From Poincaré to affine invariance: How does the Dirac equation generalize?

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Abstract

A generalization of the Dirac equation to the case of Affine symmetry, with \( \overline{SL}(4, \mathbb{R}) \) replacing \( SO(1, 3) \), is considered. A detailed analysis of a Dirac-type Poincaré-covariant equation for any spin \( j \) is carried out, and the related general interlocking scheme fulfilling all physical requirements is established. Embedding of the corresponding Lorentz fields into infinite-component \( \overline{SL}(4, \mathbb{R}) \) fermionic fields, the constraints on the \( \overline{SL}(4, \mathbb{R}) \) vector-operator generalizing Dirac’s gamma matrices, as well as the minimal coupling to (Metric-)Affine gravity are studied. Finally, a symmetry breaking scenario for \( \overline{SA}(4, \mathbb{R}) \) is presented which preserves the Poincaré symmetry. 

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1 Introduction

The Dirac equation, that describes relativistic particles with spin $\frac{1}{2}$, is a Poincaré invariant linear field equation of first order in the derivatives. Interactions can easily be introduced by the minimal coupling prescription.

A natural question to ask is whether there is a generalization of the Dirac equation in such a way that it is invariant under the affine group. This becomes important when the concept of spin shall be general-relativized. For a long time it had been wrongly believed that only single-valued representations of the Lorentz group, vectors and tensors, have a natural extension to the group $GL(n, \mathbb{R})$. However, in 1977 Y. Ne’eman has pointed out [10] that a double-covering $\overline{GL}(n, \mathbb{R})$ does exist, for proof see Ref. [12]. The latter in turn contains spinor representations. The groups $\overline{SL}(n, \mathbb{R}) \subset \overline{GL}(n, \mathbb{R})$, $n \geq 3$, are necessarily defined in infinite dimensional vector spaces. Their representations induce those of $\overline{Diff}(n, \mathbb{R})$ [12] leading to a unified description of both spinors and tensors within GR.

Our starting-point is a Poincaré invariant Dirac-type wave equation in 4-dimensional Minkowski spacetime, that is an equation of the form

$$ (i\gamma^\alpha X_\alpha \partial_\beta - \kappa)\Psi(x) = 0, $$

where $X_\alpha$ are generalized Dirac matrices. Owing to the fact that each spinorial representation of $\overline{SL}(4, \mathbb{R}) \subset \overline{GL}(4, \mathbb{R})$ contains an infinite set of Lorentz representations, the Lorentz spinor $\Psi$ will be the infinite sum of spinors $\Psi^{(j)}$. Each spinor $\Psi^{(j)}$ is chosen in such a way that it describes a spin $j$ particle and/or a resonance on a certain Regge trajectory. Hereto we will state a theorem for the derivation of the representation of a fermionic particle using the method of Gel’fand et al. [3]. This method yields matrix elements for the vector operator $X_\alpha$.

Since the spin of a particle remains invariant under Lorentz transformations, it can only be raised by an interaction force. The latter must be gravitational – or at least gravity-like, as for example the Chromogravity interaction [14], which is seen in an effective QCD approximation in the IR region and mediated by a di-gluon chromometric field $G_{\mu\nu} \sim g_{ab} A^a_\mu A^b_\nu (a, b = 1, 2, \ldots, 8)$. This is due to the fact that the gauge group of gravity “effectively” contains a tensor operator, the shear tensor, which is able to excite the spin in $\Delta j = 2$. Thus, in order to introduce spin excitations, we embed our representation into (infinitely many) $\overline{SL}(4, \mathbb{R})$ representations. This
yields a still Poincaré invariant manifold equation to which gravity can be
coupled minimally.

Our approach is in some sense complementary to other attempts [9, 1, 5, 15] to find a physical affine invariant Dirac-type equation. We start from
the Lorentz group and arrive at \( SL(4, \mathbb{R}) \) through a series of embeddings. All other approaches begin directly with the group \( SL(4, \mathbb{R}) \) or \( GL(4, \mathbb{R}) \) often lacking relevant physical considerations. For example, Mickelson [9] has proposed a \( GL(4, \mathbb{R}) \) covariant extension of the Dirac equation. However, its physical interpretation is unclear. Cant and Ne’eman [1] found a still Poincaré invariant Dirac-type wave equation for manifolds (infinite component fields). They use only a subclass of representations of \( SL(4, \mathbb{R}) \), the multiplicity-free ones. This class does not allow a \( SL(4, \mathbb{R}) \) vector operator. However, such an operator, which generalizes the Dirac matrices, is needed for an affine Dirac-type wave equation.

2 \( SL(4, \mathbb{R}) \) vector operator \( \tilde{X}_\alpha \)

For the construction of a Dirac-type equation, which is to be invariant under (special) affine transformations, we have two possibilities to derive the matrix elements of the generalized Dirac matrices \( \tilde{X}_\alpha \).

We can consider the defining commutation relations of a \( SL(4, \mathbb{R}) \) vector operator \( \tilde{X}_\alpha \),

\[
[\tilde{X}_\gamma, M_{\alpha\beta}] = ig_{\gamma\alpha} \tilde{X}_\beta - ig_{\gamma\beta} \tilde{X}_\alpha, \tag{2}
\]

\[
[\tilde{X}_\gamma, T_{\alpha\beta}] = ig_{\gamma\alpha} \tilde{X}_\beta + ig_{\gamma\beta} \tilde{X}_\alpha, \tag{3}
\]

with \( g_{\alpha\beta} \) being structure constants of \( SL(4, \mathbb{R}) \). The generators \( L_{\alpha\beta} \) of the group \( SL(4, \mathbb{R}) \) can be splitted into the Lorentz generators \( M_{\alpha\beta} := L_{\{\alpha\beta\}} \) and the shear generators \( T_{\alpha\beta} := L_{(\alpha\beta)} \). We obtain the matrix elements of the generalized Dirac matrices \( \tilde{X}_\alpha \) by solving these relations for \( \tilde{X}_\alpha \) in the Hilbert space of a suitable representation of \( SL(4, \mathbb{R}) \).

Alternatively, we can embed \( SL(4, \mathbb{R}) \) in \( SL(5, \mathbb{R}) \). Let the generators of \( SL(5, \mathbb{R}) \) be \( L_{\alpha B} \), \( A, B = 0, ..., 4 \). Then we define the \( SL(4, \mathbb{R}) \) four-vectors \( \tilde{X}_\alpha \), and \( \tilde{Y}_\alpha \) by

\[
\tilde{X}_\alpha := L_{4\alpha}, \quad \tilde{Y}_\alpha := L_{\alpha 4}, \quad \alpha = 0, 1, 2, 3. \tag{4}
\]
The operator $\tilde{X}_\alpha (\tilde{Y}_\alpha)$ obtained in this way fulfills the relations (2) and (3) by construction. It is interesting to point out that the operator $G_\alpha = \frac{1}{2}(\tilde{X}_\alpha - \tilde{Y}_\alpha)$ satisfies

$$[\tilde{G}_\alpha, \tilde{G}_\beta] = -iM_{\alpha\beta},$$  \hspace{1cm} (5)$$

thereby generalizing a property of Dirac’s $\gamma$-matrices. Since $\tilde{X}_\alpha$, $M_{\alpha\beta}$ and $T_{\alpha\beta}$ form a closed algebra, the application of $\tilde{X}_\alpha$ on the $SL(4, \mathbb{R})$ states does not lead out of the $\overline{SL}(4, \mathbb{R})$ representation Hilbert space.

In order to obtain an impression about the general structure of the matrix $\tilde{X}_\alpha$, let us consider the following embedding of three finite (tensorial) representations of $SL(4, \mathbb{R})$ into one of $SL(5, \mathbb{R})$,

$$SL(5, \mathbb{R}) \supset SL(4, \mathbb{R})$$

$$\begin{array}{c}
\begin{array}{c}
\varphi_{AB} \colon 15 \\
\varphi_{\alpha} \colon 10 \\
\varphi_{\alpha\beta} \colon 4 \\
\varphi
\end{array}
\end{array},$$  \hspace{1cm} (6)$$

where $\Box$ is the Young tableau for an irreducible vector representation of $SL(n, \mathbb{R})$, $n = 4, 5$. The effect of the application of the $SL(4, \mathbb{R})$ vector $\tilde{X}_\alpha$ on the fields $\varphi, \varphi_\alpha$ and $\varphi_{\alpha\beta}$ is

$$\tilde{X}_\alpha \otimes \varphi = \varphi_\alpha$$

$$\tilde{X}_\alpha \otimes \varphi_\alpha = \varphi_{\alpha\beta}$$

$$\tilde{X}_\alpha \otimes \varphi_{\alpha\beta} = 0.$$

Other possible Young tableaux do not appear due to the closure of the Hilbert space. Gathering these fields in a vector $\varphi_M = (\varphi, \varphi_\alpha, \varphi_{\alpha\beta})^T$, from (7) we can read off the structure of $\tilde{X}_\alpha$,

$$\tilde{X}_\alpha = \begin{bmatrix}
0 \\
\varphi_\alpha \\
\varphi_{\alpha\beta}
\end{bmatrix}$$

$$\begin{bmatrix}
\begin{array}{c}
0 \\
\varphi_\alpha \\
\varphi_{\alpha\beta}
\end{array}
\end{bmatrix}. $$  \hspace{1cm} (8)$$
It is interesting to observe that \( \tilde{X}_\alpha \) has zero matrices on the block-diagonal which implies that the mass operator \( \kappa \) in an affine invariant equation of the type (1) must vanish.

