Larmor precession and tunneling time of a relativistic neutral spinning particle through an arbitrary potential barrier

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The Larmor precession of a relativistic neutral spin-$\frac{1}{2}$ particle in a uniform constant magnetic field confined to the region of a one-dimensional arbitrary potential barrier is investigated. The spin precession serves as a clock to measure the time spent by a quantum particle traversing a potential barrier. With the help of general spin coherent state it is explicitly shown that the precession time is equal to the dwell time.

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Over the years there have been many attempts [1–3] to answer the old and fundamental question “how long does it take on average for an incident particle to tunnel through a potential barrier?” In the literature at least three main approaches have been proposed to define and evaluate this traversing time. First, one can study evolution of the wave packets through the barrier and get the phase time [4] which involves the phase sensitivity of the tunneling amplitude to the energy of the incident particle. Another definition of tunnel time is based on the determination of a set of dynamic paths. The time spent in the different paths is averaged over the set of the paths [5]. The third approach makes use of a physical clock to measure the time elapsed during the tunneling [6–9]. The various approaches corresponding to different criteria have no a clear consensus [1–3]. Consequently,

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there has been existed much controversy on the question of tunnel time. Recently a number of experiments [10–12] indicating superluminal transmission of photons through barriers has renewed interest in the subject of tunneling time. Most of investigations are focused on nonrelativistic tunneling and little work has been done toward the study of relativistic tunneling time. The tunneling time of the photons or the electromagnetic wave is usually investigated in term of the mathematical identity of Helmholtz and Schrödinger equations [13,14] and the tunnelling time of Dirac electrons is calculated in the framework of the Dirac equation in Refs. [15,16]. In our earlier paper we have reconsidered the Larmor precession of a neutral spinning particle in a general spin coherent state as a clock to measure the tunneling time through a one-dimensional rectangular barrier in the relativistic regime [17]. The reason that we choose the neutral particle instead of electron is nothing but for the sake of simplicity, since the precession of spin in magnetic field is the only effect to be considered for the Larmor clock time. The potential barrier to a neutral particle can be simply a planar film consisting of a medium characterized by a certain scattering potential which has been well used in experiments of neutron resonators and interferometers [18]. We now extend the previous investigation [17] to the case of an arbitrary potential barrier.

Larmor precession was first introduced long ago as a thought experiment designed to measure the time associated with scattering events [6]. Subsequently the method was applied to measure the tunneling time of particles penetrating barrier with a magnetic field confined to the barrier region, causing the spin of particle to precess [7]. The original scheme in Ref.[7] considered only the rotation of the spin in the plane which is perpendicular to the magnetic field. Later it was recognized that a particle tunneling through a barrier in the magnetic field does not actually perform a simple Larmor precession in a plane [9]. The main effect of the magnetic field is to align the spin along the field since the particle with spin parallel to the magnetic field has lower energy and less decay rate in barrier region than that of particle with spin antiparallel to the magnetic field. In the present paper we use the spin coherent state to obtain an equation of motion for the expectation value of spin operator in the magnetic field. We show that the relativistic neutral spin-1/2 particles perform a
simple Larmor precession in three-dimensional space and that the Larmor precession time equals the dwell time, which measures how long the matter wave remains in the potential barrier regardless of whether the particle is reflected or transmitted [19]. For the special case of symmetric potential barrier, the consistency of the dwell time, the transmission time and the reflection time is obtained, which is in agreement with the result for Schrödinger particles [1,9,20].

A relativistic neutral particle of spin $\frac{1}{2}$ with mass $m$ and magnetic moment $\mu$, moving in an external electromagnetic field denoted by the field strength tensor $F_{\mu\nu}$, is described by a four-component spinor wave function $\psi$ obeying the Dirac-Pauli equation [21]

$$[\gamma^\mu \frac{\hbar c}{i} \partial_\mu + mc^2 + \frac{1}{2} \mu \sigma^{\mu\nu} F_{\mu\nu}]\psi = 0,$$

(1)

where $c$ is the velocity of light in vacuum, $\gamma^\mu = (\gamma^0, \gamma)$ are Dirac matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

(2)

with $g^{\mu\nu}=$diag$(1, -1, -1, -1)$, and

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

(3)

It can be shown that

$$\frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} = i\alpha \cdot E - \Sigma \cdot B$$

(4)

where $E$ and $B$ are the external electric and magnetic fields, respectively, and $\alpha_i = \gamma^0 \gamma_i$, $\beta = \gamma^0$ for $i=1,2,3$. Here we make use of the Pauli representation

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}. $$

(5)

The spin operator is $S_i = \frac{\hbar}{2} \Sigma_i$ and $\sigma_i$ are the Pauli spin matrices.

