) \). It can be shown, by using the bYBe, that these are the only possible diagonal integrable boundary conditions. Therefore, if we are looking for new integrable boundary conditions with free parameters, we have to necessarily look for non-diagonal boundary conditions, that is, boundary conditions that change the \( O(N) \) index (flavor) of the incoming particle through scattering off the boundary.

In the following we will take a slightly different point of view from [2], where we used a simple two free bosons model coupled at the boundary as a guide to the possible boundary conditions for the \( O(N) \) nl\( \sigma \) model.

In [2] we have found new integrable boundary conditions that break the bulk \( O(N) \) symmetry to \( O(2) \times O(N - 2) \) at the boundary, and which depend on one free-parameter \( g \). The reason for this symmetry at the boundary is the following. Free boundary conditions (Neumann) have \( O(N) \) boundary symmetry. The diagonal boundary conditions we found previously, break the boundary symmetry to \( O(k) \times O(N - k) \). We are looking now for non-diagonal boundary conditions with a free parameter, and we may assume that in certain imitating cases, such as taking the coupling constant to 0 or \( \infty \), we should reduce to a diagonal case. Therefore, if we insist that once the boundary symmetry is broken we do not have any point in the integrable flow where it is enhanced, we should look for non-diagonal boundary conditions that are \( O(k) \times O(N - k) \) symmetric. By writing the generic boundary condition as

\[
\partial_1 n_i |_{x=0} = M_{ij} \partial_0 n_j |_{x=0}
\]  

(9)

where the indices \( i \) and \( j \) run through a subset of \( \{1, 2, \ldots, N\} \) (the “nondiagonal subset”, which could be taken to be the first \( k \) indices, for example). Under an orthogonal transformation the fields transform as \( n_i \rightarrow \tilde{n}_a = O_{ai} n_i \), where \( O \) is a \( k \times k \) orthogonal matrix. This means that \( M_{ij} \rightarrow \tilde{M}_{ab} = O_{ai} M_{ij} O_{bj} \), and if we require the boundary conditions to be \( O(k) \times O(N - k) \) symmetric, we should have \( OMO^T = M \). The only case where we can impose this condition for a non-diagonal matrix if when \( k = 2 \), since \( O(2) \) is abelian. This fixes the matrix \( M \) to be of the form \( M = g_1 I + i g_2 \sigma_2 \), where \( I \) is the identity matrix and \( \sigma_2 \) is a Pauli matrix. By inspecting the spin-4 Goldschimdt-Witten charge described above we see that if we take \( g_1 = 0 \) and \( g_2 = g \) arbitrary, the following
boundary condition is integrable:
\[
\partial_1 n_1 |_{x=0} = g \partial_0 n_2 |_{x=0} \quad \text{and} \quad \partial_1 n_2 |_{x=0} = -g \partial_0 n_1 |_{x=0} \tag{10}
\]
where we picked the first two components of the \( \vec{n} \) field without any loss of generality, and the remaining field components satisfy Dirichlet boundary condition. In a different form, this boundary condition had been studied by Corrigan and Sheng at the classical level in \([8]\), for the \( O(3) \) nonlinear sigma model.

The non-diagonal boundary conditions in 10 can be derived from the boundary lagrangian
\[ L_b = \frac{1}{2} M_{ij} \dot{n}_i \dot{n}_j, \]
which shows that \( M_{ij} \) should be anti-symmetric.

By taking \( g \to 0 \) we have diagonal boundary conditions, where 2 field components satisfy Neumann and the remaining Dirichlet, and by taking \( g \to \infty \) we recover a diagonal case again, with all field components satisfying Dirichlet boundary conditions. Therefore we have an integrable flow between diagonal boundary conditions, from \( O(N) \), corresponding to \( g = \infty \), to \( O(2) \times O(N-2) \), corresponding to \( g = 0 \).

3. The Reflection Matrix

When one tries to find an exact \( S \)-matrix for a given integrable field theory, the use of the bulk symmetries plays a crucial role, making it much easier to solve the Yang-Baxter equation. This is why we had to understand the symmetry of the boundary conditions before we could go on and try to solve the bYBe.