This can be proven for a general finite representation of \( SL(4, \mathbb{R}) \). Let us consider the action of a vector operator on an arbitrary irreducible representation \( D(g) \) of \( SL(4, \mathbb{R}) \) labeled by \([\lambda_1, \lambda_2, \lambda_3]\),

\[
[\lambda_1, \lambda_2, \lambda_3] \otimes [1, 0, 0] = [\lambda_1 + 1, \lambda_2, \lambda_3] \oplus [\lambda_1, \lambda_2 + 1, \lambda_3] \oplus [\lambda_1, \lambda_2, \lambda_3 + 1] \oplus [\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1].
\]

None of the resulting representations agrees with the representation \( D(g) \) nor with the contragradient representation \( D^T(g^{-1}) \) given by

\[
[\lambda_1, \lambda_2, \lambda_3]^c = [\lambda_1, \lambda_1 - \lambda_3, \lambda_1 - \lambda_2].
\]

For a general (reducible) representation this implies vanishing matrices on the block-diagonal of \( \tilde{X}_\alpha \) by similar argumentation as (7) led to the structure (8). Let the representation space be spanned by \( \Phi = (\varphi_1, \varphi_2, ...)^T \) with \( \varphi_i \) irreducible. Now we consider the Dirac-type equation (1) in the rest frame \( p_\mu = (E_0(0), 0, 0, 0) \) restricted to the subspace spanned by \( \varphi_i \),

\[
E_{(0)} \langle \varphi_i | \tilde{X}_0^0 | \varphi_i \rangle = \langle \varphi_i | \kappa | \varphi_i \rangle = m_i,
\]

where we assumed the operator \( \kappa \) to be diagonal. So the mass \( m_i \) and therewith \( \kappa \) must vanish since \( \langle \varphi_i | \tilde{X}_0^0 | \varphi_i \rangle = 0 \). Therefore, in an affine invariant Dirac-type wave equation, the mass generation is dynamical, i.e. it can only be evoked by an interaction. This agrees with the fact that the Casimir operator of the special affine group \( SA(4, \mathbb{R}) \) vanishes leaving the masses unconstrained [8]. So we believe that our statement also holds for infinite representations of \( SL(4, \mathbb{R}) \).

3 Prerequisites from the representation theory of the Lorentz group [3]

In the following three sections we want to find a Dirac-type equation for particles with arbitrary half-integral spin. Our main concern will be the construction of the generalized Dirac matrices \( X_\alpha \). The wave equation should
be invariant with respect to Poincaré transformations. This implies that $X_\alpha$ shall be a Lorentz vector operator satisfying

$$[X_\gamma, M_{\alpha\beta}] = ig_{\gamma\alpha} X_\beta - ig_{\gamma\beta} X_\alpha,$$

with $M_{\alpha\beta}$ being the Lorentz generators. We obtain the matrix elements of the generalized Dirac matrices $X_\alpha$ by solving these relations for $X_\alpha$ in the Hilbert space of a suitable representation of $SO(1, 3)$.

**Determination of $X_\alpha$ by the method of Gel’fand**

The representations of the Lorentz subgroup $SO(1, 3)$ can either be labeled by $\tau = [l_0, l_1]$ or by $D(j_1, j_2)$. These labels are related by

$$l_0 = j_1 - j_2, \quad l_1 = j_1 + j_2 + 1,$$

with $j_1$ and $j_2$ being the eigenvalues of the Casimir operators of $SU(2) \times SU(2) \simeq SO(1, 3)$. The total angular momentum $l$ is constrained by

$$|j_1 - j_2| \leq l \leq j_1 + j_2, \quad \text{i.e.} \quad |l_0| \leq l \leq l_1 - 1.$$

Two representations $\tau = [l_0, l_1]$ and $\tau' = [l'_0, l'_1]$ are coupled by $X_\alpha$ when

$$[l'_0, l'_1] = [l_0 \pm 1, l_1] \quad (\text{type A}),$$

$$[l'_0, l'_1] = [l_0, l_1 \pm 1] \quad (\text{type B}).$$

We depicted them by the *interlocking* scheme:

$$\tau \longleftrightarrow \tau'.$$

Assume some irreducible Lorentz representations are given. Gel’fand et al. [3] p.271-277 have solved (2) for $X_\alpha$. They find the matrix elements of $X_0$ to be of the form

$$\begin{pmatrix} j'_1 \\ l' \\ m' \end{pmatrix} \begin{pmatrix} X_0 \\ j_1 \\ j_2 \\ l \\ m \end{pmatrix} = c^{\tau\tau'}_{lm; l'm'} = c^{\tau\tau'}_l \delta_{l'l} \delta_{mm'}.$$

For $[l'_0, l'_1] = [l_0 + 1, l_1]$ the matrices $c^{\tau\tau'}_l$ ($l = |l_0|, ..., l_1 - 1$) are given by

$$c^{\tau\tau'} = c^{\tau\tau'} \sqrt{(l + l_0 + 1)(l - l_0)},$$

$$c^{\tau\tau'} = c^{\tau\tau'} \sqrt{(l + l_0 + 1)(l - l_0)}.$$
and for $[l'_0, l'_1] = [l_0, l_1 + 1]$ by
\[
\begin{align*}
    c^\tau_{\tau'} &= c^{\tau'} \sqrt{(l + l_1 + 1)(l - l_1)}, \\
    c^\tau_{l'} &= c^\tau_{l'} \sqrt{(l + l_1 + 1)(l - l_1)},
\end{align*}
\]

and $c^\tau_{\tau'} = c^{\tau'}_{\tau} = 0$ for non-interlocking representations $\tau$ and $\tau'$. $c^{\tau\tau'}$ and $c^{\tau'\tau}$ are arbitrary complex numbers. The matrix elements of $X_1, X_2$ and $X_3$ can be derived straight-forwardly from $X_0$, see [3], p. 276f.

**Requirements on the Lorentz representations**

Which class of irreducible representations are suitable for the description of fermions? Gel’fand et al. [3] impose the following requirements on the Dirac-type equation (1):

a) It shall be invariant under space reflections. An irreducible representation of the complete Lorentz group induces a representation of the proper Lorentz group. This representation is either irreducible (Case I) or it breaks up into two irreducible pieces (Case II). In the first case we have $\tau = \hat{\tau}$, where $\hat{\tau} = \pm [l_0, -l_1]$ is the *conjugate* representation of $\tau$. In the second case, $\tau \oplus \tau'$, we have $\tau' = \hat{\tau}$ and the condition $c^{\tau\tau'} = c^{\hat{\tau} \hat{\tau}'}$ for the parameters in $X_0$.

b) There shall exist a non-degenerate invariant Hermitian form. This guarantees that Eq. (1) can be derived from a Lagrangian. One requires that $\tau = \tau^*$ or $\hat{\tau} = \tau^*$, where $\tau^* = [l_0, -l_1]$ is the adjoint representation of $\tau$. For the parameters $c^{\tau\tau'}$ we have the condition $c^{\tau\tau'} = \pm c^{\hat{\tau} \hat{\tau}'}$.

The requirements a) and b) impose constraints on the labels $l_0$ and $l_1$ of the representations $\tau = [l_0, l_1]$. They are satisfied by the representations

\[
[\frac{1}{2}, l_1] \oplus [-\frac{1}{2}, l_1], \quad l_1 \text{ real}. \quad (20)
\]

c) The particle shall have positive probability (positive “charge”), i.e.

\[
\int J_0 \, d^3 x = \int \bar{\Psi} X_0 \Psi \, d^3 x > 0, \quad (21)
\]

and energy of both signs in order to describe particles and antiparticles. Gel’fand’s method guarantees this by requiring $X_0$ to have eigenvalues $\pm 1$ for states corresponding to the spin of the particle and vanishing eigenvalues for lower spin components. This will be demonstrated in the following example.
4 Determination of $X_\alpha$ exemplified at a spin 5/2 field

Let us determine the matrix elements of $X_0$ for a fermion with spin 5/2. We follow Gel’fand et al. [3] who determined this matrix for a spin 3/2 particle. A spin 5/2 particle is described by the four representations $\tau_1 = \bar{\tau}_1 = [\frac{1}{2}, \frac{3}{2}]$, $\tau_2 = [\frac{1}{2}, \frac{5}{2}]$ and $\tau_3 = [\frac{1}{2}, \frac{7}{2}]$ and their conjugate representations. $\tau_3$ describes a composite system of particles with spin 1/2, 3/2 and 5/2. The representations $\tau_1$, $\bar{\tau}_1$ and $\tau_2$ are necessary to eliminate components with spin 1/2 and 3/2 which are introduced by $\tau_3$. Fig. 1 shows the interlocking scheme$^1$ of these representations. We indicate the presence of two representations of the same type by a double arrow.