A incoming wave of relativistic neutral spin-$\frac{1}{2}$ particle polarized in an arbitrary axis impinges on a finite range barrier potential $U(x)$ that extends from $a$ to $b$. A weak uniform
constant magnetic field $B$, aligned along the z-direction and confined within the barrier region, superimposes the barrier region (see Fig.1). The Hamiltonian is seen to be

$$H_D = c\alpha_1 p_x + \beta mc^2, \quad x < a, \ x > b$$

$$H_D = c\alpha_1 p_x + \beta[(mc^2 + U(x)) - V\Sigma_3], \quad a < x < b.$$  

(6) (7)

where $V = \frac{h}{2} \omega_L$ represents the spin-field interaction, $\omega_L = \frac{2\mu B}{\hbar}$ is the Larmor frequency, and $\hbar$ is Planck’s constant (divided by $2\pi$).

In the asymptotic regions $x < a$ and $x > b$, the wave function satisfying the stationary Dirac-Pauli equation

$$H_D \psi = E\psi,$$

is

$$\psi_a = \frac{1}{\sqrt{1 + f_0^2}} \begin{pmatrix} u_1 \\ u_2 \\ f_0 u_2 \\ f_0 u_1 \end{pmatrix} e^{ik_0 x} + \begin{pmatrix} R_{U-V} u_1 \\ R_{U+V} u_2 \\ -f_0 R_{U+V} u_2 \\ -f_0 R_{U-V} u_1 \end{pmatrix} e^{-ik_0 x}, \quad x < a,$$

$$\psi_b = \begin{pmatrix} T_{U-V} u_1 \\ T_{U+V} u_2 \\ f_0 T_{U+V} u_2 \\ f_0 T_{U-V} u_1 \end{pmatrix} e^{ik_0 x}, \quad x > b.$$  

(9) (10)

where

$$f_0 = \frac{ck_0}{mc^2 + E},$$

$$k_0 = \frac{1}{c} \sqrt{E^2 - (mc^2)^2}.$$  

(11) (12)

The quantities $T_{U\pm V}$ and $R_{U\pm V}$ denote the transmission and reflection amplitudes, respectively, of an outgoing wave corresponding to the total potential energy $U(x) \pm V$.

The incoming wave, i.e., the first term on the right hand side of Eq.(9), is assumed to be a normalized spin coherent state which is an eigenstate of the spin operator $\sigma \cdot \mathbf{n}$, where
\( \mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \) denotes the arbitrary unit vector with a polar angle \( \theta \) and azimuthal angle \( \varphi \) [22]. The two components of the spinor are
\[
u_1 = \cos \frac{\theta}{2} e^{-i\varphi/2}, \quad \nu_2 = \sin \frac{\theta}{2} e^{i\varphi/2}. \tag{13} \]
From the viewpoint of scattering, the outgoing wave consists of both a reflected and a transmitted waves, which are separated from each other. The outgoing wave must be normalized to unity since the incoming wave is normalized to unity. The conservation of probability requires that the coefficients \( R_{U \pm V}, T_{U \pm V} \) satisfy the following relation
\[
(1 + f^2_0)(|T_{U \pm V}|^2 + |R_{U \pm V}|^2) = 1. \tag{14} 
\]
For the case that \( V \) is small, i.e., the probing magnetic field is weak, \( T_{U \pm V} \) and \( R_{U \pm V} \) can be expanded as a power series of \( V \) to the first order for the infinitesimal field approximation, such that
\[
T_{U \pm V} = |T_{U \pm V}| e^{i\alpha_\pm} \approx \left( |T_U| \pm V \frac{\partial |T_{U \pm V}|}{\partial V} \right) e^{i\left( \alpha_{U \pm V} \pm \frac{\partial \alpha}{\partial V} \right)},
\]
\[
R_{U \pm V} = |R_{U \pm V}| e^{i\beta_\pm} \approx \left( |R_U| \pm V \frac{\partial |R_{U \pm V}|}{\partial V} \right) e^{i\left( \beta_{U \pm V} \pm \frac{\partial \beta}{\partial V} \right)}. \tag{15} 
\]
(For the sake of brevity, we adopt the convention that the derivative \( \frac{\partial}{\partial V} \) with respect to the auxiliary potential \( V \) is taken at the zero field \( V=0 \), i.e., \( \frac{\partial}{\partial V}|_{V=0} \).) The transmitted wave and the reflected wave, \( \psi_t \) and \( \psi_r \), respectively, are
\[
\psi_t = \begin{pmatrix} T_{U-V} \nu_1 \\ T_{U+V} \nu_2 \\ f_0 T_{U+V} \nu_2 \\ f_0 T_{U-V} \nu_1 \end{pmatrix}, \quad \psi_r = \begin{pmatrix} R_{U-V} \nu_1 \\ R_{U+V} \nu_2 \\ -f_0 R_{U+V} \nu_2 \\ -f_0 R_{U-V} \nu_1 \end{pmatrix}. \tag{16} 
\]
The expectation values of spin operator for the transmitted wave in the infinitesimal field limit are
\[
\langle S_1 \rangle_t = \frac{\hbar}{2} (1 + f^2_0) |T_U|^2 \sin \theta \cos (2V \frac{\partial \alpha_U}{\partial V} + \varphi), \\
\langle S_2 \rangle_t = \frac{\hbar}{2} (1 - f^2_0) |T_U|^2 \sin \theta \sin (2V \frac{\partial \alpha_U}{\partial V} + \varphi), \\
\langle S_3 \rangle_t = \frac{\hbar}{2} (1 - f^2_0) \left( |T_U|^2 \cos \theta - V \frac{\partial |T_U|^2}{\partial V} \right). \tag{17} 
\]
The expectation values of spin operator for the reflected wave in the infinitesimal field limit are