For the purely diagonal case, the solutions of the bYBe have been found in \([9]\). They are block diagonal, \( O(k) \times O(N-k) \) symmetric, with diagonal elements \( (R_1(\theta), \ldots, R_2(\theta) \ldots) \), the first \( k \) elements corresponding to, say, Neumann, and the remaining \( N-k \) to Dirichlet. The bYBe fixes the ratio \( R_1(\theta)/R_2(\theta) \) to be
\[
\frac{R_1(\theta)}{R_2(\theta)} = \frac{c - \theta}{c + \theta} \tag{11}
\]
with \( c = -i \frac{\pi}{2} \frac{N-2k}{N-k} \). Note that there is an interesting duality by taking \( k \to N-k \), which takes \( c \to -c \), and therefore \( R_1(\theta) \leftrightarrow R_2(\theta) \).

For the boundary conditions 10, we start with the following ansatz
\[
R = \begin{pmatrix}
A(\theta) & B(\theta) & 0 & 0 & \cdots \\
-B(\theta) & A(\theta) & 0 & 0 & \cdots \\
0 & 0 & R_0(\theta) & 0 & \cdots \\
0 & 0 & 0 & R_0(\theta) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{12}
\]
This means that the first two particles can scatter onto each other with amplitude $\pm B(\theta)$, or onto themselves with amplitude $A(\theta)$. The diagonal elements correspond to the particles scattering diagonally with Dirichlet boundary conditions, with amplitude $R_0(\theta)$. Thinking in terms of the boundary lagrangian for the non-diagonally scattering particles, we see that the off-diagonal amplitudes should have opposite signs.

We can use the bYBe now, in order to compute the functions $A(\theta)$, $B(\theta)$, and $R_0(\theta)$. In the following we will quote the differential equations for $X(\theta) = A(\theta)/R_0(\theta)$ and $Y(\theta) = B(\theta)/R_0(\theta)$, obtained from the bYBe, by taking the limit where the two rapidities are equal.

The process $|A_1(\theta_1)A_i(\theta_2)| \rightarrow |A_i(-\theta_1)A_1(-\theta_2)|$, where $i$ is any of the diagonally scattering particles, gives

$$\frac{d}{d\theta} X(\theta) = \frac{X^2(\theta) - Y^2(\theta) - 1}{2\theta}.$$  \hfill (13)

The process $|A_1(\theta_1)A_i(\theta_2)| \rightarrow |A_i(-\theta_1)A_2(-\theta_2)|$ gives

$$\frac{d}{d\theta} Y(\theta) = \frac{X(\theta)Y(\theta)}{\theta}.$$ \hfill (14)

These two equations can be easily solved by the introduction of the auxiliary functions $Z_\pm(\theta) = X(\theta) \pm iY(\theta)$. We obtain

$$X(\theta) = \frac{1}{2} \left( \frac{c-\theta}{c+\theta} + \frac{c'-\theta}{c'+\theta} \right) \quad \text{and} \quad Y(\theta) = \frac{1}{2i} \left( \frac{c-\theta}{c+\theta} - \frac{c'-\theta}{c'+\theta} \right)$$ \hfill (15)

where $c$ and $c'$ are constants to be determined. Since we have only one free parameter at the boundary, we should find one equation relating $c$ and $c'$. This is accomplished by the bYBe corresponding to the process $|A_1(\theta_1)A_1(\theta_2)| \rightarrow |A_1(-\theta_1)A_2(-\theta_2)|$, from which we obtain

$$c + c' = -i\pi \frac{N-4}{N-2}$$ \hfill (16)

We have verified that with this constraint, all the other bYBe’s are satisfied. Once the ratios $X(\theta)$ and $Y(\theta)$ have been fixed, all that is left to do is to compute the overall factor for the reflection matrix, which can be done with the use of boundary unitarity and boundary crossing-symmetry, and a minimality hypothesis for the pole structure of the reflection matrix. We refer to [2] for the explicit results.