We now want to determine the compartment matrices $c_{\tau\tau'}$ which form the Dirac-type matrix $X_0^{(j=5/2)}$, see Eq. (17). From the requirement of parity invariance we obtain:

\[ c_{\tau_1\bar{\tau}_1} = c_{\bar{\tau}_1\tau_1}, \quad c_{\tau_1\bar{\tau}_1} = c_{\bar{\tau}_1\tau_1}, \quad c_{\tau_2\bar{\tau}_2} = c_{\bar{\tau}_2\tau_2}, \quad c_{\tau_3\bar{\tau}_3} = c_{\bar{\tau}_3\tau_3}, \]
\[ c_{\tau_1\tau_1} = c_{\bar{\tau}_1\bar{\tau}_1}, \quad c_{\tau_1\tau_1} = c_{\bar{\tau}_1\bar{\tau}_1}, \quad c_{\tau_2\tau_2} = c_{\bar{\tau}_2\bar{\tau}_2}, \quad c_{\tau_3\tau_3} = c_{\bar{\tau}_3\bar{\tau}_3}, \]
\[ c_{\tau_2\bar{\tau}_1} = c_{\bar{\tau}_1\tau_2}, \quad c_{\tau_3\bar{\tau}_2} = c_{\bar{\tau}_2\tau_3}, \quad c_{\tau_2\bar{\tau}_1} = c_{\bar{\tau}_1\tau_2}, \quad c_{\tau_3\bar{\tau}_2} = c_{\bar{\tau}_2\tau_3} \]  \hspace{1cm} (22)

$^1$For simplicity, arrows indicating interlockings are replaced by lines.
From the requirement of the existence of a Hermitean form we get

\[
\begin{align*}
&c^{\tau_1\dot{\tau}_1} = \bar{c}^{\tau_1\dot{\tau}_1}, \quad c^{\bar{\tau}_1\dot{\tau}_1} = \bar{c}^{\bar{\tau}_1\dot{\tau}_1}, \\
&c^{\tau_2\dot{\tau}_2} = \bar{c}^{\tau_2\dot{\tau}_2}, \quad c^{\tau_3\dot{\tau}_3} = \bar{c}^{\tau_3\dot{\tau}_3}, \\
&c^{\tau_1\bar{\tau}_1} = \pm c^{\bar{\tau}_1\dot{\tau}_1}, \quad c^{\tau_2\tau_2} = \pm c^{\bar{\tau}_2\bar{\tau}_2}, \quad c^{\tau_3\tau_3} = \pm c^{\bar{\tau}_3\bar{\tau}_3}, \\
&c^{\bar{\tau}_1\tau_1} = \pm \bar{c}^{\bar{\tau}_1\dot{\tau}_1}, \quad c^{\bar{\tau}_2\tau_2} = \pm \bar{c}^{\bar{\tau}_2\bar{\tau}_2}, \quad c^{\bar{\tau}_3\tau_3} = \pm \bar{c}^{\bar{\tau}_3\bar{\tau}_3}.
\end{align*}
\]  

(23)

Using (17) we now compute the compartment matrices \(c^{\tau\tau'}\) for \(l = 1/2, 3/2, 5/2\) while taking into account the above relations between the parameters \(c^{\tau\tau'}\). Computer algebra yields [7]:

\[
\begin{bmatrix}
\tau_3 & \hat{\tau}_3 \\
0 & 3g \\
3g & 0
\end{bmatrix}, \quad c^{\tau\tau'}_{5/2} = \begin{bmatrix}
0 & 2e & \sqrt{\frac{5}{8}f} & 0 \\
2e & 0 & 0 & \sqrt{\frac{5}{8}f} \\
-\sqrt{\frac{5}{8}f} & 0 & 0 & 2g \\
0 & -\sqrt{\frac{5}{8}f} & 2g & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tau_2 & \hat{\tau}_2 & \tau_3 & \hat{\tau}_3 \\
0 & a & 0 & b & h & 0 & 0 & 0 \\
a & 0 & b & 0 & 0 & h & 0 & 0 \\
0 & -b & 0 & c & d & 0 & 0 & 0 \\
-b & 0 & c & 0 & 0 & d & 0 & 0 \\
h & 0 & d & 0 & 0 & e & f & 0 \\
0 & h & 0 & d & e & 0 & 0 & f \\
0 & 0 & 0 & -f & 0 & 0 & g & 0 \\
0 & 0 & 0 & 0 & -f & g & 0 & 0
\end{bmatrix},
\]

\(c^{\tau\tau'}_{1/2} = \)

where

\[
a = -\frac{1}{3}; b = \frac{1}{24} \sqrt{10}; c = \frac{1}{3}; d = \frac{9}{40} \sqrt{10}; e = -\frac{1}{3}; f = \frac{4}{15} \sqrt{10}; g = \frac{1}{3}; h := 0.
\]

We excluded particles with spin 1/2 and 3/2 by requiring additionally that all eigenvalues of the matrices \(c^{\tau\tau'}_{1/2}\) and \(c^{\tau\tau'}_{3/2}\) are zero. The eigenvalues of \(c^{\tau\tau'}_{5/2}\) must be \(\pm 1\) in order to have both particles and antiparticles with spin 5/2.
The compartment matrix \( c_{1/2}' \) contains eight parameters. Three of them, namely \( e, f \) and \( g \), are already fixed by the matrices \( c_{3/2}' \) and \( c_{5/2}' \). We can set one parameter equal to zero (here \( h = 0 \)) since the requirement of vanishing eigenvalues fixes only four parameters in the matrix \( c_{1/2}' \). If we had taken just one representation of the type \( \tau_1 = [\frac{1}{2}, \frac{3}{2}] \), we would have had only \( 5 - 3 = 2 \) free parameters in \( c_{1/2}' \). However, in this case 3 parameters will be fixed by the requirement of vanishing eigenvalues.

5 Lorentz representation of a fermionic particle

The method of Gel’fand et al. can be generalized for all fermions with spin \( j \). For that we have to use an irreducible Lorentz representation which contains \( j \) as highest spin value. We refer to this as the main representation. Thereby we also introduce other spin components which must be eliminated by a set of auxiliary representations. The following theorem helps us to find these representations:

**Theorem 1** The general interlocking scheme for a particle with arbitrary half-integral spin \( j \) reads

\[
\tau_1 \quad \cdots \quad \tau_{n-4} \quad \tau_{n-3} \quad \tau_{n-2} \quad \tau_{n-1} \quad \tau_n \quad \tau_{n+1}
\]

\[
\dot{\tau}_1 \quad \cdots \quad \dot{\tau}_{n-4} \quad \dot{\tau}_{n-3} \quad \dot{\tau}_{n-2} \quad \dot{\tau}_{n-1} \quad \dot{\tau}_n \quad \dot{\tau}_{n+1}
\]

where \( \tau_i = [\frac{1}{2}, i + \frac{1}{2}] \) and \( \dot{\tau}_i = [-\frac{1}{2}, i + \frac{1}{2}] \) (\( i = 1, ..., n + 1; n = j - \frac{1}{2} \)) are finite irreducible representations of the Lorentz group. Let us denote the corresponding representation by \( \rho_j \). The number \( M_i \) of vertical arrows between \( \tau_i \) and \( \dot{\tau}_i \) is the multiplicity with which they occur in \( \rho_j \).

Remarks:

i) The representations \( \rho_j \) satisfy the requirements a) - c) of Section 3, i.e. Eq. (1) is parity invariant, derivable from a Lagrangian and describes both particles and antiparticles with spin \( j \).

ii) The spin content of the main representation \( \tau_{n+1} = [\frac{1}{2}, j+1] \) is \( \frac{1}{2}, \frac{3}{2}, ..., j \), see Eq. (14). The other representations \( \tau_1, ..., \tau_n \) are needed to eliminate lower spin values such that only a particle with spin \( j \) remains.
iii) In Eq. (1) we take the field \( \Psi^{(j)} := (\psi^{(1)}, \ldots, \psi^{(i)}, \ldots, \psi^{(n+1)})^T \), where \( \psi^{(i)} \) 
\( (i = 1, \ldots, n+1) \) denotes a spinor with (sum over all \( j \) values, see ii) 
\[
\sum_{j=1/2}^{i-1/2} 2 \frac{(2j + 1)}{m-\text{degeneracy}} = \sum_{l=1}^{i} 4j = 2i(i + 1) \quad (24)
\]
components. We note that some spinors \( \psi^{(i)} \) occur several times in \( \Psi^{(j)} \) according to their multiplicities \( M_i \).

iv) The above interlocking scheme corresponds to the representation 
\[
\rho_j := D(\tfrac{1}{2}(n + 1), \tfrac{3}{2}n) \oplus D(\tfrac{1}{2}n, \tfrac{1}{2}(n + 1)) \oplus D(\tfrac{1}{2}n, \tfrac{1}{2}(n - 1)) \oplus D(\tfrac{1}{2}(n - 1), \tfrac{1}{2}n) \oplus 2[D(\tfrac{1}{2}(n - 1), \tfrac{1}{2}(n - 2)) \oplus D(\tfrac{1}{2}(n - 2), \tfrac{1}{2}(n - 1))] \oplus 2[D(\tfrac{1}{2}(n - 2), \tfrac{1}{2}(n - 3)) \oplus D(\tfrac{1}{2}(n - 3), \tfrac{1}{2}(n - 2))] \oplus 2[D(\tfrac{1}{2}(n - 3), \tfrac{1}{2}(n - 4)) \oplus D(\tfrac{1}{2}(n - 4), \tfrac{1}{2}(n - 3))] \oplus 3[D(\tfrac{1}{2}(n - 4), \tfrac{1}{2}(n - 5)) \oplus D(\tfrac{1}{2}(n - 5), \tfrac{1}{2}(n - 4))] \oplus \ldots \oplus M_1[D(\tfrac{1}{2}, 0) \oplus D(0, \tfrac{1}{2})].
\] (25)

We will now prove the theorem. The representation (25) shows that the multiplicities \( M_i \) do not follow a simple rule. The proof provides an algorithm for the determination of these multiplicities.