\[
\langle S_1 \rangle_r = \frac{\hbar}{2}(1 + f_0^2)|R_U|^2 \sin \theta \cos(2V \frac{\partial \beta_U}{\partial V} + \varphi), \\
\langle S_2 \rangle_r = \frac{\hbar}{2}(1 - f_0^2)|R_U|^2 \sin \theta \sin(2V \frac{\partial \beta_U}{\partial V} + \varphi), \\
\langle S_3 \rangle_r = \frac{\hbar}{2}(1 - f_0^2) \left( |R_U|^2 \cos \theta - V \frac{\partial |R_U|^2}{\partial V} \right). 
\]

Equations (17) and (18) show that the spin performs a Larmor precession around the z-axis. To see the spin Larmor precession explicitly we may take the sum of expectation values of spin in the reflected and transmitted states. We then have

\[
\langle S_1 \rangle = \frac{\hbar}{2} \sin \theta \cos \left((1 + f_0^2)|T_U|^2 2V \frac{\partial \alpha_U}{\partial V} + (1 + f_0^2)|R_U|^2 2V \frac{\partial \beta_U}{\partial V} + \varphi\right), \\
\langle S_2 \rangle = \frac{\hbar}{2} \frac{(1 - f_0^2)}{(1 + f_0^2)} \sin \theta \sin \left((1 + f_0^2)|T_U|^2 2V \frac{\partial \alpha_U}{\partial V} + (1 + f_0^2)|R_U|^2 2V \frac{\partial \beta_U}{\partial V} + \varphi\right), \\
\langle S_3 \rangle = \frac{\hbar}{2} \frac{(1 - f_0^2)}{(1 + f_0^2)} \cos \theta, 
\]

where we have used Eq. (14) and the following power series with respect to the small quantity \( V \),

\[
(1 + f_0^2) \left(|T_U|^2 \cos(2V \frac{\partial \alpha_U}{\partial V} + \varphi) + |R_U|^2 \cos(2V \frac{\partial \beta_U}{\partial V} + \varphi)\right) \\
\approx \cos \left((1 + f_0^2)|T_U|^2 2V \frac{\partial \alpha_U}{\partial V} + (1 + f_0^2)|R_U|^2 2V \frac{\partial \beta_U}{\partial V} + \varphi\right) 
\]

and

\[
(1 + f_0^2) \left(|T_U|^2 \sin(2V \frac{\partial \alpha_U}{\partial V} + \varphi) + |R_U|^2 \sin(2V \frac{\partial \beta_U}{\partial V} + \varphi)\right) \\
\approx \sin \left((1 + f_0^2)|T_U|^2 2V \frac{\partial \alpha_U}{\partial V} + (1 + f_0^2)|R_U|^2 2V \frac{\partial \beta_U}{\partial V} + \varphi\right). 
\]