Note that if $c = c'$ the off-diagonal amplitudes $\pm Y(\theta)$ vanish, and we recover a diagonal scattering problem. The other instance where $Y(\theta)$ vanishes is when $|c|, |c'| \rightarrow \infty$. In the first case the ratio $X(\theta) = \frac{e^{-\theta}}{e^\theta}$ with $c = -i\pi \frac{N-4}{2(N-2)}$, which corresponds precisely to the case where
the first two field components satisfy Neumann boundary conditions and the remaining $N - 2$ Dirichlet. This is the reason why we chose the diagonally scattering field components to satisfy Dirichlet boundary conditions. In this case the solution for the reflection matrix is $O(2) \times O(N - 2)$ symmetric. By looking at the explicit form of the boundary conditions (10), we see that this corresponds to $g = 0$. In the second case $X(\theta) = 1$, which means that the reflection matrix is proportional to the identity, and therefore $O(N)$ symmetric. This corresponds to taking $g \to \infty$, and therefore, to all components satisfying Dirichlet boundary conditions.

We can introduce the following convenient parametrization: $c = -i \frac{N - 4}{2N} + \xi(g)$ and $c' = -i \frac{N - 4}{2N} - \xi(g)$, where $\xi(g)$ is an unknown function of the boundary coupling constant. The two cases described in the preceding paragraph correspond to $\xi(0) = 0$ and $\xi(g \to \infty) \to \infty$. This establishes an integrable flow between different diagonal boundary conditions.

One could be tempted at trying a generalization of the ansatz (12), with more than one non-diagonal block, corresponding to more than one pair of particles being coupled at the boundary. By using the bYBe it can be shown that there are no solutions of this type [2].

In [2] we found other solutions to the bYBe for the $O(2N) \text{nl}\sigma$ model, but were not able to link them to any boundary conditions. Another special case is the $O(2) \text{nl}\sigma$ model. Naively one could be lead to think that the $O(2) \text{nl}\sigma$ model is equivalent to a massless free boson, through a mapping $(n_1, n_2) \to (\cos(\theta), \sin(\theta))$, but this is not the case, and after a more careful analysis, it can be shown that the $O(2) \text{nl}\sigma$ model is equivalent to the sine-Gordon model at $\beta^2 = 8\pi$, which describes the Kosterlitz-Thouless point of the classical XY model. The solution we found depends on three parameters, instead of two as in the boundary conditions found by Ghoshal and Zamolodchikov in [1]). The resolution of this discrepancy is that we are looking at the sine-Gordon model at a special value of the coupling constant, and as already remarked in ([1]), at these special points there are more solutions than the ones found for the general case.

4. conclusions

We have found new integrable boundary conditions for the $O(N) \text{nl}\sigma$ model, which depend on one free parameter $g$. These boundary conditions break the bulk $O(N)$ symmetry to $O(2) \times O(N - 2)$, and by taking the limits $g \to 0$ and $g \to \infty$ we recover the diagonal solutions studied
previously. This establish an integrable flow between two different sets of boundary conditions.

Recently Mackay and Short [10] have studied the principal chiral model with a boundary and found an interesting relationship between their boundary conditions and the theory of symmetric spaces. Their solutions, though, are quite different from ours, and some work should be done in trying to clarify their relationship.

As natural follow-up problems, one should try to find explicitly the function $\xi(g)$ in the reflection matrices, and to study the boundary thermodynamic Bethe ansatz equations.

An interesting direction to pursue would be to extend these results to the $SO(N)$ Gross-Neveu (GN) model. Since the $S$-matrix for the elementary fermions of the GN model is equivalent to the one for the $O(N)$ nl-$\sigma$ model, up to a CDD factor, we certainly can find solutions of the bYBe of the form (12) for the GN model too.

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Notes

1. If a quantity $\phi$ transforms as $\phi \rightarrow \exp(p\alpha)\phi$ under a Lorentz boost, that is, the rapidity variable $\theta \rightarrow \theta + \alpha$, we say that $\phi$ has Lorentz weight $p$.

2. This choice of boundary condition for the remaining field components will become clear when we discuss the boundary Yang-Baxter equation.

3. The $\{A_i(\theta)\}$ are the usual Faddeev-Zamolodchikov operators

References