**Proof of the theorem**

First, let us assume that a diagram\(^2\) of the type of Theorem 1 is given and we want to verify whether it has the right number of multiplicities or not. We have to assure that there are enough parameters to fix in each compartment matrix since then we are able to set the parameters such that the eigenvalues of the compartment matrices are all zero and \( \pm 1 \) for the compartment matrix with the highest \( l \)-value, respectively.

This can be achieved in the following way. We write the highest \( l \)-value 
\( (= l_1 - 1) \) of each representation \( \tau_i \) next to it, see e.g. Fig. 1 and Fig. 2. Note

\(^2\) For the following we rotate it in 90° anti-clockwise.
that the partial diagram above an \( l \)-value contains all the information of the

number of parameters of the compartment matrix \( c_{i}^{\tau \tau'} \) with this \( l \)-value. It
determines the number of free parameters \( A_{l} \) and the number of parameters
\( B_{l} \) which will be fixed by the method of Gel’fand.

**Determination of \( A_{l} \):**

Observe that each interlocking gives rise to one parameter \( c_{\tau \tau'} \). Actually,
each interlocking gives rise to two parameters, \( c_{\tau \tau'} \) and \( c_{\tau' \tau} \), cf. Eqs. (18) and (19). However, for the class of representations given by Eq.(20), we have [3] p.320 \( \tau = \tau^{*}, \dot{\tau} = \dot{\tau}^{*} \) and thus \( c_{\tau \tau'} \) = \( \pm c_{\tau' \tau}^{*} \tau^{*} = \dot{c}_{\tau \tau}^{*} \) since we have to take into
account the requirement that our Dirac-type equation shall be derivable from
a Lagrangian, cf. [3] p.292. In other words, \( c_{\tau \tau'} \) and \( c_{\tau' \tau} \) are related. Thus
by counting the interlockings of a partial diagram we obtain the number \( A_{l} \)
of parameters in the compartment matrix \( c_{i}^{\tau \tau'} \).

The number of interlockings \( A_{l} \) can be obtained by counting the arrows
in a diagram. Horizontal arrows are weighted differently than vertical ones.
The following rules can be used to determine these weights.

**Rule 1 (vertical arrows):** Each vertical arrow is weighted by \( n \cdot m \),
whereby \( n \) and \( m \) are the multiplicities of the horizontal arrows which adjoin
it. Vertical arrows between dotted representations are weighted by zero.

Example: The following diagram shows a 2-fold and a 3-fold horizon-
tal arrow. We weight the vertical arrow by 6 since we get the parameters
\( c_{\tau_1 \tau_2}, c_{\tau_2 \tau_1}, c_{\tau_1' \tau_2}, c_{\tau_1' \tau_2}, c_{\tau_1'' \tau_1} \) and \( c_{\tau_1'' \tau_1} \), i.e. 6 interlockings of type B.

\[
\begin{array}{c c c}
\tau_2, \tau_2' & 6 & 0 \\
\tau_1, \tau_1', \tau_1'' & \hline & \hline
\end{array}
\]

Note that, because of parity invariance, we have \( c_{\tau \tau'} = c_{\dot{\tau} \dot{\tau}'} \) and therefore
we must not count arrows between dotted representations. This is why we
weight them by zero.

**Rule 2 (horizontal arrows):** The number of interlockings (of type A)
given by a \( n \)-fold arrow is

\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}.
\]
Example: Let us consider a three-fold arrow (represented by three lines). We simply count the mutual interlockings of the representations $\tau_i$ and $\dot{\tau}_i$.

Here we have altogether six parameters: three parameters $c^{\tau \tau}, c^{\tau \tau'}, c^{\tau \tau''}$, two parameters $c^{\tau' \tau}, c^{\tau' \tau''}$ and one parameter $c^{\tau'' \tau''}$ indicated by arrows with arrowheads in the above figure. Due to parity invariance ($c^{\tau \tau'} = c^{\tau \tau'}$) there are no further free parameters. A forth representation $\tau''''$ would interlock with each of the four conjugate representations $\dot{\tau}_i, \dot{\tau}_i', \dot{\tau}_i''$, and $\dot{\tau}_i'''$ and we would obtain four further interlockings. Clearly, a $n + 1$-fold arrow has $n + 1$ more interlockings than the $n$-fold one.

Finally, we apply

**Rule 3:** $A_l$ is the sum of the weights of all arrows in a partial diagram minus the numbers $B_{l'}$ with $l < l' \leq j$.

When we sum up all weights we get the number of interlockings and therewith the number of free parameters in the compartment matrix $c^{\tau \tau'}_l$. We have to subtract $B_{l'}$ ($\forall l' > l$) since these are the number of parameters which have already been fixed by the compartment matrices $c^{\tau \tau'}_l$ and are not at our disposal any more.

**Determination of $B_l$:**

**Rule 4:** $B_l$ is the number of horizontal arrows in a partial diagram — $n$-fold arrows are counted $n$ times.

This gives us the number of irreducible representations contributing to the compartment matrix $c^{\tau \tau'}_l$ and therewith its dimension $n = 2B_l$. The corresponding characteristic polynomial in $\lambda$ is then of order $n$ and has the form

$$P(\lambda) = \lambda^n + c_{n-2}\lambda^{n-2} + \cdots + c_2\lambda^2 + c_0. \quad (27)$$

Since the eigenvalues of $X_0$ (and therewith those of $c^{\tau \tau'}_l$) are $\pm\lambda_1, \pm\lambda_2, \ldots$ [2] p.144, the characteristic polynomial only contains even powers of $\lambda$. The constants $c_i$ depend on the parameters of $c^{\tau \tau'}_l$. In order to get vanishing eigenvalues, we set the $n/2$ constants $c_i = 0$ ($i = 0, 2, \ldots, n - 2$). These relations fix $n/2 = B_l$ parameters.
If for some \( l \) the number \( A_l \) of parameters which are at our disposal is less than the number of parameters \( B_l \) which will be fixed then the interlocking scheme fails. In this case we apply

**Rule 5:** Assume the multiplicity \( M_{l+\frac{1}{2}} \) of the representation \( \tau_{l+\frac{1}{2}} \) is \( n \). If \( A_l < B_l \), increase the multiplicity \( M_{l+\frac{1}{2}} \) in 1 by replacing the \( n \)-fold by a \( n + 1 \)-fold arrow in the diagram and check \( A_l \) and \( B_l \) again.

We can always introduce so many representations \( \tau_{l+\frac{1}{2}} \) until \( A_l \geq B_l \). Each new representation \( \tau_{l+\frac{1}{2}} \) increases \( B_l \) in one. However, \( A_l \) increases in

\[
\frac{(n + 1)((n + 1) + 1)}{2} + (n + 1) m - \frac{n(n + 1)}{2} - n m = n + 1 + m > 1,
\]

where \( n \) is the multiplicity of \( \tau_{l+\frac{1}{2}} \) and \( m \) that of \( \tau_{l+\frac{1}{2}+1} \), i.e. we can always achieve that \( A_l \geq B_l \).

**Algorithm for obtaining the multiplicities**

Now we are prepared to construct a diagram which has the right multiplicities. We start from an “empty” diagram, i.e. a diagram with simple arrows everywhere. This corresponds to \( j = \frac{1}{2} \) squares. Using rules 1 and 2, we write the weights next to each arrow and determine \( A_l \) and \( B_l \) according to rules 3 and 4. We begin at the top horizontal arrow of the diagram: \( A_{l=j} = 1 \) and \( B_{l=j} = 1 \) (o.k. since \( A_l \geq B_l \)). Next we evaluate the partial diagram for \( l = j - 1 \): \( A_{l=j-1} = 3 - 1 = 2 \) and \( B_{l=j-1} = 2 \) (o.k.). Then \( A_{l=j-2} = 5 - 2 - 1 = 2 \) and \( B_{l=j-2} = 3 \) (not o.k.). Therefore, we apply rule 5, i.e. we set \( M_{l+\frac{1}{2}=j-\frac{1}{2}} = 1 \rightarrow 2 \), and get \( A_{l=j-2} = 8 - 2 - 1 = 5 \) and \( B_{l=j-2} = 4 \) (o.k.). In this way we go ahead until we reach the bottom arrow.

As in the spin-\( \frac{3}{2} \)-case [3] p.347f, the energy has both signs and the charge is positive definite since the compartment matrix with the highest \( l \)-value can always be chosen to have eigenvalues \( \pm 1 \).

Result: By the introduction of enough auxiliary fields it is always possible to construct a wave equation (1) which fulfills the wished properties. □

In this proof we assumed in Rule 4 that all coefficients \( c_i \) of the characteristic polynom of a compartment matrix do not vanish \( (c_i \neq 0) \) and that they are all different \( (c_i \neq c_j) \). Of course, it might be that some \( c_i \) are zero in advance or that two or more \( c_i \) are equal. Then the relations fix less than \( n/2 \) parameters. However, the examples show that this is usually not the case. But this is difficult to prove. So, strictly speaking, we can only prove
that a scheme works, but not that another one fails. To prove the latter we have to compute also all characteristic polynomials and check whether there are $c_i$ which coincide or vanish.

As a final remark we mention that there exists a non-minimal solution for the multiplicities. In the appendix we prove that a representation $\rho_j$ with multiplicities $M_i = n + 2 - i$ for the representations $\tau_i \ (i = 1, \ldots, n + 1)$ can be used for the description of a particle with spin $j$. So the multiplicities in our minimal solution given by the above algorithm increase slower than linearly.