Equations (19) are formally the same as the Larmor precession equation of spin \( \mathbf{S} \) in a uniform constant magnetic field. To see this let us consider a relativistic neutral spin-\( \frac{1}{2} \) particle in a uniform constant magnetic field \( B \) along the z-direction in the absence of potential barrier. The Larmor precession is obtained by solving the Heisenberg equation.
\[
\frac{d}{dt} S(t) = \frac{1}{i\hbar} [S(t), H_s]
\]  
(22)

with the spin Hamiltonian

\[
H_s = -\frac{1}{2} \hbar \omega_L \beta \Sigma_3.
\]  
(23)

If the initial wave function is given by the spin coherent state

\[
\psi_i = \frac{1}{\sqrt{1 + f_0^2}} \begin{pmatrix} u_1 \\ u_2 \\ f_0 u_2 \\ f_0 u_1 \end{pmatrix},
\]  
(24)

the expectation values of the spin at time \( t \) are

\[
\langle S_1(t) \rangle = \frac{\hbar}{2} \sin \theta \cos(-\omega_L t + \varphi),
\]

\[
\langle S_2(t) \rangle = \frac{\hbar}{2} \frac{1 - f_0^2}{1 + f_0^2} \sin \theta \sin(-\omega_L t + \varphi),
\]

\[
\langle S_3(t) \rangle = \frac{\hbar}{2} \frac{1 - f_0^2}{1 + f_0^2} \cos \theta.
\]  
(25)

Comparing Eqs.(19) and (25), we see that for the infinitesimal magnetic field the Larmor tunneling time \( \tau_L \) is

\[
\tau_L = (1 + f_0^2) |T_U|^2 \left( -\hbar \frac{\partial \alpha_U}{\partial V} \right) + (1 + f_0^2) |R_U|^2 \left( -\hbar \frac{\partial \beta_U}{\partial V} \right)
\]  
(26)

which is just the average time scale over the transmission and reflection channels defined as the dwell time \( \tau_D \) [1,23],

\[
\tau_L = \tau_D.
\]  
(27)

The transmitted time \( \tau_T \) and the reflected time \( \tau_R \) are identified from Eq. (26) as

\[
\tau_T = -\hbar \frac{\partial \alpha_U}{\partial V}, \quad \tau_R = -\hbar \frac{\partial \beta_U}{\partial V},
\]  
(28)

which are exactly in accordance with the Larmor time \( \tau_L^U \) and \( \tau_L^R \) for the Schrödinger particles introduced by Büttiker in Ref. [9].
For the special case of symmetric potential barriers, i.e., \( U(x) = U(-x) \), the scattering phases \( \alpha_U \) and \( \beta_U \) satisfy the relation [23]

\[
\alpha_U = \frac{\pi}{2} + \beta_U.
\] (29)

From Eq. (26) and the probability conservation Eq. (14) we obtain

\[
\tau_L = \tau_D = \tau_T = \tau_R,
\] (30)

which is the same result as obtained for the symmetric rectangular potential barrier [17].

In Ref. [17] we have shown that the transmission time \( \tau_T \) of the relativistic neutral particles described by plane waves is given by

\[
\tau_T = \frac{f_0}{c^2 k} \frac{4dk \xi E(k^2 - f_0^2 \xi^2) + \hbar(ck^2 + E\xi)(k^2 + f_0^2 \xi^2) \sinh\left(\frac{4dk}{\hbar}\right)}{4f_0^2 \xi^2 k^2 + (k^2 + f_0^2 \xi^2)^2 \sinh^2\left(\frac{2dk}{\hbar}\right)}
\] (31)

for the symmetric rectangular potential barrier with the width \( 2d \) and height \( U_0 \), where

\[
\xi \equiv \frac{1}{c}(mc^2 + U_0 + E), \quad k = \frac{1}{c} \sqrt{(mc^2 + U_0)^2 - E^2}.
\] (32)

We have demonstrated with numerical estimation that this transmission time can be much smaller than the time for the particle to penetrate a constant magnetic field without a barrier, which implies apparent superluminal tunneling.

To summarize, we have presented a general proof that the dwell time equals the Larmor precession time for a relativistic neutral spinning particle penetrating an arbitrary potential barrier. We also extend the equality to include the transmission time \( \tau_T \) (i.e., Eq.(30)) explicitly for a symmetric potential barrier.

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Figure Caption:

Fig.1 Spin-$\frac{1}{2}$ particles, polarized in an arbitrary direction head towards a potential barrier $U(x)$ in the range $a<x<b$ with a small magnetic field $B$, aligned in the direction of the z-axis.
Fig. 1