**Comparison with the approach of Singh and Hagen [20]**

The aim of our approach is the same as that of Singh-Hagen, though achieved by a completely different method: We have found a Dirac-type equation, derivable from a Lagrangian, which replaces the Rarita-Schwinger scheme of a fermion, see [20] Eq. (2). Singh-Hagen found a set of first-order differential
equations which does the same. Up to spin 9/2 particles both approaches use the same Lorentz representations, cf. the representation (25) with that in [20].

The interlocking scheme for a spin-11/2-particle is shown in Fig. 2. The number next to each arrow is the number of interlockings which are induced by it, use rules 1 and 2. The above described method yields three times the representation \( \tau_1 \) in contradiction to what Singh-Hagen [20] claim. If we took \( \tau_1 \) only twice, as they do, we would obtain \( A_{1/2} = 29 - 21 = 8 \) and \( B_{1/2} = 10 \) (\( A < B \), not o.k.). Therefore, we have to introduce a third \( \tau_1 \) representation and obtain \( A_{1/2} = 34 - 21 = 13 \geq B_{1/2} = 11 \) (o.k.).

6 “Reggeization”

We want to find the Lorentz representation of the resonances on hadronic Regge trajectories. These resonances can be classified by the group \( \overline{SL}(4, \mathbb{R}) \) [11]. When plotted in a Chew-Frautschi diagram, the Regge trajectories show a linear relation between the square of the mass \( M \) of a resonance and its spin \( J \),

\[
J = \alpha(0) + \alpha' M^2, \tag{29}
\]

where \( \alpha(0) \) sets the low-energy scale, about \( 1 \text{GeV} \), and \( \alpha' \) is the slope of the trajectories, about \( 0.9 \text{(GeV)}^{-2} \) (numerical values for the first three flavors).

The extra-ordinary linearity of these trajectories suggests that the higher spin resonances should rather be described as excitations of the lowest state of a multiplet than by independent wave equations. For such a description we define the “Regge” representation as the direct sum of the representations \( \rho_j \) given by Theorem 1,

\[
\rho := \bigoplus_{j = \frac{1}{2}}^{\infty} \rho_j. \tag{30}
\]

The corresponding infinite-component spinor is \( \Psi := (\Psi^{(1/2)}, \Psi^{(3/2)}, ...)^T \).

The representation \( \rho \) describes two exchange-degenerate Regge trajectories at once: the lowest state of the first one has spin \( \frac{1}{2} \), the other one spin \( \frac{3}{2} \). They obey the \( \Delta J = 2 \) rule, e.g. for spinors \( \{J\} = \{\frac{1}{2}, \frac{3}{2}, \ldots\} \) and \( \{J\} = \{\frac{3}{2}, \frac{5}{2}, \ldots\} \). We could also consider just one Regge trajectory. There is
Figure 3: \((j_1, j_2)\)-content of the Regge representation.

no crucial difference since the same irreducible Lorentz representations are used.

The irreducible representations in (25) are depicted in Fig. 3. All of them lie within the zone of non-trivial action of \(X_\alpha\). For example, the representation \(D\left(\frac{1}{2}, 0\right) \oplus D\left(0, \frac{1}{2}\right)\) is indicated by a filled and an open circle at \((j_1, j_2) = (\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\).

We can now apply the method of Gel’fand in order to determine the matrices \(X_0^{(j)}\) for each particle with spin \(j\). \(X_0^{(1/2)}\) is equal to \(\gamma_0\) used in the conventional Dirac equation. The \(X_0\) matrix (and therewith \(X_a (a = 1, 2, 3)\)) corresponding to the Regge representation \(\rho\) is of the blockdiagonal form

\[
X_0 = \begin{bmatrix}
X_0^{(1/2)} & & \\
& X_0^{(3/2)} & \\
& & X_0^{(5/2)} \\
& & & \ddots
\end{bmatrix}.
\]

Thus (1) becomes an infinite set of decoupled equations describing free Regge resonances.

We now couple the representations \(\rho_j\) in order to introduce spin excitations of the resonances. Physically such excitations of the spin can only be induced by an interaction force since the spin value does not change as long as
the particle only undergoes Lorentz transformations. Two neighbouring resonances on the Regge trajectories differ in their spin value in 2. We need an operator which interlocks the representations \( \tau = \left[ \frac{1}{2}, l_1 \right] \) and \( \tau' = \left[ \frac{1}{2}, l_1 + 2 \right] \). It turns out that this can be done by the shear operators of the group \( SL(4, \mathbb{R}) \). Since the latter can only act on \( SL(4, \mathbb{R}) \) manifolds \( \Psi \), we have to embed the Regge representation into a representation of this group.

7 (Non-)multiplicity-free representations of \( SL(4, \mathbb{R}) \)

7.1 Multiplicity-free representations

Can we embed the Regge representation into a multiplicity-free \((k_i = 0, i = 1, 2)\) representation of \( SL(4, \mathbb{R}) \)? The multiplicity-free representations are well known. They have been classified in [17]. We will not repeat this here, but we strongly recommend to study them before going ahead.

According to Harish-Chandra\(^3\) [4] the representations \( U(g) \), \( g \in G \) of a noncompact group \( G \) can be defined in a homogeneous Hilbert space \( H = \{ f(k) | k \in K \} \) over the maximal compact subgroup \( K \subset G \). Then \( U(g) \) is a continuous mapping from \( G \) into the set of linear transformations on \( H \) given by

\[
U(g)f(k) = \exp[\alpha(h(k, g))]f(k \cdot g),
\]

where \( g \in G \), \( k \in K \), \( e^h \in A \) and \( A \) is the Abelian subgroup. The maximal compact subgroup of \( SL(4, \mathbb{R}) \) is \( SO(4) \simeq SU(2) \times SU(2) \). After the application of the deunitarizing automorphism \( A \) [17], the eigenvalues of its Casimir operators, \( j_1 \) and \( j_2 \), can be identified with those of the Lorentz group since \( SO(4)_A \simeq SO(1, 3) \). Each representation of \( SL(4, \mathbb{R}) \) contains Lorentz submultiplets \((j_1, j_2)\). All these submultiplets are called the \((j_1, j_2)\)-content of a \( SL(4, \mathbb{R}) \) representation.

The Lorentz \((j_1, j_2)\) submultiplets can be excited by means of the shear operator \( Z_{\alpha \beta} \) \((\alpha, \beta = 0, \pm 1)\), which is in its turn a \((1, 1)\) irreducible tensor operator of the Lorentz group. From its matrix representation in the general case, see (34) below, we deduce that its action can be visualized by a ‘Union

\(^3\)For a summary of the representation theory of noncompact groups developed by Harish-Chandra see also [16] Sec. 3.
forbidden

Figure 4: \((j_1, j_2)\)-content of some multiplicity-free representations of \(SL(4, \mathbb{R})\).

Jack', for details see [5] Ch. 4.5. In Fig. 5 this is demonstrated for the point \((7/2, 1)\). Due to the properties of the 3-j-symbols in the multiplicity-free case, we just have ‘×’-like transitions between Lorentz submultiplets such that the lattice is divided into eight sublattices [17] Fig. 1. Four of them, \(L_5, L_6, L_7\) and \(L_8\), could be relevant for the embedding of the Regge representation. They are drawn in Fig. 4. However, not all of their Lorentz submultiplets belong to an invariant lattice, i.e. to a multiplicity-free representation of \(SL(4, \mathbb{R})\). We crossed them out in Fig. 4. By comparison with Fig. 3 we see that the irreducible Lorentz representations \(D(n, n - 1/2) \oplus D(n - 1/2, n)\) \((n = 1, 2, ...\) of the Regge representation cannot be embedded into any of the multiplicity-free representations of \(SL(4, \mathbb{R})\). Only those with \(n = 0, 2, 4, ...\) are contained in the lattices \(L_5\) and \(L_6\) and could be embedded into the \(SL(4, \mathbb{R})\) representation \(D^{\text{disc}}(\frac{1}{2}, 0)_A \oplus D^{\text{disc}}(0, \frac{1}{2})_A\).

Moreover, we face another problem with multiplicity-free representations of \(SL(4, \mathbb{R})\). It is shown in [1] App. A that no multiplicity-free representation (except for the sum of ladder representations which are of no use here) admits an \(SL(4, \mathbb{R})\) vector, i.e. \((\frac{3}{2}, \frac{1}{2})\), operator \(\tilde{X}_\alpha\).

Indeed, for finite (tensorial) representations this can easily be seen by using Young tableaux. The tensor product of \(\tilde{X}_\alpha\), represented by \(\Box\), and a multiplicity-free (all are ladder type) representation results in the sum of a
multiplicity-free and a non-multiplicity-free representation,

\[
\begin{array}{c}
\text{multiplicity-free} \otimes \text{multiplicity-free} = \text{multiplicity-free} \oplus \text{non-multiplicity-free}
\end{array}
\]  \hspace{2cm} (33)

Consequently, the application of a $\mathcal{SL}(4, \mathbb{R})$ vector operator $\tilde{X}_\alpha$ naturally leads to non-multiplicity-free representations. In the case of spinorial (infinite-dimensional) representations, we point out two relevant facts: (i) these representations are not of the ladder type, and (ii) the tensor product of the vector representation ($\tilde{X}_\alpha$) and a multiplicity-free spinorial irreducible representation does not contain any representation of the latter type. Thus, it is not possible to restrict on multiplicity-free representations alone.

### 7.2 Non-multiplicity-free representations

Some results for the general case can be found in [18, 19]. Here the representations are non-multiplicity-free, i.e.
the label $k_i \neq 0$ ($i = 1, 2$). The generators of $\mathcal{SL}(4, \mathbb{R})$, the Lorentz and shear generators, $M_{\alpha\beta}$ and $T_{\alpha\beta}$, can be replaced by the spherical tensors $J^{(1)}_{\alpha}$, $J^{(2)}_{\alpha}$, and $Z_{\alpha\beta}$ ($\alpha, \beta = 0, \pm 1$) [17].

The matrix elements of the $SU(2)$ generators $J^{(1)}_{\alpha}$ and $J^{(2)}_{\alpha}$ are well known from angular momentum theory. The matrix elements of the shear generators $Z_{\alpha\beta}$ ($\alpha, \beta = 0, \pm 1$) read [18]

\[
\langle j_1' j_2' \mid Z_{\alpha\beta} \mid j_1 j_2 \rangle = (-1)\hat{j}_1' - \hat{m}_1' \times \hat{j}_2' - \hat{m}_2' \times \langle j_1' j_2' \mid j_1 j_2 \rangle \times (34)
\]

with the reduced matrix element

\[
\langle j_1' j_2' \mid Z \mid j_1 j_2 \rangle = \frac{(-1)^{\hat{j}_1' - k_1'}(-1)^{\hat{j}_2' - k_2'}}{\sqrt{(2\hat{j}_1' + 1)(2\hat{j}_2' + 1)(2j_1 + 1)(2j_2 + 1)}} \times \left[ e + 4 - j_1' (j_1' + 1) + j_1 (j_1 + 1) - j_2' (j_2' + 1) + j_2 (j_2 + 1) \right] \\
\times \left( \begin{array}{cc}
\hat{j}_1' & 1 \\
-k_1' & 0
\end{array} \right) \left( \begin{array}{cc}
\hat{j}_2' & 1 \\
-k_2' & 0
\end{array} \right) \times \left( \begin{array}{cc}
1 & j_1 \\
0 & k_1
\end{array} \right) \left( \begin{array}{cc}
1 & j_2 \\
0 & k_2
\end{array} \right)
\]
In the Appendix we relate the 15 generators $L_{\alpha\beta} = M_{\alpha\beta} + T_{\alpha\beta}$ to the spherical tensors $J^{(1)}_{\alpha}$, $J^{(2)}_{\alpha}$ and $Z_{\alpha\beta}$ ($\alpha, \beta = 0, \pm 1$).

Note some differences to the multiplicity-free case. Since the operator $Z_{\alpha\beta}$ induces ‘$\times$’-like and ‘$+$’-like transitions between Lorentz submultiplets (‘Union Jack’), we just have four sublattices. Two of them, $L_1(1/2, 0)$ and $L_2(0, 1/2)$, which are important for the embedding, are depicted in Fig. 5. Since a state is characterized by $|j_1 j_2 k_1 k_2\rangle$ and not just by $|j_1 j_2\rangle$ (quantum numbers $m_1$ and $m_2$ are ignored), we should keep in mind that we actually deal with a four-dimensional lattice. Therefore, each dot in Fig. 5 can represent more than one Lorentz submultiplet. The small-printed number next to each dot is the multiplicity of the Lorentz subrepresentation $D(j_1, j_2)$.

\begin{align*}
- (c + k_1 - k_2) & \begin{pmatrix}
  j_1' & 1 & j_1 \\
  -k_1' & 1 & k_1
\end{pmatrix}
\begin{pmatrix}
  j_2' & 1 & j_2 \\
  -k_2' & 1 & k_2
\end{pmatrix} \\
- (c - k_1 + k_2) & \begin{pmatrix}
  j_1' & 1 & j_1 \\
  -k_1' & -1 & k_1
\end{pmatrix}
\begin{pmatrix}
  j_2' & 1 & j_2 \\
  -k_2' & 1 & k_2
\end{pmatrix} \\
+ (d + k_1 + k_2) & \begin{pmatrix}
  j_1' & 1 & j_1 \\
  -k_1' & 1 & k_1
\end{pmatrix}
\begin{pmatrix}
  j_2' & 1 & j_2 \\
  -k_2' & 1 & k_2
\end{pmatrix} \\
+ (d - k_1 - k_2) & \begin{pmatrix}
  j_1' & 1 & j_1 \\
  -k_1' & -1 & k_1
\end{pmatrix}
\begin{pmatrix}
  j_2' & 1 & j_2 \\
  -k_2' & -1 & k_2
\end{pmatrix}
\end{align*}

In the Appendix we relate the 15 generators $L_{\alpha\beta} = M_{\alpha\beta} + T_{\alpha\beta}$ to the spherical tensors $J^{(1)}_{\alpha}$, $J^{(2)}_{\alpha}$ and $Z_{\alpha\beta}$ ($\alpha, \beta = 0, \pm 1$).

Note some differences to the multiplicity-free case. Since the operator $Z_{\alpha\beta}$ induces ‘$\times$’-like and ‘$+$’-like transitions between Lorentz submultiplets (‘Union Jack’), we just have four sublattices. Two of them, $L_1(1/2, 0)$ and $L_2(0, 1/2)$, which are important for the embedding, are depicted in Fig. 5. Since a state is characterized by $|j_1 j_2 k_1 k_2\rangle$ and not just by $|j_1 j_2\rangle$ (quantum numbers $m_1$ and $m_2$ are ignored), we should keep in mind that we actually deal with a four-dimensional lattice. Therefore, each dot in Fig. 5 can represent more than one Lorentz submultiplet. The small-printed number next to each dot is the multiplicity of the Lorentz subrepresentation $D(j_1, j_2)$.
Determination of the multiplicities

We want to find the multiplicities of the Lorentz submultiplets of $SL(4,\mathbb{R})$ representations. As an example, let us determine those of the lattice $L_1(\frac{1}{2}, 0)$. From the properties of the 3-j-symbols in the matrix representation of $Z_{\alpha \beta}^\alpha$, we know that $k'_1 - k_1 = \pm 1$ and $k'_2 - k_2 = \pm 1$. This allows ‘×’-like transitions in the $k_1$-$k_2$-lattice. It can thus be divided into eight sublattices in an analogous way as the $j_1$-$j_2$-lattice was divided in the multiplicity-free case.

We now choose two $k_1$-$k_2$-lattices such that they would form the lattice $L_1(\frac{1}{2}, 0)$, if the lattices were a $j_1$-$j_2$-lattice instead of $k_1$-$k_2$-ones. Thus the two relevant $k_1$-$k_2$-lattices are those shown in Fig. 6: one is represented by open circles, the other one by closed circles.

Now, we can ask which $(j_1, j_2)$ submultiplets of $L_1(\frac{1}{2}, 0)$ contain a specific pair $(k_1, k_2)$. In other words, we want to determine the number of states

$$|j_1 \quad j_2\rangle$$

for a given pair $(k_1, k_2)$. Hereto we have to remember the conditions $j_1 \geq |k_1|$ and $j_2 \geq |k_2|$. This means that $(k_1, k_2)$ determines the minimal value of a sublattice in the $j_1$-$j_2$-lattice in which all $(j_1, j_2)$ submultiplets contain $(k_1, k_2)$. In Fig. 7 we show two examples: the $(j_1, j_2)$-sublattices for $(k_1, k_2) = (1/2, 0)$ and $(3/2, 1)$. 

Figure 6: Two of eight $k_1$-$k_2$-lattices.
In order to determine the number of a certain Lorentz submultiplet, i.e. the multiplicity of \((j_1, j_2)\), in principle, we have to determine the sublattices of the type as in Fig. 7 for all pairs \((k_1, k_2)\) of the \(k_1\)-\(k_2\)-lattices shown in Fig. 6. Then we count the number of sublattices which contain this \((j_1, j_2)\) value. For short, we can also consider just \((k_1, k_2) = (j_1, j_2)\) in the \(k_1\)-\(k_2\)-lattice and count all the circles which lie inside the rectangle with the edges \((k_1, k_2) = \{(0,0), (j_1, 0), (0, j_2), (j_1, j_2)\}\) since all of them lead to \((j_1, j_2)\)-sublattices which contain this specific \((j_1, j_2)\) value. In Fig. 6 this is shown for \((j_1, j_2) = (7/2, 3)\). Its multiplicity is thus 16. This is the small-printed number next to the component \((7/2, 3)\) in Fig. 5.

We end up with a simple formula for the multiplicity \(m\) of a Lorentz submultiplet \((j_1, j_2)\),

\[
m = (j_1 + a) \times (j_2 + b),
\]

(36)

where \(a = b = \frac{1}{2}\) for half-integral and \(a = b = 1\) for integral \(j_1, j_2\) values.
8 Embedding of a Regge representation in a $\mathbb{SL}(4, \mathbb{R})$ representation

The $(j_1, j_2)$-content of the Regge representation is shown in Fig. 3. For its embedding we need a series of $\mathbb{SL}(4, \mathbb{R})$ which contains the $j_1$-$j_2$-lattices $L_1(\frac{1}{2}, 0)$ and $L_2(0, \frac{1}{2})$, see Fig. 5. The possible values of the complex representation labels $c, d, e$ in (34) are [18, 19]

\begin{align*}
A) & \quad c_1 = 0, \quad e_2 \in \mathbb{R}, \\
B_1) & \quad d_1 = 0, \quad d_2 \in \mathbb{R}, \\
B_2) & \quad d_1 = k_1 + k_2, \quad d_2 = 0; \quad k_1 + k_2 = \frac{1}{2}, \frac{3}{2}, \ldots, \\
B_3) & \quad 0 < d_1 < 1, \quad d_2 = 0; \quad k_1 + k_2 = 0, \pm 2, \pm 4, \ldots, \\
B_4) & \quad 0 < d_1 < \frac{1}{2}, \quad d_2 = 0; \quad k_1 + k_2 \equiv \frac{1}{2} \text{(mod 2)} \text{ or } \frac{3}{2} \text{(mod 2)}, \\
C_1) & \quad c_1 = 0, \quad c_2 \in \mathbb{R}, \\
C_2) & \quad c_1 = k_1 - k_2, \quad c_2 = 0; \quad k_1 - k_2 = \frac{1}{2}, \frac{3}{2}, \ldots, \\
C_3) & \quad 0 < c_1 < 1, \quad c_2 = 0; \quad k_1 - k_2 = 0, \pm 2, \pm 4, \ldots, \\
C_4) & \quad 0 < c_1 < \frac{1}{2}, \quad c_2 = 0; \quad k_1 - k_2 \equiv \frac{1}{2} \text{(mod 2)} \text{ or } \frac{3}{2} \text{(mod 2)}.
\end{align*}

These are chosen such that the representations are unitary and that there exists a positive scalar product. A series of $\mathbb{SL}(4, \mathbb{R})$ is fixed by any combination of $(A)$, $(B_i)$ and $(C_j) (i, j = 1, 2, 3, 4)$. For each series one can determine the $k_1$-$k_2$-sublattices. In principle, there are eight lattices

\begin{align*}
L_1 & = L(0, 0), \quad L_2 = L(\frac{1}{2}, \frac{1}{2}), \quad L_3 = L(0, 1) = L(1, 0), \\
L_4 & = L(\frac{1}{2}, \frac{3}{2}) = L(\frac{3}{2}, \frac{1}{2}), \quad L_5 = L(\frac{1}{2}, 0), \quad L_6 = L(0, \frac{1}{2}), \\
L_7 & = L(0, \frac{3}{2}), \quad L_8 = L(\frac{3}{2}, 0).
\end{align*}

In Fig. 8 only the minimal values $(k_1, k_2)$ of these lattices are plotted. All other points of the $k_1$-$k_2$-lattices can be obtained by performing ‘×’-like transitions starting from the minimal values $(k_1, k_2)$. For the combination $AB_1C_1$, e.g., we have neither restrictions on $k_1$ nor on $k_2$. Thus all eight lattices are allowed, see the first diagram in the upper left corner of Fig. 8. While for the series $AB_1C_i$ and $AB_2C_1$ ($i = 2, 3, 4$) there is just one constraint, for the remaining series $k_1$ and $k_2$ have to satisfy two constraints.

Knowing the allowed $k_1$-$k_2$-lattices, we can determine the $(j_1, j_2)$-content. Each point $(k_1, k_2)$ denotes all allowed $(j_1, j_2)$, i.e. $j_1 \geq |k_1|$ and $j_2 \geq |k_2|$.
Figure 8: The $k_1$-$k_2$-lattice can be divided into eight sublattices. The 16 diagrams show the possible sublattices for each series.
Altogether we find nine series, cf. Fig. 8, which admit the \( k_1 \)-\( k_2 \)-lattices \( L_5, L_6, L_7, \) and \( L_8 \). These lattices lead to the relevant \( j_1 \)-\( j_2 \)-lattices \( L_1(1/2,0) \) and \( L_2(0,1/2) \) of Fig. 5. For example, we could choose the so called principal series - the combination \( AB_1C_1 \):

\[
\pi = D_{\text{prin}}^{\text{SL}(4,\mathbb{R})}(c_2, d_2, e_2; (\frac{1}{2}, 0)) \oplus D_{\text{prin}}^{\text{SL}(4,\mathbb{R})}(c_2, d_2, e_2; (0, \frac{1}{2})).
\] (39)

However, each series, corresponding to one of the combination \( AB_iC_j \) \((i, j \neq 3) \) (9 possibilities), can be taken for the embedding.

9 Dirac-type field equations, minimal coupling of gravity and symmetry breaking

In this final section we want to review the steps toward affine generalization of the Dirac equation as well as its coupling to gravity. Furthermore, we propose a spontaneous symmetry breaking scenario of the \( \mathcal{SA}(4,\mathbb{R}) \) gauge symmetry down to the Poincaré one.

We started in flat 4-dimensional Minkowski spacetime. In Sections 3 to 5 we showed how Gel’fand’s method to derive gamma matrices can be generalized to obtain Dirac-type equations for fermions with arbitrary spin \( j \),

\[
(i\rho^{\alpha\beta}X^{(j)}(\partial_{\beta} - m^{(j)}) \Psi^{(j)} = 0.
\] (40)

The matrix \( X^{(j)}_\alpha \) can be constructed by applying Gel’fand’s method to the representation \( \rho_j \) given by Eq. (25).

In Section 6 we summed up these representations over all half-integral spin values, cf. Eq. (30), in order to describe systems such as two exchange-degenerate Regge trajectories. Spin excitations of the Regge resonances can then be introduced by minimal coupling of the Christoffel-type connection of Chromogravity. This connection is in its turn given in terms of the chromometric field \( G_{\alpha\beta} \), i.e. in the anholonomic notation it reads

\[
\Gamma_{\beta\gamma}^{(i)} = \frac{1}{2} G^{\alpha\delta}(\partial_{\gamma}G_{\beta\delta} + \partial_{\delta}G_{\gamma\delta} - \partial_{\delta}G_{\beta\gamma}).
\] (41)

The corresponding curved space Dirac-type equation is given by

\[
(iX^{\alpha}e_i^\alpha D_i - \kappa) \Psi = 0,
\] (42)
with the holonomic covariant derivative defined by

\[ D_i = \partial_i + \Gamma_{i\alpha\beta}^{(1)} L^{\alpha\beta}. \]  

(43)

This equation is invariant with respect to local Poincaré transformations.

Since the Regge resonances can be classified by the group $\overline{SL}(4, \mathbb{R})$, in Section 8 we embedded the Regge representation $\rho$ into a suitable representation of $\overline{SL}(4, \mathbb{R})$. Note that the spin content of a genuine world spinor field is described by the $\overline{SL}(4, \mathbb{R})$ representations as well. Formally, we are now allowed to replace the Lorentz spinor $\Psi$ in (42) by the manifold $\Psi^{(SL)}$ spanning the representation space of a representation of the series $\pi$ defined in (39). Thus, we obtain a manifold description that is suitable for either an effective baryonic field of Regge recurrences or for a world spinor field of affine gravity.

As argued in Section 2, in a completely affine wave equation the mass term vanishes, i.e. the equation has to be of the form

\[ i\tilde{X}^\alpha e_i^\alpha D_i \Psi^{(SL)} = 0 \]  

(44)

with the $\overline{SL}(4, \mathbb{R})$ vector operator $\tilde{X}_\alpha$ defined by (4). In the gravity case, the covariant derivative $D_i$ now contains a full affine connection which we take from metric-affine gravity (MAG) [5].

Note that in an equation of the form of Eq. (42) we have not specified the mass term $\kappa$ so far. In order to gain (42) from (44), we propose, along the lines of Ref. [13] a symmetry breaking scenario of the $\overline{SA}(4, \mathbb{R})$ which preserves the Poincaré symmetry. It is the minimal spontaneous symmetry breaking scheme in which, besides the infinite-component $\Psi(x)$ field, we introduce an additional 10-component second-rank symmetric $\overline{SL}(4, \mathbb{R})$ field $\varphi_{\alpha\beta}(x)$. The $\varphi_{\alpha\beta}$ field is the minimal field that (i) has non-trivial $\overline{SL}(4, \mathbb{R})$ transformation properties and (ii) it contains a Lorentz scalar component, $\varphi^{(0,0)}(x) = \eta^\alpha^\beta \varphi_{\alpha\beta}(x)$, thus preserving the Lorentz symmetry in the process of spontaneous breaking of the $\overline{SL}(4, \mathbb{R})$ symmetry. The Lorentz decomposition of the $\varphi_{\alpha\beta}(x)$ field is $\varphi_{\alpha\beta}(x) = \varphi_{\alpha\beta}^{(0,0)}(x) + \varphi_{\alpha\beta}^{(1,1)}(x)$, where $\varphi_{\alpha\beta}^{(1,1)}(x)$ is the traceless 9-component field.

We consider the Lagrangian

\[ \mathcal{L} = \mathcal{L}_{MAG} + \overline{\Psi} \tilde{X}^\alpha e_i^\alpha D_i \Psi^{(SL)} + \frac{1}{2} \eta^\alpha^\beta e_i^\alpha e_j^\beta (D_i \varphi^{\gamma\delta})(D_j \varphi^{\gamma\delta}) \]

\[ - \mu_M \frac{\overline{\Psi} \varphi^{\gamma\delta} \varphi_{\gamma\delta} \Psi^{(SL)}}{2} - \frac{\mu^2}{2} \varphi^{\gamma\delta} \varphi_{\gamma\delta} - \frac{\lambda}{4} (\varphi^{\gamma\delta} \varphi_{\gamma\delta})^2, \]

(45)
which describes manifold $\Psi$, 10-component field $\varphi_{\alpha\beta}$, their mutual interaction, as well as their affine gravity interactions. Here $\varphi_{\alpha\beta}$ interacts with the manifold with strength $\mu_M$ and $\mathcal{L}_{\text{MAG}}$ is the most general MAG Lagrangian given by Eq. (10) in [6]. Provided $\mu^2 < 0$, one finds a non-trivial vacuum expectation value determined by

$$\lambda \langle 0 | \varphi^{\gamma\delta} \varphi_{\gamma\delta} | 0 \rangle + \mu^2 = 0. \quad (46)$$

We perform a suitable $SL(4, \mathbb{R})$ transformation in the space of field components, such that $\varphi_{\gamma\delta} = \varphi_{(0,0)\gamma\delta}$, and obtain the nontrivial vacuum expectation value for the Lorentz scalar component, $v \equiv \langle 0 | \varphi^{(0,0)} | 0 \rangle = \sqrt{-\mu^2/\lambda}$.

Taking $\varphi_{\alpha\beta}(x) = (v + \chi^{(0,0)}(x)) \eta_{\alpha\beta} + \varphi_{\alpha\beta}^{(1,1)}(x)$, we find a massive scalar field $\chi^{(0,0)}$, and a set of 9 massless Goldstone fields $\varphi_{\alpha\beta}^{(1,1)}$, while the spinorial manifold acquires the mass as well, i.e.

$$m(\chi^{(0,0)}) = \sqrt{-2\mu^2}, \quad m(\varphi^{(1,1)}) = 0, \quad m(\Psi) = \mu_M v^2 = \kappa. \quad (47)$$

Let us parametrize now $\varphi_{\alpha\beta}$ as follows,

$$\varphi_{\alpha\beta}(x) = (v + \chi^{(0,0)}(x)) \eta_{\alpha\beta} + \varphi_{\alpha\beta}^{(1,1)}(x) e^{i \chi^{(1,1)}_{\gamma\delta} T_{\gamma\delta}}, \quad (48)$$

where $T_{\gamma\delta}$ are the shear generators. After the gauge transformation $U = \exp(-i \chi^{(1,1)}_{\gamma\delta} T_{\gamma\delta})$, the connection fields become (infinitesimally)

$$\Gamma'_{i(\alpha\beta)} = \Gamma_{i(\alpha\beta)} - \frac{1}{v} \partial_i \chi^{(1,1)}_{\alpha\beta}, \quad (49)$$

while the 9 Goldstone fields $\chi^{(1,1)}_{\alpha\beta}$ get absorbed by the symmetric part of the connection $\Gamma_{i(\alpha\beta)}$ which is associated with nonmetricity. The latter in turn becomes massive, i.e. $M(\Gamma_{i(\alpha\beta)}) \neq 0$. The antisymmetric part of the connection, which is associated with spin, remains massless, i.e. $M(\Gamma_{[i(\alpha\beta)]}) = 0$.

We can, furthermore, make use of the nonlinear symmetry realizations and find explicitly matrix elements of the Lorentz vector $X^\alpha_{AB}$ in terms of matrix elements of the $SL(4, \mathbb{R})$ vector $\tilde{X}^\alpha_{\tilde{A}B}$, i.e.,

$$X^\alpha_{AB} \equiv E^C_A X^\alpha_{CD} E^D_B, \quad \tilde{X}^\tilde{A}_{\tilde{B}} = \exp(i \chi^{(1,1)}_{\alpha\beta} T_{\alpha\beta}) \tilde{X}^\tilde{A}_{\tilde{B}},$$

$$\Psi_A = E^\tilde{A}_A \Psi_{\tilde{A}}. \quad (50)$$
where $E^A_B$, is the nonlinear symmetry realizer. The (tracefree part of the) MAG-metric tensor $g_{\alpha\beta}$ can be defined from the Goldstone fields $\chi^{(1,1)}_{\alpha\beta}$ as

$$g_{\alpha\beta} := r^\mu \alpha r^\nu \beta \eta_{\mu\nu}, \quad r^\mu \alpha := \exp\left(\frac{i}{2} \chi^{(1,1)}_{\alpha\beta} T^{\alpha\beta}\right)^\mu \alpha \quad (51)$$

as suggested by the nonlinear realization of the local affine group [21].

To summarize, we break spontaneously the $\text{SL}(4, \mathbb{R})$ symmetry down to the Lorentz symmetry, the fermionic fields acquire nontrivial mass, and all quantities of an equation of the form given by Eq. (42) are explicitly given in terms of the quantities of Eq. (44).

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**Appendix**

**Transition from spherical to Cartesian tensors**

It is often useful to relate the Cartesian generators $L_{\alpha\beta} = M_{\alpha\beta} + T_{\alpha\beta}$ of $\text{SL}(4, \mathbb{R})$ to the spherical tensors $J^{(1)}_\alpha, J^{(2)}_\alpha$ and the double tensor $Z_{\alpha\beta}$ ($\alpha, \beta = 0, \pm 1$). The inverse of Eq. (2.3) in [17] yields the generators of the maximal compact subgroup $\text{SO}(4)$,

$$M_{ab} = \varepsilon_{abc}(J^{(1)}_c + J^{(2)}_c), \quad (52)$$

$$T_{0a} = J^{(1)}_a - J^{(2)}_a. \quad (53)$$

The relation between the spherical vector $J_{0,\pm}$ and the Cartesian vector $J_a$ are well-known.

We decompose the double tensor $Z_{\alpha\beta}$ of rank $(1, 1)$ with respect to the rotation group, $\text{SO}(4) \supset \text{SO}(3)$, $D^{(1)} \times D^{(1)} = D^{(0)} \oplus D^{(1)} \oplus D^{(2)}$, and obtain the three corresponding tensors

$$Z^{(k)}_\gamma = \sum_{\alpha,\beta} Z_{\alpha\beta}(11\alpha\beta|11k\gamma), \quad (54)$$
cf. Eq. (35.2) in [22], with rank $k = 0, 1, 2$ ($\gamma = -k, ..., +k$) and the Clebsch-Gordon coefficient $(11\alpha\beta|11k\gamma)$. The tensors $Z^{(0)}_\gamma$, $Z^{(1)}_\gamma$, and $Z^{(2)}_\gamma$ have 1, 3, and 5 independent components which we now relate to the Cartesian tensor $Z_{ab}$,

\[ Z_{31} = -\frac{1}{2}(Z^{(2)}_{+1} - Z^{(2)}_{-1} + Z^{(1)}_{+1} + Z^{(1)}_{-1}), \]
\[ Z_{13} = -\frac{1}{2}(Z^{(2)}_{+1} - Z^{(2)}_{-1} - Z^{(1)}_{+1} - Z^{(1)}_{-1}), \]
\[ Z_{23} = \frac{i}{2}(Z^{(2)}_{+1} + Z^{(2)}_{-1} + Z^{(1)}_{+1} - Z^{(1)}_{-1}), \]
\[ Z_{32} = \frac{i}{2}(Z^{(2)}_{+1} + Z^{(2)}_{-1} - Z^{(1)}_{+1} + Z^{(1)}_{-1}), \]
\[ Z_{12} = -\frac{i}{2}(Z^{(2)}_{+2} - Z^{(2)}_{-2} + \sqrt{2}Z^{(1)}_{0}), \]
\[ Z_{21} = -\frac{i}{2}(Z^{(2)}_{+2} - Z^{(2)}_{-2} - \sqrt{2}Z^{(1)}_{0}), \]
\[ Z_{11} = \frac{1}{2}(Z^{(2)}_{+2} + Z^{(2)}_{-2}) - \frac{1}{\sqrt{6}}Z^{(2)}_{0} - \frac{1}{\sqrt{3}}Z^{(0)}_{0}, \]
\[ Z_{22} = \frac{1}{2}(Z^{(2)}_{+2} + Z^{(2)}_{-2}) - \frac{1}{\sqrt{6}}Z^{(2)}_{0} - \frac{1}{\sqrt{3}}Z^{(0)}_{0}, \]
\[ Z_{33} = \frac{2}{\sqrt{6}}Z^{(2)}_{0} - \frac{1}{\sqrt{3}}Z^{(0)}_{0}. \] (55)

$Z_{ab}$ is related to the spatial shear tensor $T_{ab}$ and to the boosts $M_{0c}$ according to

\[ Z_{ab} = T_{ab} + \varepsilon_{abc}M_{0c}. \] (56)

**Non-minimal solution for the multiplicities**

For the multiplicities $M_i = n + 2 - i$ of the representations $\tau_i$ $(i = 1, ..., n+1)$, we have to show that $A_l \geq B_l$ for all $l = \frac{1}{2}, ..., j$.

**Proof by induction:** Since $A_j = B_j = 1$, $A_l \geq B_l$ for $l = j$. Now, assume $A_l \geq B_l$. Using Rules 1 to 4, we obtain

\[ A_{l-1} = A_l + M_l(M_l + 1) + \frac{(M_l + 1)(M_l + 2)}{2} - B_l \]
\[ \geq (M_l + 1)(\frac{3}{2}M_l + 1) \] (57)
\[ B_{l-1} = B_l + M_l + 1 \] (58)
with \( M_i \) being the multiplicity of \( \tau_i, i = l + 1/2 \), and \( M_{i-1} = M_i + 1 \) the multiplicity of \( \tau_{i-1} \).

Now \( A_{l-1} \geq B_{l-1} \) follows since

\[
(M_i + 1) \frac{3}{2} M_i \geq B_i = \sum_{k=i}^{n+1} M_k = \frac{1}{2} M_i (M_i + 1). \tag{59}
\]

References


